Asymptotic Relationships between Systems of Volterra Equations.

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Summary. - *Stability type results concerning continuous solutions of perturbed systems of Volterra equations as well as theorems concerning the asymptotic equivalence of such perturbed and unperturbed systems are established.*

1. **Introduction.** - Consider the perturbed system of VOLTERRA equations

$$(1.1) \quad x(t) = f(t) - \int_0^t a(t, s)[x(s) + g(s, x(s))] ds \quad (t \geq 0),$$

where x, f, g are n -component vector functions, a is a n by n matrix, and the perturbation g is regarded as small in various senses; in all cases $g(t, x) = o(|x|)$ uniformly in t as $|x| \rightarrow 0$ will be sufficient, except in Theorem 4. Our purpose is to compare in various ways solutions of (1.1) with those of the unperturbed linear system

$$(1.2) \quad y(t) = f(t) - \int_0^t a(t, s)y(s) ds.$$

In doing this we generalize stability type results for continuous solutions in [1], [2], where a stronger hypothesis than the one used here is required of g . However, here we give up uniqueness. This is done in Theorems 1 and 2. Our main result, Theorem 3, provides a criterion for the asymptotic equivalence of (1.1), (1.2). The first part of Theorem 3 partially extends a result of A. STRAUSS [3; Theorem 2], where a considerably stronger hypothesis is required of the kernel $a(t, s)$, than is needed here; on the other hand we require an additional hypothesis of the nonlinear terms g in the general case.

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The second part is apparently new. In the important special case considered in Theorem 4:

$$|g(t, x)| \leq \lambda(t) |x|^\sigma \quad (0 \leq \sigma < 1)$$

where $\lambda(t)$ is a bounded continuous function on $0 \leq t < \infty$ tending to zero as $t \rightarrow \infty$, the additional hypothesis needed to prove the first part of Theorem 3 is dropped and the proof is carried out by an elementary argument based on obtaining an a priori estimate for solutions of (1.1). Theorem 3 and 4 may be regarded as a generalization to VOLTERRA equations of classical results on asymptotic equivalence due primarily to N. LEVINSON [4] and H. WEYL [5] (see especially F. BRAUER and J. S. W. WONG [6] for a recent and general form of these results) in ordinary differential equations. The author is grateful to F. BRAUER for pointing out the most recent version of these results to him, and to R. K. MILLER and P. RABINOWITZ for useful discussions concerning Theorems 3 and 4.

The main tool in proving our results is the following form of the SCHAUDER-TYCHONOFF fixed point theorem. Its elementary proof may be found in COPPEL [7].

THEOREM. — *Let $C(J)$ denote the set of functions which are continuous on the interval J (bounded or unbounded). Let F denote the subset of these functions $x(t)$ such that $|x(t)| \leq \mu(t)$ for all $t \in J$, where μ is a fixed positive continuous function on J . Let T be a mapping of F into itself such that*

(i) *T is continuous (in the sense that if $x_n \in F$, $n = 1, 2, \dots$, $x_n \rightarrow x$ uniformly on compact subsets of J , then $Tx_n \rightarrow Tx$ uniformly on compact subsets of J)*

(ii) *the functions in the image set TF are equicontinuous and bounded at each point of J .*

Then the mapping T has at least one fixed point in F . (It is important to notice that the functions in TF need not be uniformly equicontinuous on J).

2. Summary of Results. — We first require, as in [1] and [3] several concepts and formulae which will be used throughout. The *resolvent kernel* $r(t, s)$ corresponding to the kernel $a(t, s)$ in (1.1) is defined as the (unique) solution of the *resolvent system*

$$(2.1) \quad r(t, s) = a(t, s) - \int_s^t a(t, u)r(u, s)du;$$

this solution exists (and is locally integrable in (t, s)) if the kernel $a(t, s)$ is locally integrable in (t, s) for $0 \leq s \leq t < \infty$. The solution of the linear system (1.2) is expressible as

$$(2.2) \quad y(t) = f(t) - \int_0^t r(t, s)f(s)ds$$

under various hypotheses concerning the given function f . Moreover, it can be shown that the system (1.1) is equivalent to the integral equation

$$(2.3) \quad x(t) = y(t) - \int_0^t r(t, s)g(s, x(s))ds,$$

where $y(t)$ is given by (2.2). We remark that the proof of the equivalence is a rather long but elementary exercise involving the use of (2.1). In going from (1.1) to (2.3) use is made of the fact that if a, r satisfy (2.1) then

$$\int_s^t a(t, u)r(u, s)du = \int_s^t r(t, u)a(u, s)du.$$

The motivation for introducing the resolvent kernel, and the equivalent system (2.3) as well as various hypotheses concerning $r(t, s)$, notably (2.4) below, which follow as well as several applications and examples may be found in [1] and [3].

