Singularly perturbed elliptic boundary value problems

I. Poisson kernels and potential theory.

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Summary. - *The paper treats elliptic operators of the form $L(\varepsilon\partial_1, \dots, \varepsilon\partial_n)$, where L is a polynomial in n variables of order $2m_1$, and ε is a small parameter. Solutions u_ε of $Lu=0$ in a half space satisfying conditions $B_j(\varepsilon\partial_1, \varepsilon\partial_2, \dots, \varepsilon\partial_n)u = \varepsilon\gamma_j\varphi_j(x)$ ($j = 1, \dots, m_1$) on the boundary are constructed and estimated using Hölder norms, Poisson kernels, and an elaborate potential theory. Properties of the interior limit $u_0 = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x)$ are studied. The paper is preparatory to a detailed investigation of Schauder estimates for such problems with variable coefficients.*

Introduction.

Differential equations with small parameters multiplying some or all of the highest order derivatives (singularly perturbed differential equations) have been the object of considerable interest in recent decades, both because of the intriguing problems associated with them, and because of their relevance in engineering and physical applications. Regarding this last point, one need only mention the problem of slightly viscous flow past a rigid obstacle; in fact this was the problem that gave rise to the concept of «boundary layer» through Prandtl's famous investigations in 1904. The terminology «boundary layer» is still used almost universally to describe the abrupt change near boundaries exhibited by solutions of many singular perturbation problems.

Over the years, a great deal of mathematical knowledge has developed concerning singularly perturbed partial differential equations. The first rigorous study of problems in this area was possibly that of WASOW in 1946 [10]. A now famous paper by VISHIK and LYUSTERNIK [9] treating various classes of problems in both ordinary and partial differential equations appeared in 1957. Of the results appearing since then, we mention here only a few of those having to do with elliptic problems. Second order elliptic boundary value problems were studied in great detail by LEVINSON in [7] and by ECKHAUS and JÄGER in [2]. A nonlinear fourth order problem

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arising in plate theory was investigated by FIFE in [3]. The convergence in various spaces as $\varepsilon \rightarrow 0$ of solutions of elliptic singular perturbation problems was studied by HUET in [6] and in her previous papers, as well as by GREENLEE [5], FRIEDMAN [4], and TON [8].

The aim of the present series is a thorough study of a class of singularly perturbed elliptic boundary value problems based on SCHAUDER estimates (pointwise estimates in HÖLDER norms). The power of these estimates is such that in the past they have been instrumental in the existential treatment of nonlinear problems. In the present context, it is anticipated that they will be useful in obtaining precise asymptotic results for nonlinear singularly perturbed problems. This program was carried out successfully for a particular nonlinear fourth order problem in [3].

This first paper treats only problems with constant coefficients in a half-space. The extension to variable coefficients and other domains will be based on these results and will be given in later papers of the series. The elliptic partial differential operators treated here have the form $L(\varepsilon D)$, where ε is a small parameter, $L(\xi)$ is a polynomial (not homogeneous) in n variables $\xi = (\xi_1, \dots, \xi_n)$, and D denotes the vector differential operator $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$.

The boundary operators also involve ε and assume a similar form. The class of problems treated here is delineated in Sec. 1 by certain assumptions on the polynomial L and the polynomials defining the boundary conditions. For example, a strengthened ellipticity condition and a strengthened complementing condition, analogous to those in [1], are imposed.

It is interesting to compare this class with the similar, but different, class of problems to which the methods of VISHIK and LYUSTERNIK apply. A description of this latter class is given in Sec. 2, together with a derivation of the asymptotic series resulting from the application of their methods to this class (in their original paper, only DIRICHLET boundary conditions were treated). It should be remembered that the intent of the present papers is different from that of VISHIK and LYUSTERNIK; they are concerned with developing formal asymptotic series for the solution and providing a proof of the validity of such series when a priori estimates are available. Our main purpose, on the other hand, is to derive precise a priori estimates with minimal regularity assumptions on the data. A great deal of other information about the solutions will be obtained in the process.

In Sec. 3 we construct POISSON kernels, in terms of which solutions of the boundary value problem (homogeneous equation and inhomogeneous boundary data) can be given. FOURIER transform methods are used. Several constructions are possible here; but we endeavor to obtain the kernels with best possible behavior for large values of their arguments.

Growth and decay estimates, both for large and small values of their arguments, are derived for the POISSON kernels in Sec. 4. It is here that these kernels are seen to decompose naturally into an «interior» and a «boundary layer» part. This induces a decomposition of the solution into two parts, one of them exhibiting the sharp transition near the boundary characteristic of singular perturbation problems, and the other dominant in the «interior»; i. e., in regions bounded away from the boundary.

Sec. 5 deals with a rather complicated type of potential theory suited to problems with small parameters. In this theory the nature of the POISSON kernels for large values of their argument is as important as that for small values.

In Sec. 6 the potential theory is applied to yield estimates, in HÖLDER norms and maximum norms, of solutions in the half-space. Sec. 7 is devoted to an examination of the interior limit; i. e., of the function $u^0(x)$ obtained as the limit as $\varepsilon \rightarrow 0$ of the solution at points not on the boundary. A representation in terms of auxiliary kernels is obtained for u^0 , and the question of what boundary conditions it satisfies is studied.

1. - Notation and assumptions.

1.1. Differential operators.

We shall generally write constant coefficient differential operators in n variables $\bar{x} = (x_1, \dots, x_n)$ in the form $P\left(\frac{1}{i}D_x\right)$, where $D_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$. Then $P(\bar{\xi})$, where $\bar{\xi} = (\xi_1, \dots, \xi_n)$, is the operator's characteristic polynomial. We shall distinguish the last variable, setting $x = (x_1, \dots, x_{n-1})$, $y = x_n$, $\xi = (\xi_1, \dots, \xi_{n-1})$ and $i\eta = \xi_n$. Thus (x, y) denotes a point in n -space, and a characteristic polynomial may be written as $P(\bar{\xi}) = P(\xi, i\eta)$.

We shall be dealing with solutions of the equation

$$(1.1) \quad L\left(\frac{\varepsilon}{i}D_x\right)u(\bar{x}) = 0$$

in the half-space $y = x_n > 0$, satisfying certain boundary conditions on the plane $y = 0$. Here L is a polynomial of even degree $2m_1$, and ε is a small real parameter. We assume the degree of the lowest order terms in L is also even, and denote it by $2m_0$. Finally we assume $m_1 > m_0 \geq 0$. Our principle hypothesis on L is a type of strong ellipticity:

ASSUMPTION I. - There is a constant $E > 0$ such that

$$(1.2) \quad \operatorname{Re} L(\bar{\xi}) \geq E(|\bar{\xi}|^{2m_1} + |\bar{\xi}|^{2m_0})$$

for all real vectors $\bar{\xi}$.