Unless stated explicitly otherwise we assume the following:

(H_1) $r(t, s)$ is locally integrable in (t, s) for $0 \leq s \leq t < \infty$.

(H_2) $g(t, x)$ is continuous in (t, x) for $0 \leq t < \infty, |x| < \infty$ and $g(t, 0) = 0$.

(H_3) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|g(t, x)| \leq \varepsilon|x|$ uniformly in t , whenever $|x| \leq \delta$. Here $||$ is any convenient norm in Euclidean n -space.

Let BC denote the class of bounded and continuous functions on $0 \leq t < \infty$. If $\varphi \in BC$ define

$$\|\varphi\| = \sup_{0 \leq t < \infty} |\varphi(t)|.$$

THEOREM 1. - Suppose $y(t) \in BC$ is a solution of the linear system in (1.2). Let there exist a constant $B > 0$ such that

$$(2.4) \quad \int_0^t |r(t, s)| ds \leq B \quad (t \geq 0)$$

and let

$$(2.5) \quad \lim_{h \rightarrow \infty} \left[\int_t^{t+h} |r(t+h, s)| ds + \int_0^t |r(t+h, s) - r(t, s)| ds \right] = 0$$

for each $t > 0$. Then for every λ , $0 < \lambda < 1$, there exists a $\varepsilon_0 > 0$ such that $0 < \varepsilon \leq \varepsilon_0$ and $\|y\| \leq \lambda \varepsilon$ together imply that there exists at least one solution $x(t)$ of the perturbed system (1.1) such that $x(t) \in BC$ and $\|x\| \leq \varepsilon$.

The condition $\|y\| \leq \lambda \varepsilon$ is naturally satisfied by requiring $f \in BC$ and $\|f\| \leq \lambda \varepsilon (1 + B)^{-1}$, as may be seen from (2.2). Thus Theorem 1 may be interpreted as a stability result in which f plays the role of initial conditions.

If g happens to satisfy a LIPSCHITZ condition (with a constant M that is arbitrary as opposed to the situation in [1]) the solution $x(t)$ of Theorem 1 is unique.

As an immediate consequence of Theorem 1 we have the following result which may be regarded as an asymptotic stability theorem.

COROLLARY 1.1. - If in addition to the hypotheses of Theorem 1 $\lim_{t \rightarrow \infty} y(t) = 0$ and for each $T > 0$

$$(2.6) \quad \lim_{t \rightarrow \infty} \int_0^T |r(t, s)| ds = 0,$$

then the solution $x(t)$ satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Note that $\lim_{t \rightarrow \infty} f(t) = 0$, (2.2), and (2.6) imply that $\lim_{t \rightarrow \infty} y(t) = 0$.

Theorem 1 and Corollary 1.1 extend Theorem 3 and Corollary 3.1 of [1] at the expense of giving up uniqueness. Several applications discussed in [1], in particular the example (Theorem 6) hold in the present, more general, setting. Theorem 1 can also be proved by applying a general result recently established independently by MILLER [8].

We next extend by a similar technique the result established in [2]. This concerns the VOLTERRA-FREDHOLM system

$$(2.7) \quad x(t) = f(t) + \int_0^t v(t, s)g(s, x(s))ds + \int_0^\infty k(t, s)h'(s, x(s))ds$$

for $t \geq 0$. Note that this is not a perturbation of a linear system; (2.7) arises naturally in studying boundary value problems on the infinite half line. The functions f, g, h are given n -vectors, while v and k are given n by n matrices. We shall not introduce the resolvent kernel; instead we shall make a fairly strong hypothesis concerning the kernel v and k as we did in [2].

THEOREM 2. - *Let $f \in BC$ and let the kernel v, k satisfy H_1 . Let v satisfy (2.4), (2.5) and let there exist a constant $K > 0$ such that*

$$(2.8) \quad \int_0^\infty |k(t, s)| ds \leq K < \infty \quad (0 \leq t < \infty);$$

further, let

$$(2.9) \quad \lim_{h \rightarrow 0} \int_0^\infty |k(t+h, s) - k(t, s)| ds = 0$$

for every $t > 0$. Let g, h satisfy H_3 . There exists a number $\varepsilon_0 > 0$ such that to every $0 < \varepsilon \leq \varepsilon_0$ there corresponds a $\delta > 0$ with the property that $\|f\| \leq \delta$ implies the existence of a solution $x(t) \in BC$ of (2.7) such that $\|x\| \leq \varepsilon$. If in addition $\lim_{t \rightarrow \infty} f(t) = 0$ and if v also satisfies (2.6) and if k also satisfies

$$(2.10) \quad \lim_{t \rightarrow \infty} \int_0^\infty |k(t, s)| ds = 0,$$

then the solution $x(t)$ satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

As in Theorem 1, if both g and h satisfy a LIPSCHITZ condition the solution x in Theorem 2 is unique. Notice that Theorem 2 holds if either v or k in (2.7) are identically zero. However, Theorem 2 is a strictly nonlinear result. There are various ways to see that Theorem 2 does not hold in general for linear systems. Take for instance $k(t, s) \equiv 0$, $f(t) \equiv f_0$ (a constant), $g(t, x) \equiv \rho x$ where ρ is a fixed constant, and

$$v(t, s) \equiv \exp[-2(t-s)] - \exp[-(t-s)].$$

Then v and k satisfy the hypotheses of Theorem 2, but g does not. The unique solution of (2.7) in this case is

$$x(t) = c_1 \exp(\rho_1 t) + c_2 \exp(\rho_2 t) + \frac{2}{\rho + 2} f_0 \quad (\rho \neq -2),$$

where c_1, c_2 are fixed constants (depending on ρ_1, ρ_2, f_0). If $\rho < -2$ one has $\rho_1 > 0$, $\rho_2 < 0$, $c_1 \neq 0$ and the solution $x(t)$ is not bounded on $0 \leq t < \infty$.

(If $\rho = -2$, the corresponding solution is also unbounded). Examples similar to this one, but for integrodifferential equations were constructed by LEVIN [9; p. 372].

Theorem 2 not only extends the result of [2], but the present proof also corrects an error in the proof in [2], where it was asserted that the integral operator B defined by the kernel $k(t, s)$ is completely continuous on the space $S(\varepsilon)$ of functions $\varphi \in BC$ and $\|\varphi\| \leq \varepsilon$. This, however, is not the case and therefore the Lemma of KRASNOSELSKII used in [2] is not applicable (moreover, it was not needed since an independent proof of the result stated in [2] is easily constructed).

Our main result generalizes Theorem 1 and Corollary 1.1 above, as well as Theorem 2 of [3] in several ways. In place of H_2, H_3 we now suppose

$$H_4 \left\{ \begin{array}{l} g(t, x) \text{ is continuous in } (t, x) \text{ for } 0 \leq t < \infty, |x| < \infty \text{ and} \\ |g(t, x)| \leq \begin{cases} \lambda(t)|x| & \text{if } |x| \geq 1 \\ \lambda(t) & \text{if } |x| < 1 \end{cases} \\ \text{where } \lambda(t) \geq 0 \text{ is a fixed bounded continuous function on } [0, \infty) \\ \text{such that } \lim_{t \rightarrow \infty} \lambda(t) = 0 \text{ and such that } \|\lambda\| B \leq \frac{1}{2} \text{ where } B > 0 \text{ is} \\ \text{the a priori constant defined in (2.4).} \end{array} \right.$$

THEOREM 3. - *Let g satisfy H_4 ; let the resolvent kernel $r(t, s)$ satisfy H_1 , (2.4), (2.5), (2.6). Let $y(t) \in BC$ be a fixed solution of the unperturbed system (1.2). Then there exists a solution $x(t) \in BC$ of the perturbed system (1.1) such that*

$$(2.11) \quad \lim_{t \rightarrow \infty} (x(t) - y(t)) = 0.$$

Conversely, given a solution $x(t) \in BC$ of (1.1) there exists a solution $y(t) \in BC$ (1.2) such that (2.11) holds; i.e. the systems (1.1) and (1.2) are asymptotically equivalent.

The remarks in [3] concerning the implications of Theorem 3 remain, of course, valid. In particular H_4 includes as a special case the condition

$$|g(t, x)| \leq \lambda(t)|x|^\alpha \quad (0 \leq \alpha \leq 1)$$

where $\lambda(t)$ is as in H_4 . However, the condition $\lim_{t \rightarrow \infty} \lambda(t) = 0$ cannot be improved to $\int_0^\infty \lambda(t) dt < \infty$, as shown by Example 3 of [3].

The first part of Theorem 3 partially generalizes Theorem 2 of [3] because in [3] the original kernel $a(t, s)$ must satisfy a much more stringent hypothesis (see H_1 in [3]) than is needed here. The most serious of these is

$$\sup_{0 \leq t \leq T} \int_0^t |a(t, s)| ds < \infty$$

for every $T > 0$. This hypothesis need not be satisfied for the prototype example $a(t, s) = A(t - s)B(s)$ discussed in [1, p. 682], where $A(t)$ is completely monotonic on $0 < t < \infty$ and $B(t) \geq 0$ is of class BC , and to which our results are applicable.