Let $L_1(\bar{\xi})$ and $L_0(\bar{\xi})$ be the homogeneous polynomials consisting of the terms of L of orders $2m_1$ and $2m_0$ respectively. Dividing (1.2) by $|\bar{\xi}|^{2m}$ and letting $|\bar{\xi}| \rightarrow \infty$ we find that L_1 is elliptic; similarly dividing $|\bar{\xi}|^{2m}$ and letting $\bar{\xi} \rightarrow 0$, we find that L_0 is elliptic.

We now write $L(\xi, \eta) \equiv L(\xi, i\eta)$, with similar definitions for L_0 and L_1 . Thus (1.1) becomes

$$(1.3) \quad L\left(\frac{\varepsilon}{i}D_x, -\varepsilon D_y\right)u = 0.$$

The behavior of the roots $\eta_i(\xi)$ of $L(\xi, \eta) = 0$ assumes a prominent role in what follows. We shall be interested in those roots which have positive real parts. In this connection there is the well-known root condition [1], a consequence of ellipticity when $n > 2$ and when the differential operator concerned is homogeneous. When $n = 2$ we impose it as an extra condition on L_0 and L_1 .

ASSUMPTION II: For each real vector ω , $|\omega| = 1$, there exist exactly m_1 roots $\gamma_i(\omega)$ ($i = 1, \dots, m_1$) of $L_1(\omega, \gamma) = 0$ with positive real part, and exactly m_0 roots $\alpha_i(\omega)$ ($i = 1, \dots, m_0$) of $L_0(\omega, \alpha) = 0$ with positive real part.

The next assumption says that the $\eta_i(\xi)$ have no branch points for real $\xi \neq 0$, and may be extended to be analytic functions for certain complex values of ξ near the origin.

ASSUMPTION III: The roots $\eta_i(\xi)$ are analytic as functions of the real variable ξ . Furthermore there is a neighborhood of the origin and a deleted neighborhood of infinity in the complex σ plane such that $\eta_i(\sigma\omega)$ are analytic functions of σ and of the real vector ω for σ in these neighborhoods and ω in a neighborhood of the unit sphere $|\omega| = 1$.

The properties of the roots which are of most interest to us are summarized in the following lemma, to be proved below.

LEMMA 1.1. - *Exactly half of the $\eta_i(\xi)$ (which we take as the first half) have positive real part for real $\xi \neq 0$. The following asymptotic properties hold, where the β_i ($i = m_0 + 1, \dots, m_1$) are constants with positive real parts*

$$(1.4) \quad \eta_i(\rho\omega) = \rho\gamma_i(\omega) + O(1) \quad (\rho \rightarrow \infty, \rho \text{ real}), \quad i = 1, 2, \dots, m_1$$

$$(1.5) \quad \eta_i(\rho\omega) = \rho\alpha_i(\omega) + O(\rho^2) \quad (\rho \rightarrow 0, \rho \text{ real}), \quad i = 1, 2, \dots, m_0$$

$$(1.6) \quad \eta_i(\rho\omega) = \beta_i + O(\rho) \quad (\rho \rightarrow 0, \rho \text{ real}), \quad i = m_0 + 1, \dots, m_1.$$

PROOF. - The proof is based on the following known fact about the dependence of roots of polynomials on parameters: Let $P_1(t)$ (not a constant) and $P_2(t)$ be polynomials in the single variable t , and let $t_1^1, t_2^1, \dots, t_m^1$ be the roots of P_1 (not necessarily distinct). Then among the roots of $P_1(t) + \varepsilon P_2(t)$ for small enough ε , there are m , which we call $t_i(\varepsilon)$ (not necessarily distinct) such that $\lim_{\varepsilon \rightarrow 0} t_i(\varepsilon) = t_i^1$. The same result holds if P_2 is allowed to be a polynomial in the two variables ε and t .

In applying this result we suppose ξ to be real and set $\rho = |\xi|$, $\xi = \rho\omega$, $\zeta_i = \eta_i/\rho$ (then ω is a unit vector in E^{n-1}). Then

$$(1.7) \quad 0 = L(\xi, \eta_i(\xi)) = L(\rho\omega, \rho\zeta_i) = \rho^{2m_1} L_1(\omega, \zeta_i) + \dots + \rho^{2m_0} L_0(\omega, \zeta_i).$$

Setting $\varepsilon = \rho^{-1}$, dividing (1.7) by ρ^{2m_1} , and applying the above result, we find that $\lim_{\rho \rightarrow \infty} \zeta_i(\rho\omega)$ exists for all i , and the limits are roots γ of $L_1(\omega, \gamma) = 0$. We order the roots so that $\frac{\eta_i(\rho\omega)}{\rho} \rightarrow \gamma_i(\omega)$ for $i = 1, \dots, m_1$. This together with the analytic nature of the η_i expressed in Assumption III yields (1.4). It also proves that for large $|\xi|$ exactly half of the η_i have positive real part. This must therefore be true for all real ξ by continuity of the η_i with respect to ξ together with the fact that Assumption I guarantees that for no real $\xi \neq 0$ do any of the $\eta_i(\xi)$ have zero real part. This proves the first statement in the lemma.

To prove (1.5) we similarly divide (1.7) by ρ^{2m_0} and set $\varepsilon = \rho$. Then by possibly reordering the roots η_i we obtain (1.5).

The question remains as to what happens to η_i for $i > m_0$ as $\rho \rightarrow 0$. Now $L(0, \eta)$ is a polynomial which we may write as $a\eta^{2m_0} + \dots + b\eta^{2m_1}$, with $b \neq 0$ by virtue of the ellipticity of L_0 . Thus $\eta = 0$ is a root of multiplicity exactly $2m_0$, and the other roots (call them β_i) are different from 0. Thus exactly $2m_0$ of roots η_i approach 0 as $\rho \rightarrow 0$; these have already been accounted for as those such that $\eta_i \sim \rho\alpha_i$. The remaining ones must approach the $2(m_1 - m_0)$ roots β_i , of which (using the final statement of the lemma) half must have positive real part. Call that half β_i , $i = m_0 + 1, \dots, m_1$. Then (1.6) follows.

1.2. - Boundary operators.

Let $B_i(\xi, \eta)$ ($i = 1, 2, \dots, m_1$) be polynomials of degrees μ_i , and γ_i be certain integers. We shall be concerned with solutions $u(x, y)$ of (1.3) in the half-space $y > 0$ satisfying

$$(1.8) \quad B_i\left(\frac{\varepsilon}{i} D_x, \varepsilon D_y\right) u \Big|_{y=0} = \varepsilon^{-\gamma_i} \varphi_j(x).$$

We impose the following «strengthened complementing condition» on B_j and L :

ASSUMPTION IV: We define $L^+(\xi, \tau) \equiv \sum_{k=1}^{m_1} (\tau - \eta_k(\xi))$ and $L_1^+(\omega, \tau) \equiv \sum_1^{m_1} (\tau - \gamma_i(\omega))$. As polynomials in τ , the functions $B_j(\xi, \tau)$ are linearly independent mod $L^+(\xi, \tau)$,

- (1) for each real $\xi \neq 0$, and
- (2) for each complex ξ in some deleted neighborhood of the origin.