Finally, we return to the special case

$$(2.12) \quad |g(t, x)| \leq \lambda(t) |x|^\sigma \quad (0 \leq \sigma < 1)$$

where $\lambda(t) \geq 0$ is as in (H_4) , but the condition $\|\lambda\| B \leq \frac{1}{2}$ is dropped.

THEOREM 4. - *Let g satisfy (2.12) and let the remaining assumptions of Theorem 3 be satisfied. Then the systems (1.1) and (1.2) are asymptotically equivalent in the sense of Theorem 3.*

3. Proof of Theorem 1 and Corollary 1.1. - Fix a number $\beta > 0$ such that $\beta B < 1 - \lambda$. By H_3 choose the number $\varepsilon_0 > 0$ in the statement of Theorem 1 such that

$$(3.1) \quad |g(t, x)| \leq \beta |x| \quad (|x| \leq \varepsilon_0)$$

uniformly in t . Let $0 < \varepsilon \leq \varepsilon_0$ and consider the set

$$S_\varepsilon = \{\varphi \in BC : \|\varphi\| \leq \varepsilon\}.$$

Define the operator R by the relation

$$(3.2) \quad (R\varphi)(t) = y(t) - \int_0^t r(t, s)g(s, \varphi(s))ds$$

for any function $\varphi \in BC$. We assert:

- (i) R maps S_ε into itself for $0 \leq \varepsilon \leq \varepsilon_0$.

For, using (2.4), (3.1)

$$\begin{aligned} |R\varphi(t)| &\leq |y(t)| + \int_0^t |r(t, s)| \beta |\varphi(s)| ds \\ &\leq \|y\| + \beta \sup_{0 \leq s \leq t} |\varphi(t)| B \leq \lambda \varepsilon + \beta \varepsilon B. \end{aligned}$$

Thus

$$\|R\varphi\| \leq \lambda \varepsilon + (1 - \lambda) \varepsilon = \varepsilon \quad (0 \leq \varepsilon \leq \varepsilon_0),$$

and $RS_\varepsilon \subseteq S_\varepsilon$.

(ii) *The mapping R is continuous on S_ε .*

For, if $\varphi_n \in S_\varepsilon$ and $\{\varphi_n\} \rightarrow \varphi$ uniformly on every compact subset of $[0, \infty)$, then from (3.2), (2.4)

$$\begin{aligned} |(R\varphi)(t) - (R\varphi_n)(t)| &\leq \int_0^t |r(t, s)| |g(s, \varphi(s)) - g(s, \varphi_n(s))| ds \\ &\leq \sup_{0 \leq s \leq t} |g(s, \varphi(s)) - g(s, \varphi_n(s))| B \end{aligned}$$

which tends to zero as $n \rightarrow \infty$, uniformly on compact subsets of $[0, \infty)$ by (H_2) .

(iii) *The functions in the image set RS_ε , $0 \leq \varepsilon \leq \varepsilon_0$, are equicontinuous and bounded at every point of $[0, \infty)$.*

For, the boundedness is already proved in (i). For the pointwise equicontinuity let $t_0 \in [0, \infty)$ and let $t > t_0$ (the same argument applies to $t < t_0$). Then from (3.2) one has

$$\begin{aligned} (R\varphi)(t) - (R\varphi)(t_0) &= y(t) + \int_0^{t_0} r(t_0, s) g(s, \varphi(s)) ds \\ &\quad - \int_0^t r(t, s) g(s, \varphi(s)) ds \\ &= y(t) - y(t_0) + \int_0^t [r(t_0, s) - r(t, s)] g(s, \varphi(s)) ds \\ &\quad - \int_{t_0}^t r(t, s) g(s, \varphi(s)) ds. \end{aligned}$$

Thus

$$|(R\varphi)(t) - (R\varphi)(t_0)| \leq |y(t) - y(t_0)| \\ + \sup_{0 \leq s \leq t} |g(s, \varphi(s))| \left[\int_0^{t_0} |r(t_0, s) - r(t, s)| ds + \int_{t_0}^t |r(t, s)| ds \right],$$

and by H_3 , (3.1) one easily has

$$|(R\varphi)(t) - (R\varphi)(t_0)| \leq \\ |y(t) - y(t_0)| + \beta \varepsilon_0 \left[\int_0^{t_0} |r(t_0, s) - r(t, s)| ds + \int_{t_0}^t |r(t, s)| ds \right].$$

Let $\eta > 0$ be given. Choose $\delta_1 > 0$ so that

$$|y(t) - y(t_0)| \leq \frac{\eta}{2} \quad \text{for } |t - t_0| \leq \delta_1.$$

By (2.5) choose $\delta_2 > 0$ so that

$$\int_0^{t_0} |r(t, s) - r(t_0, s)| ds + \int_{t_0}^t |r(t, s)| ds \leq \frac{\eta}{2(1 + \beta \varepsilon_0)} \\ \text{for } |t - t_0| \leq \delta_2.$$

Let $\delta = \min(\delta_1, \delta_2)$ (since δ_1, δ_2 may depend on t so does δ). Then

$$|(R\varphi)(t) - R\varphi(t_0)| \leq \eta \quad \text{whenever } |t - t_0| \leq \delta,$$

establishing the pointwise equicontinuity of the functions in RT_ε .

Thus by the SCHAUDER-TYCHONOFF fixed point theorem there exists a function $\tilde{\varphi} \in S_\varepsilon$ such that $R\tilde{\varphi} = \tilde{\varphi}$ or such that

$$\tilde{\varphi}(t) = y(t) - \int_0^t (r(t, s) g(s, \tilde{\varphi}(s))) ds \quad (0 \leq t < \infty)$$

and $\|\tilde{\varphi}\| \leq \varepsilon$. Since (2.3) and (1.1) are equivalent systems, this completes the proof of Theorem 1.

To prove Corollary 1.1 we may define

$$S_0 = \{ \varphi : \varphi \in S_\varepsilon \text{ and } \lim_{t \rightarrow \infty} \varphi(t) = 0 \}.$$

Evidently S_0 is a closed subset of S_ε under the uniform norm.

Thus it suffices to show that $RS_0 \subseteq S_0$, i.e. $\lim_{t \rightarrow \infty} (R\varphi)(t) = 0$ whenever $\lim_{t \rightarrow \infty} \varphi(t) = 0$.

For a proof of this elementary fact see [1, p. 685]. Alternatively, suppose $\lim_{t \rightarrow \infty} x(t) \neq 0$, where $x(t)$ is a solution of (2.3) in S_ε , and let

$$\mu = \limsup_{t \rightarrow \infty} |x(t)| > 0.$$

Fix a number γ , where $1 - \lambda < \gamma < 1$ and where λ is the number in Theorem 1. Choose $T > 0$ so large that $|x(t)| \leq \mu/\gamma$ whenever $t \geq T$. Then (2.3), (3.1) imply that

$$\begin{aligned} |x(t)| &\leq |y(t)| + \int_0^T |r(t, s)| |g(s, x(s))| ds + \int_T^t |r(t, s)| |g(s, x(s))| ds \\ &\leq |y(t)| + \beta \varepsilon \int_0^T |r(t, s)| ds + \beta \frac{\mu}{\gamma} \int_T^t |r(t, s)| ds. \end{aligned}$$

Taking the lim sup one obtains from the hypothesis and (2.4), (2.6)

$$\mu \leq 0 + 0 + \beta \frac{\mu}{\gamma} B < \frac{\mu}{\gamma} (1 - \lambda) < \mu$$

which is a contradiction and the proof of Corollary 1.1 is complete.

4. Proof of Theorem 2. - For $\varphi \in S_\varepsilon$ define the operator T by the relation

$$(4.1) \quad (T\varphi)(t) = f(t) + \int_0^t v(t, s)g(s, \varphi(s)) ds + \int_0^\infty k(t, s)h(s, \varphi(s)) ds.$$

We again show that T satisfies the hypothesis of the SCHAUDER-TYCHONOFF fixed point theorem. We assert

(i) $TS_\varepsilon \subseteq S_\varepsilon$ for $0 \leq \varepsilon \leq \varepsilon_0$.

Fix $\zeta > 0$ such that $K\zeta < 1$. By H_3 pick a number $\sigma_1 > 0$ such that $|h(t, x)| \leq \zeta|x|$ for $|x| \leq \sigma_1$, uniformly in t . Let $\gamma = \frac{1 - K\zeta}{2B}$ and choose a number $\sigma_2 > 0$ such that $|g(t, x)| \leq \gamma|x|$ whenever $|x| \leq \sigma_2$, uniformly in t .

Choose $\varepsilon_0 = \min(\sigma_1, \sigma_2)$ and for $0 \leq \varepsilon \leq \varepsilon_0$ choose $\delta \leq (1 - K\zeta) \frac{\varepsilon}{2}$.

Then for $\varphi \in S_\varepsilon$ and $\|f\| \leq \delta$ one has from (4.1)

$$|(T\varphi)(t)| \leq \|f\| + \int_0^t |v(t, s)| \gamma |\varphi(s)| ds + \int_0^\infty |k(t, s)| \zeta |\varphi(s)| ds,$$

so that, using (2.4) and (2.8),

$$\|(T\varphi)(t)\| \leq \delta + \frac{1 - K\zeta}{2B} \cdot \varepsilon B + \zeta \varepsilon K \leq \varepsilon,$$

and thus $TS_\varepsilon \subseteq S_\varepsilon$.