Furthermore for each real $\omega \neq 0$, the polynomials $B'_i(\omega, \tau)$ are linearly independent mod $L_1^+(\omega, \tau)$, where B'_j consists of the highest order terms of B_j .

2. - The Vishik-Lyusternik procedure.

2.1. - Assumptions.

Although the main purpose of this series of papers is to develop SCHAUDER estimates for the problem (1.1), (1.12), it is of interest to see how the expansion method of VISHIK and LYUSTERNIK [9] may be applied to problems of this type. This method is essentially a way to construct approximate solutions, for small ε , in the form of a series in powers of ε . Each coefficient in this series is a function defined as the solution of some problem in which the parameter ε does not appear. Some of these coefficients are of «interior» type, i.e., their qualitative properties near the boundary are no different from their properties away from the boundary. Other coefficients are of «boundary layer type»; these decay rapidly for small ε as one moves away from the boundary, but are very significant within a strip with width of the order ε near the boundary. Specifically, these latter are functions of x and $\zeta = y/\varepsilon$ only, and decay exponentially as $\zeta \rightarrow \infty$. Given sufficient smoothness of the data of the problem, and given a priori estimates (such as we shall establish in this paper, or such as are pursued in [9] for the DIRICHLET problem), one may prove that the formal series constructed by this method do approximate the exact solution of the problem, for small ε .

In contrast to this, SCHAUDER estimates applied directly to the problem without the VISHIK-LYUSTERNIK procedure yield pointwise estimates both in a boundary layer and in the interior, under minimal smoothness assumptions on the data.

The procedure described below is for the case of constant coefficients; but it may clearly be adapted (see [9]) to the case of variable coefficients and nonplanar boundary, provided the coefficients and boundary are smooth enough.

The VISHIK-LYUSTERNIK procedure operates under assumptions on the operators L and B_j which are related to, but somewhat different from, those given in the preceding section. We list the new assumptions here:

I'. Assumption 1, pag. 102,

II'. Assumption II, pag. 102,

III'. For each l , $1 \leq l \leq m_1$, let $B_\Pi(\xi)$ denote that homogeneous polynomial consisting of the highest order terms only of $B_l(\xi)$, and $B_{l0}(\xi)$ that consisting only of the lowest order terms. The operators $B_{l0}\left(\frac{1}{i}D_x, -D_y\right)$ ($l = 1, \dots, m_0$) are to satisfy the complementing condition with respect to $L_0\left(\frac{1}{i}D_x, -D_y\right)$.

IV'. The γ_l are distinct integers and are equal respectively to the degrees of $B_{l0}(\xi)$.

V'. The polynomials $B_l(0, -z)$, $l = m_0 + 1, \dots, m_1$, are linearly independent mod $\Pi(z - \beta_i)$, the product being over all zeros β_i of $L(0, \beta)$ with negative real parts.

2.2. - Expansion of operators.

Since $L\left(\frac{\varepsilon}{i}D_x, -\varepsilon D_y\right)$ is a polynomial in ε , we may write it in the form

$$(2.1) \quad \varepsilon^{-2m_0}L\left(\frac{\varepsilon}{i}D_x, -\varepsilon D_y\right) = L_0\left(\frac{1}{i}D_x, -D_y\right) + \sum_{j=1}^{2(m_1-m_0)} \varepsilon^j L_j\left(\frac{1}{i}D_x, -D_y\right).$$

Here of course L^0 is the operator L_0 defined in Section 2.1, and $L^{2(m_1-m_0)} \equiv L_1$. Similarly

$$(2.2) \quad \varepsilon^{-\gamma_l} B_l\left(\frac{\varepsilon}{i}D_x, -\varepsilon D_y\right) = B_l^0\left(\frac{1}{i}D_x, -D_y\right) + \sum_{j=1}^{\gamma_l-\gamma_l} \varepsilon^j B_l^j\left(\frac{1}{i}D_x, -D_y\right).$$

where $B_l^0 = B_{l0}$ and $B^{\gamma_l-\gamma_l} = B_\Pi$.

We now «stretch» the variable y by defining $\zeta = \frac{y}{\varepsilon}$. Performing this substitution in the above operators and again expanding with respect to ε , we obtain

$$(2.3) \quad L\left(\frac{\varepsilon}{i}D_x, -\varepsilon D_y\right) = L\left(\frac{\varepsilon}{i}D_x, -D_\zeta\right) \equiv \Lambda_0(D_\zeta) + \sum_{j=1}^{2m_1} \varepsilon^j \Lambda_j(D_x, D_\zeta).$$

where of course $\Lambda_0(D_\zeta) = L(0, -D_\zeta)$. Similarly

$$(2.4) \quad B_l\left(\frac{\varepsilon}{i}D_x, -D_\zeta\right) = \Gamma_{l0}(D_\zeta) + \sum_{j=1}^{\mu_l} \varepsilon^j \Gamma_{lj}(D_x, D_\zeta).$$

2.3. - Approximate solutions.

We seek approximate solutions of the problem

$$(2.5a) \quad L\left(\frac{\varepsilon}{i}D_x, -\varepsilon D_y\right)u = 0, \quad y > 0$$

$$(2.5b) \quad B_l\left(\frac{\varepsilon}{i}D_x, -\varepsilon D_y\right)u \Big|_{y=0} = \varepsilon^{\gamma_l} \varphi_l(x)$$

in the form

$$(2.6) \quad u \approx U + V,$$

where

$$(2.7a) \quad U = \sum_{j=0}^{\mu} \varepsilon^j u_j(x, y),$$

$$(2.7b) \quad V = \varepsilon \sum_{j=0}^{\nu} \varepsilon^j v_j(x, \zeta)$$

and $q = \gamma_{m_0+1}$. The integers μ and ν are for the moment arbitrary.