(ii) *T is continuous on S_ε .*

This fact is proved exactly as in the proof of (ii) Theorem 1.

(iii) *The function in the image set TS_ε , $0 \leq \varepsilon \leq \varepsilon_0$, are bounded and equicontinuous at every point of $[0, \infty)$.*

The boundedness is already established in (i). Let $t_0 \in [0, \infty)$ and let $t > t_0$ (the same argument applies to $t < t_0$). Then one easily shows, much as in the proof of (iii) Theorem 1:

$$\begin{aligned} |(T\varphi)(t) - (T\varphi)(t_0)| &\leq |f(t) - f(t_0)| + \gamma \varepsilon_0 \left\{ \int_0^t |v(t, s) - v(t_0, s)| ds \right. \\ &\quad \left. + \int_0^t |v(t, s)| ds \right\} + \zeta \varepsilon_0 \int_0^\infty |k(t, s) - k(t_0, s)| ds. \end{aligned}$$

Let $\eta > 0$ be given. Choose $\delta_1 > 0$ so small that

$$|f(t) - f(t_0)| \leq \eta/3 \text{ whenever } |t - t_0| \leq \delta_1;$$

choose $\delta_2 > 0$ (by (2.5)) so small that

$$\left| \int_0^{t_0} |v(t, s) - v(t_0, s)| ds + \int_{t_0}^t |v(t, s)| ds \right| \leq \frac{\eta}{3(1 + \gamma \varepsilon_0)}$$

whenever $|t - t_0| \leq \delta_2$; finally choose $\delta_3 > 0$ so small that (by (2.9))

$$\int_0^\infty |k(t, s) - k(t_0, s)| ds \leq \frac{\eta}{3(1 + \zeta \varepsilon_0)} \text{ whenever } |t - t_0| \leq \delta_3.$$

Then for $|t - t_0| < \delta$ one has $|(T\varphi)(t) - (T\varphi)(t_0)| \leq \eta$, where $\delta = \min(\delta_1, \delta_2, \delta_3)$. Thus by the SCHAÜDER-TYCHONOFF fixed point theorem there exists a function $x(t) \in S_\varepsilon$ such that $Tx = x$ (i.e. $x(t)$ satisfies (2.7)) and $\|x\| \leq \varepsilon$. This establishes the first statement in Theorem 2. The proof that $\lim_{t \rightarrow \infty} x(t) = 0$, under the additional hypothesis stated, may be carried out in two ways, much the same as that of Corollary 1.1.

5. Proof of Theorem 3. - Let $y(t)$ be a bounded solution of (1.2) with $\|y\| \leq K$, for some $K > 0$. We first establish the existence of a bounded solution $x(t)$ of (1.1). Let R be the operator defined by (3.2) acting on the space BC and let

$$S_K = \{\varphi : \varphi(t) \in BC, \quad \|\varphi\| \leq 2K\}.$$

We assert:

$$(i) \quad RS_K \subset S_K.$$

For, if $\|\varphi\| \leq 2K$, then from (3.2), (2.4) and H_4 one has

$$\begin{aligned} |(R\varphi)(t)| &\leq |y(t)| + \int_0^t |r(t, s)| |g(s, \varphi(s))| ds \\ &\leq \|y\| + \int_0^t |r(t, s)| \lambda(s) |\varphi(s)| ds \leq K + 2K\|\lambda\| \leq 2K. \end{aligned}$$

Thus $\|R\varphi\| \leq 2H$. (This is the only place where assumption $\|\lambda\| B \leq \frac{1}{2}$ is used).

$$(ii) \quad R \text{ is continuous on } S_K.$$

This is proved exactly as (ii) in the proof of Theorem 1.

(iii) The functions in the image set RS_K are bounded and equicontinuous at every point of $[0, \infty)$. The boundedness has already been proved in (i). For the equicontinuity one has from (3.2) and H_4 for $t > t_0$ (same argument applies to $t < t_0$):

$$\begin{aligned} |(R\varphi)(t) - (R\varphi)(t_0)| &\leq |y(t) - y(t_0)| + \\ &\sup_{0 \leq s \leq t} |g(s, \varphi(s))| \left[\int_0^{t_0} |r(t_0, s) - r(t, s)| ds + \int_{t_0}^t |r(t, s)| ds \right] \\ &\leq |y(t) - y(t_0)| + 2\Delta K \left[\int_0^{t_0} |r(t_0, s) - r(t, s)| ds + \int_{t_0}^t |r(t, s)| ds \right], \end{aligned}$$

where $\Lambda = \|\lambda\|$. Given $\varepsilon > 0$ choose $\delta_1 > 0$ such that

$$\|y(t) - y(t_0)\| \leq \frac{\varepsilon}{2} \text{ for } |t - t_0| \leq \delta_1$$

and choose $\delta_2 > 0$ such that (see (2.5))

$$\int_0^{t_0} |r(t, s) - r(t_0, s)| ds + \int_{t_0}^t |r(t, s)| ds \leq \frac{\varepsilon}{4K\Lambda} \text{ for } |t - t_0| \leq \delta_2.$$

Then for $|t - t_0| \leq \delta = \min(\delta_1, \delta_2)$ one has $\|(R\varphi)(t) - (R\varphi)(t_0)\| \leq \varepsilon$ which proves the equicontinuity.

Thus by the SCHAUDER-TYCHONOFF theorem the mapping R has a fixed point in S_K , i.e. there exists a function $\tilde{\varphi} \in S_K$ such that

$$\tilde{\varphi}(t) = y(t) - \int_0^t r(t, s) g(s, \tilde{\varphi}(s)) ds \quad (0 \leq t < \infty),$$

and because of the equivalence of (2.3) and (1.1), φ is a bounded solution of (1.1).

We next prove that $\lim_{t \rightarrow \infty} (\varphi(t) - y(t)) = 0$. Let $\varepsilon > 0$ be given. Since $\lambda(t) \rightarrow 0$ as $t \rightarrow \infty$ choose $T > 0$ such that

$$5.1) \quad |\lambda(t)| \leq \frac{\varepsilon}{4KB} \quad (t \geq T)$$

where B is the constant in (2.4). Choose $T_1 \geq T$ such that (see (2.6))

$$5.2) \quad \int_0^T |r(t, s)| ds \leq \frac{\varepsilon}{4\Lambda K} \quad (t \geq T_1).$$

From (2.3), (H_4) and $\|\tilde{\varphi}\| \leq 2K$ one has for $t \geq T$

$$\begin{aligned} |\tilde{\varphi}(t) - y(t)| &\leq \int_0^T |r(t, s)| |\lambda(s)| |\tilde{\varphi}(s)| ds + \int_T^t |r(t, s)| |\lambda(s)| |\tilde{\varphi}(s)| ds \\ &\leq 2K\Lambda \int_0^T |r(t, s)| ds + 2K \sup_{T \leq t \leq \infty} \lambda(t) \int_0^t |r(t, s)| ds. \end{aligned}$$

Using (5.2) to estimate the first term and (5.1) (2.4) the second term one obtains

$$|\tilde{\varphi}(t) - y(t)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (t \geq T_1).$$

Since $\varepsilon > 0$ is arbitrary this proves the assertion and completes the proof of the first part.

To prove the converse let $u(t)$ be a bounded solution of (1.1) (i.e. of (2.3)). Define

$$(5.3) \quad v(t) = u(t) + \int_0^t r(t, s) g(s, u(s)) ds.$$

Evidently by (H_4) and (2.4)

$$|v(t)| \leq \|u\| + \Lambda B \|u\| < \infty \quad (0 \leq t < \infty),$$

so that v is bounded. We assert

$$(i) \quad \lim_{t \rightarrow \infty} (v(t) - u(t)) = 0.$$

Let $\varepsilon > 0$ be given. Let $\|u\| = K$. Define T, T_1 as in (5.1), (5.2). Then from (5.3), H_4 one has for $t \geq T$

$$|v(t) - u(t)| \leq \int_0^T |r(t, s)| |\lambda(s)| |u(s)| ds + \int_T^t |r(t, s)| |\lambda(s)| |u(s)| ds,$$

and from (2.4), (2.6), (5.1), (5.2) for $t > T_1$

$$|v(t) - u(t)| \leq \Lambda K \frac{\varepsilon}{4K\Lambda} + \frac{\varepsilon}{4KB} KB = \frac{\varepsilon}{2}$$

Since $\varepsilon > 0$ is arbitrary this proves the assertion.

$$(ii) \quad v(t) \text{ is a solution of (1.2).}$$

For, using successively (5.3), FUBINI'S theorem, (2.1), (1.1), and again (5.3) one has:

$$\begin{aligned} f(t) &= \int_0^t a(t, s) v(s) ds \\ &= f(t) - \int_0^t a(t, s) u(s) ds - \int_0^t a(t, s) \left\{ \int_0^s r(s, \sigma) g(\sigma, u(\sigma)) d\sigma \right\} ds \end{aligned}$$

$$\begin{aligned}
&= f(t) - \int_0^t a(t, s) u(s) ds - \int_0^t \left\{ \int_\sigma^t a(t, s) r(s, \sigma) ds \right\} g(\sigma, u(\sigma)) d\sigma \\
&= f(t) - \int_0^t a(t, s) u(s) ds - \int_0^t a(t, \sigma) g(\sigma, u(\sigma)) d\sigma + \int_0^t r(t, \sigma) g(\sigma, u(\sigma)) d\sigma \\
&= u(t) + \int_0^t r(t, \sigma) g(\sigma, u(\sigma)) d\sigma = v(t),
\end{aligned}$$

which shows that v satisfies (1.2).