We obtain the u_j and v_j by first substituting (2.7) into (2.5) and representing the operators in the form (2.1), (2.3) when applied to U and (2.2), (2.4) when applied to V :

$$(2.8a) \quad 0 \approx \varepsilon^{-2m_0} L(U + V) = \left(\sum_0 \varepsilon^j L_j\right) \left(\sum_0 \varepsilon^k u_k\right) + \varepsilon^{q-2m_0} \left(\sum_0 \varepsilon^j \Lambda_j\right) \left(\sum_0 \varepsilon^k v_k(x, \zeta)\right),$$

$$(2.8b) \quad \begin{aligned} \varphi_l(x) \approx \varepsilon^{-\gamma_l} B_l(U + V) \Big|_{y=0} &= \left(\sum_0 \varepsilon^j B_l^j\right) \left(\sum_0 \varepsilon^k u_k\right) \Big|_{y=0} \\ &+ \varepsilon^{q-\gamma_l} \left(\sum_0 \varepsilon^j \Gamma_{lj}\right) \left(\sum_0 \varepsilon^k v_k\right) \Big|_{\zeta=0}. \end{aligned}$$

We equate coefficients of like powers of ε on the two sides of the $\approx$ sign to obtain

$$(2.9) \quad L^0 u_k = - \sum_{\substack{\alpha+\beta=k \\ \alpha \geq 1}} L^\alpha u_\beta,$$

$$(2.10) \quad \Lambda_0 v_k = - \sum_{\substack{\alpha+\beta=k \\ \alpha \geq 1}} \Lambda_\alpha v_\beta,$$

$$(2.11) \quad B_l^0 u_k|_{y=0} = - \sum_{\substack{\alpha+\beta=k \\ \alpha \geq 1}} B_l^\alpha u_\beta|_{y=0} = \sum_{\alpha+\beta+q-\gamma_l=k} \Gamma_{l\alpha} v_\beta|_{\zeta=0} + \delta_{k0} \varphi_l$$

for $q > \gamma_l$, and

$$(2.12) \quad \Gamma_{l0} v_k|_{\zeta=0} = - \sum_{\substack{\alpha+\beta=k \\ \alpha \geq 0}} \Gamma_{l\alpha} v_\beta|_{\zeta=0} - \sum_{\alpha+\beta-q+\gamma_l=k} B_l^\alpha u_\beta|_{y=0} + \delta_{k+q-\gamma_l,0} \varphi_l$$

for $q \leq \gamma_l$.

The specification of $q = \gamma_{m_0+1}$ implies that (2.11) constitutes m_0 boundary conditions on u_k ; $l = 1, 2, \dots, m_0$. If the right sides are known, then (2.9), (2.11) form a well posed boundary value problem for u_k (note Assumption III', Section 2.1). It will also be shown that (2.10), (2.12), together with the requirement $v_k \rightarrow 0$ as $\zeta \rightarrow \infty$, form a well posed problem for the v_k , provided the right sides are known. Granting this, we see that one may determine the functions in the order $u_0, v_0, u_1, v_1, u_2, v_2, \dots$.

We now investigate the possibility of solving (2.10), (2.12) when the right sides are known. First we note that Λ_0 and Γ_{l0} are ordinary differential operators in ζ , so that the variable x appears only as a parameter on the right sides. Next we note that the polynomial $\Lambda_0(t) = L(0, -t)$ has exactly $m_1 - m_0$ roots with negative real parts. In fact, these are precisely the constants β_i in (1.6). Let $\mathbf{C}$ be a contour in the complex z -plane which lies strictly to the left of the imaginary axis, and encloses the $m_1 - m_0$ constants β_i . Let

$$P(z) \equiv \prod_{i=m_0+1}^{m_1} (z + \beta_i),$$

so $P(z)$ divides $\Lambda_0(z) = L(0, -z)$.

LEMMA. - Functions $g_{jk}(x, z)$ can be found which are analytic functions of z inside $\mathbf{C}$, and such that

$$(2.13) \quad v_k(x, y) \equiv \sum_{j=1}^{k+1} \int_{\mathbf{C}} \frac{g_{jk}(x, z) e^{\zeta z}}{(P(z))^j} dz$$

satisfy (2.10), (2.12), and the condition $v_k(x, \zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$

PROOF. - The last property is of course clear from the nature of $\mathbf{C}$. We work inductively with the index k . Set ψ_{ik} = right side of (2.13). Then for $k = 0$ the problem reduces to

$$(2.14) \quad \Lambda_0 v_0 = 0, \quad \zeta > 0$$

$$(2.15) \quad \Gamma_{10} v_0|_{\zeta=0} = \psi_{10}(x)$$

$$(2.16) \quad v_0^\sharp \rightarrow 0 \quad \text{as} \quad \zeta \rightarrow \infty.$$

The function $v_0 = \int_C \frac{g_{10}(x, z) e^{\zeta z} dz}{P(z)}$ clearly satisfies (2.14) and (2.16); furthermore (2.15) reduces to

$$\psi_{10}(x) = \int_C \frac{g_{10}(x, z) \Gamma_{10}(z) dz}{P(z)}, \quad l = m_0 + 1, \dots, m.$$

We seek g_{10} in the form of a polynomial $g_{10}(x, z) = \sum_{j=0}^{m_1-m_0-1} a_j(x) z^j$; whence we have the system

$$\sum_j a_j(x) c_{jl} = \psi_{10}(x),$$

where

$$c_{jl} = \int_C \frac{z^j \Gamma_{10}(z) dz}{P(z)}.$$

The fact (Assumption V') that the $\Gamma_{10}(z)$ are linearly independent mod $P(z)$ implies the matrix c_{jl} is nonsingular; the a_j and thereby the $g_{10}(x, z)$ are thus entirely determined. This proves the lemma for the case $k=0$. Now assume it true for $k < s$, and consider the case $k=s$. The right side of (2.10) may be written

$$-\int_C \sum_{\substack{\alpha+\beta=s \\ \beta \leq s-1}} \frac{\Lambda_\alpha(D_x, z) \sum_{j=1}^{\beta+1} g_{j\beta}(x, z) e^{\zeta z}}{(P(z))^j} dz = \sum_{j=1}^s \int_C \frac{h_{js}(x, z) e^{\zeta z} dz}{(P(z))^j},$$

where $h_{js}(x, z) = \sum_{\beta=j-1}^{s-1} \Lambda_{s-\beta}(D_x, z) g_{j\beta}(x, z)$ are known functions. Using this and setting (2.13) into (2.10), we come to the equation

$$\sum_{j=1}^{s+1} \int_C \frac{g_{js}(x, z) \Lambda_0(z) e^{\zeta z} dz}{(P(z))^j} = \sum_{j=1}^s \int_C \frac{h_{js}(x, z) e^{\zeta z} dz}{(P(z))^j}.$$

Factoring $\Lambda_0(z) = Q(z)P(z)$ where Q^{-1} is analytic within $\mathbf{C}$, we have

$$\sum_{j=0}^s \int_C \frac{g_{j+1,s}(x, z) Q(z) e^{\zeta z} dz}{(P(z))^j} = \sum_{j=1}^s \int_C \frac{h_{js}(x, z) e^{\zeta z} dz}{(P(z))^j}.$$

Thus we have only to set

$$g_{js}(x, z) = \frac{h_{j-1, s}(x, z)}{Q(z)}$$

for $j = 2, 3, \dots, s+1$. This leaves only $g_{1s}(x, z)$ to be determined; this is found from the boundary conditions (2.11);

$$\int_C \frac{\Gamma_{10}(z) \sum_{j=1}^{s+1} g_{js}(x, z)}{(P(z))^j} dz = \psi_{1s}(x),$$

or

$$\int_C \frac{g_{js}(x, z) \Gamma_{10}(z) dz}{P(z)} = \psi_{1s}(x) - \sum_{j=2}^{s+1} \int_C \frac{\Gamma_{10}(z) g_{js}(x, z) dz}{(P(z))^j},$$

where the right side is known. From the latter system of $m_1 - m_0$ equations, g_{1s} is found in the same way that g_{10} was found above. This completes the proof of the lemma.