This completes the proof of Theorem 3.

6. Proof of Theorem 4. - Let $y(t) \in BC$ be a solution of (1.2). Let $x(t)$ be a local, continuous solution of (1.1) (equivalently of (2.3)) existing on some interval $0 \leq t \leq \tau$. The existence of such a solution can be established in several (classical) ways; assumption (2.5) insures the continuity of the solution. Let

$$\|x\|_\tau = \sup_{0 \leq t \leq \tau} |x(t)|.$$

We show that $\|x\|_\tau$ satisfies an a priori bound independent of τ .

From (2.3), (2.4) and (2.13) one has

$$\begin{aligned}
\|x\|_\tau &\leq \|y\| + \int_0^\tau |r(\tau, s)| \lambda(s) |x(s)|^\sigma ds \\
&\leq \|y\| + \|x\|_\tau^\sigma \Lambda B,
\end{aligned}$$

where $\Lambda = \|\lambda\|$. Using YOUNG'S inequality for positive real numbers a and b :

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \left(\frac{1}{p} + \frac{1}{q} = 1 \right),$$

with $p = \frac{1}{\sigma}$, $q = \frac{1}{1-\sigma}$, $0 < \sigma < 1$, one obtains

$$\|x\|_\tau \leq \|y\| + \sigma \|x\|_\tau + (1 - \sigma)(B\Lambda)^{\frac{1}{1-\sigma}}$$

Thus

$$(6.1) \quad \|x\|_\tau \leq (1 - \sigma)^{-1} \|y\| + (B\Lambda)^{\frac{1}{1-\sigma}}.$$

If $\sigma = 0$, the estimate (6.1) also holds, as may be seen directly from (2.3). Clearly, the right hand side of (6.1) is independent of τ .

A standard continuation argument for VOLTERRA equations (see eg [10; lemma 1.2]) shows that the local solution can be continued to the right to the interval $0 \leq t \leq \infty$, and that the continuation, which we again call $x(t)$ for simplicity, satisfies the a priori bound (6.1). In particular $x(t) \in BC$ and $\|x\| \leq K$, where K is the right hand side of (6.1).

To complete the first part of the proof we assert that $x(t) - y(t) \rightarrow 0$ as $t \rightarrow \infty$. Let $\varepsilon > 0$ be given. Since $\lambda(t) \rightarrow 0$ choose $T > 0$ such that

$$(6.2) \quad \lambda(t) \leq \frac{\varepsilon}{2K^\sigma B} \quad (t \leq T)$$

where B is defined by (2.4). Choose $T_1 \geq T$ such that (by (2.6))

$$(6.3) \quad \int_0^T |r(t, s)| ds \leq \frac{\varepsilon}{2\Lambda K^\sigma} \quad (t \geq T_1).$$

From (2.3) and (2.12) one has for $t \geq T$

$$|x(t) - y(t)| \leq \int_0^T |r(t, s)| \lambda(s) |x(s)|^\sigma ds + \int_T^t |r(t, s)| \lambda(s) |x(s)|^\sigma ds.$$

Since $\|x\| \leq K$ this yields

$$|x(t) - y(t)| \leq \Lambda K^\sigma \int_0^T |r(t, s)| ds + K^\sigma \sup_{T \leq t < \infty} \lambda(t) \int_0^t |r(t, s)| ds.$$

Using (6.3) to estimate the first term and (6.2), (2.4) to estimate the second term one obtains

$$|x(t) - y(t)| \leq \Lambda K^\sigma \frac{\varepsilon}{2\Lambda K^\sigma} + K^\sigma \frac{\varepsilon}{2K^\sigma B} B = \varepsilon \quad (t \geq T_1).$$

Since $\varepsilon > 0$ is arbitrary, this proves the assertion and completes the first part of the proof.

To prove the converse, namely that the existence of a bounded solution $u(t)$ of (1.1) implies the existence of a bounded solution $v(t)$ of (1.2) such that $u(t) - v(t) \rightarrow 0$ as $t \rightarrow \infty$, proceeds exactly as in the second part of the proof of Theorem 3, Section 5, using (6.2), (6.3) in place of (5.1), (5.2) respectively. This completes the proof of Theorem 4.

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