The characterization of V as being of «boundary layer type» is clearly justified, since as a function of $\zeta = \frac{y}{\varepsilon}$, it decays exponentially as $\zeta \rightarrow \infty$. In fact, using residues one may express (2.13) as a combination of functions $a_i(x, \zeta)e^{\beta_i \zeta}$, where $\operatorname{Re} \beta_i < 0$ and a_i are polynomials in ζ .

3. - Poisson kernels for the parameterless half-space problem

The objective in this section will be to construct Poisson kernels for problem

$$(3.1) \quad L\left(\frac{1}{i} D_x, -D_y\right)u = 0, \quad y > 0$$

$$(3.2) \quad B_j\left(\frac{1}{i} D_x, -D_y\right)u \Big|_{y=0} = \varphi_j(x), \quad j = 1, \dots, m_1,$$

which is the problem introduced in Section 1, with $\varepsilon = 1$. By Poisson kernels one means functions $K_j(x, y)$, $j = 1, \dots, m_1$, such that for a wide class of

functions $\varphi_j(x)$, the convolution

$$u(x, y) = \sum_j \int K_j(x - \tau, y) \varphi_j(\tau) d\tau$$

is a solution of (3.1), (3.2).

3.1. - Preliminary constructions.

First we mention some simple but useful consequences of Lemma 1.1.

LEMMA 3.1. - *There exists real numbers $\alpha, \beta > 0$ such that*

$$(3.3) \quad \operatorname{Re} \eta_i(\rho\omega) \geq 3\alpha\rho$$

$$(3.4) \quad \operatorname{Re} \beta_i \geq 3\beta$$

for all $i \leq m_1$, real ρ , and real ω with $|\omega| = 1$.

PROOF. - Eqs. (1.5) and (1.6) together with the positive property of α_i and β_i show that $\operatorname{Re} \eta_i(\rho\omega)/\rho$ is bounded from below for all $|\omega| = 1$ and real ρ in some interval $(0, \rho_0)$. Similarly (1.4) shows this ratio to be bounded in some interval (ρ_1, ∞) . Finally the continuity and positivity of $\eta_i(\xi)$ show it to be bounded in the remaining finite interval $[\rho_0, \rho_1]$. Call the common lower bound 3α . This proves (3.3); (3.4) is obvious.

LEMMA 3.2. - *For every real $a > 0$ there is a bounded simple closed contour C_a in the complex z -plane, lying to the left of the line $\operatorname{Re} z = -2\alpha$, enclosing and bounded away from the set*

$$\left\{ z : z = \frac{\eta_i(\rho\omega)}{\rho}, i = 1, \dots, m_1, \rho \geq a, |\omega| = 1 \right\}.$$

PROOF. - An immediate consequence of Lemma 3.1.

REMARK. - Lemma 1.1 gives rise to a natural factorization of the operator $L(\xi, -D_y)$ into a part involving those roots η_i with positive real parts when ξ is real, and the remainder. Specifically, we define

$$(3.5a) \quad L^+(\xi, -D_y) = \prod_{k=1}^{m_1} (-D_y - \eta_k(\xi))$$

$$(3.5b) \quad L_1^+(\omega, -D_\tau) = \prod_{k=1}^{m_1} (-D_\tau - \gamma_k(\omega))$$

$$(3.5c) \quad L_0^+(\omega, -D_\tau) = \sum_{k=1}^{m_0} (-D_\tau - \alpha_k(\omega)).$$

Clearly any function $g(y)$ satisfying $L^+g = 0$ will also satisfy $Lg = 0$.

The Poisson kernels will be given later in terms of auxiliary functions G_i , N_i , which we proceed to construct.

LEMMA 3.3. - *There exist functions $G_i(\sigma, \omega, y)$, $i = 1, 2, \dots, m_1$, defined and analytic for the complex variable σ in a neighborhood of the origin, for all real unit vectors ω , and for all real $y \geq 0$, such that*

(1) *for each σ and ω , they are linearly independent (as functions of y) solutions of*

$$L^+(\sigma\omega, -D_y)G_i = 0;$$

(2) *for $i = 1, 2, \dots, m_0$, they may be expressed in the form*

$$G_i(\sigma, \omega, y) = e^{-\alpha\sigma y} \widehat{G}_i(\sigma, \omega, \sigma y),$$

where α is the constant in (3.3) and the functions $G_i(\sigma, \omega, \tau)$ and their derivatives are bounded as follows, when $\sigma = \rho$ is real:

$$(3.6) \quad |D^k \widehat{G}_i(\rho, \omega, \tau)| \leq E_k$$

for $\rho \geq 0$, $|\omega| = 1$, $\tau \geq 0$. Here D^k stands for any derivative with respect to ρ , ω , and/or τ of order k , and E_k are some positive constants depending only on L^+ and k ;

(3) *for $\sigma = \rho$ real and $i = m_0 + 1, \dots, m_1$, the G_i may be expressed in the form*

$$(3.7) \quad G_i(\rho, \omega, y) = e^{-(\beta y + \alpha \rho y)} \widehat{G}_i(\rho, \omega, \rho y)$$

where the $\widehat{G}_i$ and their derivatives have the same bounds as those in (2) above.

PROOF. - If the roots $\eta_i(\sigma\omega)$ are distinct for σ in a neighborhood of the origin, we could simply define $G_i(\sigma, \omega, y) = e^{-\eta_i(\sigma\omega)y}$. However, the following construction works in any case.

We define (for complex σ in a neighborhood of the origin)

$$\widehat{L}^1(\sigma, \omega, z) = \sum_{i=1}^{m_0} \left(z - \frac{\eta_i(\sigma\omega)}{\sigma} \right),$$

$$G_k(\sigma, \omega, y) \equiv \int_C \frac{z^{k-1} e^{\sigma y z}}{\widehat{L}^1(\sigma, \omega, z)} dz, \quad k = 1, 2, \dots, m_0,$$

where $\mathbf{C}$ is a contour enclosing and bounded away from the set of numbers $\left\{ \frac{\eta_i(\sigma\omega)}{\sigma}, i = 1, \dots, m_0, |\omega| = 1, |\sigma| < a \right\}$ where a is a positive number to be determined below. Because of (3.3) it is clear that $\mathbf{C}$ may be chosen to lie entirely to the left of the vertical line $\operatorname{Re} z = -2\alpha$. Then (3.6) follows immediately from the representation

$$\widehat{G}_k(\rho, \omega, \tau) = \int_{\mathbf{C}} \frac{z^{k-1} e^{\tau(z+\alpha)} dz}{\widehat{L}^1(\sigma, \omega, z)},$$

and the fact that on $\mathbf{C}$, $|e^{\tau(z+\alpha)}| \leq e^{-\alpha\tau}$.

Similarly let $\widehat{L}^2(\sigma, \omega, z) = \prod_{i=m_0+1}^{m_1} (z - \eta_i(\sigma\omega))$, and let $\mathbf{C}'$ enclose the roots $\{\eta_i(\sigma, \omega), |\omega| = 1, |\sigma| \leq a, i = m_0 + 1, \dots, m_1\}$. By taking a small enough, we may insure the possibility of having $\mathbf{C}'$ lie to the left of the line $\operatorname{Re} z = -2\beta$. Then we could take

$$G_i(\sigma, \omega, y) = \int_{\mathbf{C}'} \frac{z^{i-m_0-1} e^{zy} dz}{\widehat{L}^2(\sigma, \omega, z)}$$

$$\widehat{G}_i(\rho, \omega, y) = \int_{\mathbf{C}'} \frac{z^{i-m_0-1} e^{(z+\beta+\alpha\rho)\tau/\rho} dz}{\widehat{L}^2(\rho, \omega, z)}.$$

For $z \in \mathbf{C}'$ and $\rho \leq a$, we have

$$|e^{(z+\beta+\alpha\rho)\tau/\rho}| \leq a^{-(\beta+\alpha\rho)\tau/\rho} = e^{-(\beta\tau/\rho)+\alpha\tau} \leq e^{-(\beta/a)+\alpha}\tau.$$

Further restrict a so that $-\frac{\beta}{a} + \alpha \leq -\alpha$, and (3.7) is established.

LEMMA 3.4. - *There exist functions $N_k(\rho, \omega, \zeta)$, $k = 1, \dots, m_1$, with the properties*

(1) *for each ρ and ω , they are linearly independent polynomials in ζ of degrees $\leq m_1 - 1$.*

(2) *they are analytic in ρ and ω ,*

(3) *the limits $\lim_{\rho \rightarrow \infty} N_k(\rho, \omega, \zeta) = N_k(\infty, \omega, \zeta)$ exist,*

(4) *for any $a > 0$, $\rho > 0$ $|\omega| = 1$,*

$$(3.8) \quad \int_{\mathbf{C}_a} \frac{N_k(\rho, \omega, \zeta) \rho^{-\nu_k} B_k(\rho\omega, \rho\zeta)}{\prod_i \left(\zeta - \frac{\eta_i(\rho\omega)}{\rho} \right)} d\zeta = \delta_{kl}.$$

PROOF. - An analogous construction, based on the complementing condition, was given in [1]. In fact using that construction, we infer the existence of functions satisfying (1), (2), and (4). Only (3) must be checked. Setting $\varepsilon = \frac{1}{\rho}$, we have

$$\rho^{-\nu_i} B_i(\rho\omega, \rho\zeta) = B'_i(\omega, \zeta) + O(\varepsilon),$$

$$\Pi_i \left(\frac{\zeta - \eta_i(\rho\omega)}{\rho} \right) = L_1^+(\omega, \zeta) + O(\varepsilon).$$

Let $N_k(\infty, \omega, \zeta)$ be the polynomial satisfying

$$\int_{\mathcal{C}_a} \frac{N_k(\infty, \omega, \zeta) B'_i(\omega, \zeta)}{L_1^+(\omega, \zeta)} d\zeta = \delta_{ki}.$$

Such exist, according to the [1] construction and the last statement in Assumption IV). Again, the construction shows that the N_k are continuous in ε at $\varepsilon = 0$. This proves (3).

3.2. - Construction of the Poisson kernels.

We shall construct Poisson kernels $K_j(x, y)$ with the property that

$$(3.9) \quad u(x, y) = \sum_{j=1}^{m_1} K_j(x - \tau, y) \varphi_j(\tau) d\tau$$

is a solution of (3.1), (3.2).

For purposes of construction we introduce an infinitely differentiable cut-off function $\chi(\rho)$ defined for positive real ρ such that

$$\chi(\rho) = \begin{cases} 1, & 0 \leq \rho \leq \frac{1}{2}, \\ 0, & \rho \geq 1. \end{cases}$$

Recalling $\rho = |\xi|$ and $\omega = \frac{\xi}{\rho}$, we seek kernels in the form ($y > 0$)

$$(3.10) \quad K_j(x, y) = (2\pi)^{-n+1} \int_{|z|=a} \frac{dz}{2\pi iz} \int \chi(\rho/a) \left(\frac{\rho+z}{\rho} \right)^{n-1} A_{jk}(\xi + z\omega) G_k(\rho+z, \omega, y) \\ \cdot e^{ix \cdot (\xi + z\omega)} d\xi + (2\pi)^{-n+1} \int_{\mathcal{C}_a} d\zeta \int (1 - \chi(\rho/a)) \rho^{-\nu_j} N_j(\rho, \omega, \zeta) \left(\Pi \left(\zeta - \right.$$

$$\left. - \frac{\eta_i(\rho\omega)}{\rho} \right) \Big)^{-1} e^{\rho(ix, \omega + \zeta y)} d\xi \equiv K_{j0}(x, y) + K_{j1}(x, y),$$

where the number α and functions A_{jk} are to be determined. For the definition of $\mathbf{C}_\alpha$ see Lemma 3.2. There will be no convergence problems in the second integral because for $y > 0$, the integrand decays exponentially as $\xi \rightarrow \infty$. The first contour integral over $|z| = \alpha$ avoids any possible singularity A_{jk} will have at the origin.

It is clear from the construction of the G 's that for $y > 0$ all the K satisfy $L\left(\frac{1}{i}D_x, -D_y\right)K_j = 0$. We choose the functions A_{jk} with regard to the boundary conditions.

We set $A^{kl}(\sigma\omega, y) \equiv B_l(\sigma\omega, -D_y)G_k(\sigma, \omega, y) \equiv b_l(\sigma\omega, -D_y)G_k(\sigma, \omega, y)$, where (as polynomials in τ) $b_l(\xi, \tau) \equiv B_l(\xi, \tau) \pmod{L^+}$ and the order of $b_l \leq m_1 - 1$. By Lemma 3.3 we know that these functions are analytic in a neighborhood of the σ -origin. Our first restriction on the number α is that it be small enough so the set of values $\{\sigma = \rho + z, 0 \leq \rho \leq \alpha, |z| \leq \alpha\}$ is included in that neighborhood. Further restrictions will be imposed later. The matrix $A^{kl}(\sigma\omega, 0)$ is invertible for $\sigma \neq 0$; for if not there would be a combination $\sum_k c_k(\sigma\omega)A^{kl}(\sigma\omega, 0) = 0$ for all l . Writing this out in terms of the b_l , which by Assumption IV are linearly independent, for $\sigma \neq 0$, and using uniqueness of initial value problems for ordinary differential equations, we see that this would imply $\sum c_k G_k \equiv 0$, contradicting the linear independence of the G_k guaranteed in Lemma 3.3.

We choose $A_{jk}(\sigma\omega)$ to be the matrix inverse to $A^{kl}(\sigma\omega, 0)$. Its elements are then analytic in a deleted neighborhood of the σ -origin. Then

$$(3.11) \quad \lim_{\sigma \rightarrow 0} \sum_k A_{jk}(\sigma\omega)A^{kl}(\sigma\omega, y) = \delta_{jl},$$

the limit being uniform for σ in compact subsets of the deleted neighborhoods referred to.

LEMMA 3.5. - Let $K_j(x, y)$ be defined by (3.10), and let $\varphi_j(x)$, $j = 1, \dots, m_1$, be continuous functions with compact support. Then

$$(3.12) \quad \lim_{y \rightarrow 0} B_l\left(\frac{1}{i}D_x, -D_y\right) \int K_j(x - \tau, y) \varphi_j(\tau) d\tau = \delta_{jl} \varphi_j(x).$$

PROOF. - It will be sufficient to prove the lemma for $x = 0$. We consider the parts K_{j0} and K_{j1} separately. Accordingly

$$B_l\left(\frac{1}{i}D_x, -D_y\right) \int K_{j0}(x - \tau, y) \varphi_j(\tau) d\tau \Big|_{x=0} = \int B_l\left(\frac{1}{i}D_x, -D_y\right) K_{j0}(x - \tau, y) \varphi_j(\tau) \Big|_{x=0}.$$

We interchange the integrations with respect to τ here and z in (3.10), thus obtaining

$$\begin{aligned} & \sum_k (2\pi)^{-n+1} \int_{|z|=a} \frac{dz}{2\pi iz} \int d\xi \left(\frac{\rho+z}{\rho} \right)^{n-1} \chi(\rho/a) A_{jk}(\xi+z\omega) A^{kl}(\xi+z\omega, y) \int \varphi_j(\tau) e^{-i\tau(\xi+z\omega)} d\tau \\ &= \sum_k (2\pi)^{-\frac{(n-1)}{2}} \int_{|z|=a} \frac{dz}{2\pi iz} \int d\xi \left(\frac{\rho+z}{\rho} \right)^{n-1} \chi(\rho/a) A_{jk}((\rho+z)\omega) A^{kl}((\rho+z)\omega, y) \widehat{\varphi}_j((\rho+z)\omega), \end{aligned}$$

where $\widehat{\varphi}_j$ is the FOURIER transform of φ_j . We take the limit of this as $y \rightarrow 0$ to obtain from (3.11)

$$(3.13) \quad (2\pi)^{-(n-1)/2} \delta_{jl} \int \chi(\rho/a) \widehat{\varphi}_l(\xi) d\xi.$$

Similar operations may be performed on $B_l \left(\frac{1}{i} D_x, -D_y \right) \int K_{jl} \varphi_j d\tau$. In fact, the expression for K_{jl} decreases exponentially as $\rho \rightarrow \infty$ for $y > 0$, so the differential operator may be brought inside and integrations interchanged. We use (3.8) to then obtain

$$\lim_{y \rightarrow 0} B_l \int K_{jl} \varphi_j d\tau = (2\pi)^{-(n-1)/2} \int (1 - \chi(\rho/a)) \delta_{jl} \widehat{\varphi}_j(\xi) d\xi.$$

Combining this with (3.13) we obtain

$$\lim_{y \rightarrow 0} B_l \int K_j \varphi_j d\tau = \delta_{jl} (2\pi)^{-(n-1)/2} \int \widehat{\varphi}_j(\xi) d\xi = \delta_{jl} \varphi(0)$$

by the inversion theorem. This proves the lemma.

4. - Estimation of the Poisson kernels.

In this section we derive precise pointwise estimates for the Poisson kernels constructed in the previous section, together with their derivatives. In fact, we estimate $P \left(\frac{1}{i} D_x, -D_y \right) K_j(x, y)$, where $P(\xi, \eta)$ is any polynomial. It will be fixed in the discussion to follow. We denote its degree by m .

4.1. - Another representation, and definition of κ_{j1} .

Following the representation (3.10), it is natural to express

$$(4.1) \quad P\left(\frac{1}{i} D_x, -D_y\right) K_f(x, y) = L_{j0}(x, y) + L_{j1}(x, y) + L_{j2}(x, y)$$

where

$$(4.2a) \quad L_{j0} = (2\pi)^{-n+1} \int \frac{dz}{2\pi iz} \int_{|z|=a} \chi(\rho/a) \left(\frac{\rho+z}{\rho}\right)^{n-2} \sum_{k=1}^{m_0} A_{jk}(\xi+z\omega) P(\xi+z\omega, -D_y) G_k(\rho+z, \omega, y) e^{ix(\xi+z\omega)} d\xi;$$

$$(4.2b) \quad L_{j1} = [\text{the same expression with the summation going instead from } m_0+1 \text{ to } m_1];$$

$$(4.2c) \quad L_{j2} = P\left(\frac{1}{i} D_x, -D_y\right) K_{j1}(x, y).$$

We define κ_{j0} to be the largest integer for which the function

$$\rho^{-\kappa_{j0}} \sum_{j=1}^{m_0} A_{jk}(\rho\omega) P(\rho\omega, -D_y) G_k(\rho, \omega, y)$$

is bounded as $\rho \downarrow 0$ uniformly for all unit vectors ω and for $y \geq 0$. [These integers κ_{j0} , and the integers κ_{j1} to be defined below, are crucial for the estimates to be derived in this section.] In view of statement (2) of Lemma 3.3, we may express

$$(4.3) \quad \sum_{k=1}^{m_0} A_{jk}(\sigma\omega) P(\sigma\omega, -D_y) G_k(\sigma, \omega, y) \equiv \sigma^{\kappa_{j0}} \psi_{j0}(\sigma, \omega, \sigma y) e^{-\alpha\sigma y},$$

where for $\sigma = \rho$ real, ψ_{j0} satisfies

$$(4.4) \quad |\Psi_{j0}(\rho, \omega, \tau)| \leq E e^{-\alpha\tau}$$

for some constant E .

Similarly let κ_{j1} be the largest integer for which

$$\rho^{-\kappa_{j1}} \sum_{k=m_0+1}^{m_1} A_{jk}(\rho\omega) P(\rho\omega, -D_y) G_k(\rho, \omega, y)$$

is bounded as $\rho \downarrow 0$; then from statement (3) of Lemma 3.3, we may write

$$(4.5) \quad \sum_{k=m_0+1}^{m_1} A_{jk}(\sigma\omega) P(\sigma\omega, -D_y) G_k(\sigma, \omega, y) = \sigma^{\kappa_{j1}} \Psi_{j1}(\sigma, \omega, \sigma y) e^{-(\beta y + \alpha\sigma y)},$$

$$|\Psi_{j1}(\rho, \omega, \tau)| \leq E e^{-\alpha\tau}.$$

We may thus rewrite (4.2a, b) as

$$(4.6a) \quad L_{j0} = (2\pi)^{-n+1} \int \frac{dz}{2\pi iz} \int_{|\omega|=1} d\omega \int_0^a \chi(\rho/a)(\rho+z)^{\alpha_{j0}+n-2} \Psi_{j0}(\rho+z, \omega, (\rho+z)y) e^{i(\rho+z)(x \cdot \omega + ixy)} d\rho,$$

$$(4.6b) \quad L_{j1} = e^{-\beta y} (2\pi)^{-n+1} \int \frac{dz}{2\pi iz} \int_{|\omega|=1} d\omega \int_0^a \chi(\rho/a)(\rho+z)^{\alpha_{j1}+n-2} \Psi_{j1}(\rho+z, \omega, (\rho+z)y) e^{i(\rho+z)(x \cdot \omega + ixy)} d\rho.$$

4.2. - Statement of the results.

This section is devoted to proving the following theorem. Recall that μ_j is the order of B_j .

THEOREM 4.1. - *Assume $m + n - 2 - \mu_j \geq 0$. Then the following estimate holds for some C, C_0, C_1 , where we have set $r^2 = |x|^2 + y^2$.*

$$\left| P\left(\frac{1}{i}D_x, -D_y\right) K_j(x, y) \right| \leq \begin{cases} C[r^{1-n-\alpha_{j0}}(1 + C_0 \log r) + r^{1-n-\alpha_{j1}}e^{-\beta y}(1 + C_1 \log r)] & \text{for } r \geq \frac{1}{2}, \\ Cr^{1-n+\mu_j-m}, & \text{for } r \leq 2 \end{cases}$$

where $C_i = 0$ if $\alpha_{ji} + n - 2 \geq 0$, $i = 0, 1$.

4.3. - Preliminary lemmas.

Several lemmas will be useful for the purpose of estimating the L_{j1} . In what follows, the same symbol C will be used to denote various different constants. The dependence of C on certain parameters is sometimes denoted by subscripts. The proofs of the first two lemmas will be found in the appendix.

LEMMA 4.2. - *Let $\varphi(\rho, \omega, \tau)$ be an infinitely differentiable function defined for real ρ , $0 \leq \rho \leq 1$; real unit $(n-1)$ -vectors $\omega = (\omega_1, \dots, \omega_{n-1})$, and real $\tau \geq 0$. Suppose each derivative of φ is bounded in absolute value by an expression $Ce^{-\gamma\tau}$ for some $\gamma > 0$ where C may depend on the order of differentiation. Also suppose that φ vanishes for ρ near 1. Then for any integer $k \geq 0$, real $(n-1)$ -vector x , and real number $y \geq 0$,*

$$(4.7) \quad \left| \int_{|\omega|=1} d\omega \int_0^1 \rho^k \varphi(\rho, \omega, \rho y) e^{i\rho(x \cdot \omega + ixy)} d\rho \right| \leq C_k (|x|^2 + y^2)^{-(k+1)/2}.$$

LEMMA 4.3. - Let $\varphi(\rho, \omega)$ be an infinitely differentiable function defined for real ρ , $1 \leq \rho < \infty$, unit vectors ω . Suppose that φ and each of its derivatives with respect to components of ω has variation in ρ bounded uniformly in ω . Also suppose that the ρ -derivatives satisfy $|D_\rho^N \varphi(\rho, \omega)| \leq C_N \rho^{-N-1}$, $N \geq 1$. Finally suppose that φ vanishes for ρ near 1. Then for any $k \geq 0$, there exists a constant C_k such that for real $(n-1)$ -vectors x , and real numbers $y > 0$,

$$(4.8) \quad \left| \int_{|\omega|=1} d\omega \int_1^\infty \rho^k \varphi(\rho, \omega) e^{i\rho(x \cdot \omega + ixy)} d\rho \right| \leq C_k (|x|^2 + y^2)^{-(k+1)/2}.$$

LEMMA 4.4. - For any integer $k > 0$ and real ρ , $0 < \rho < a$,

$$\int_{|z|=a} (\rho + z)^{-k} \frac{dz}{z} = 0.$$

PROOF. - Immediate, by computation of residues.

DEFINITION. - Let $f(\rho)$ be infinitely differentiable near the origin for $\rho > 0$. Then for any integer $k \geq 0$, we define the symbol

$$(4.9) \quad [f(\rho)]_k^\rho \equiv f(\rho) - \sum_{n=0}^{k-1} \frac{\rho^n}{n!} f^{(n)}(0) \equiv \frac{\rho^k}{k!} f^{(k)}(\widehat{\rho}),$$

where $0 < \widehat{\rho} < \rho$.

COROLLARY TO LEMMA 4.4. - If $x_{j0} + n - 2 < 0$,

$$(4.10) \quad L_{j0}(x, y) = (2\pi)^{-n+1} \int_{|\omega|=1} d\omega \int \chi(\rho/a) \rho^{x_{j0}+n-2} [\Psi_{j0}(\rho, \omega, \rho y) e^{i\rho(x \cdot \omega + ixy)}]_{-(x_{j0}+n-2)}^\rho d\rho$$

A similar formula holds for L_{j1} if $x_{j1} + n - 2 < 0$.

PROOF. - Note that

$$\begin{aligned} \Psi_{j0}(\sigma, \omega, \sigma y) e^{i\sigma(x \cdot \omega + ixy)} &= [\Psi_{j0}(\sigma, \omega, \sigma y) e^{i\sigma(x \cdot \omega + ixy)}]_{-(x_{j0}+n-2)}^\sigma \\ &+ [\text{a certain number of terms of the form} \\ &\quad a(x, \omega, y)(\rho + z)^\nu \text{ where } \nu + x_{j0} + n - 2 < 0]. \end{aligned}$$

By Lemma 4.4, these latter terms contribute nothing to the integral in (4.6a). Hence the function in the integrand of (4.6a) may be bracketed as above. This leaves an integrand which is regular in ρ and z for ρ in a

neighborhood of the origin and for $|z| \leq a$. After interchanging the order of the integration with respect to ρ and z in the resulting integral, we obtain the formula (4.10) given.

LEMMA 4.5. - *Let w be a complex variable with imaginary part y . Let $f(\rho, \tau)$ be infinitely differentiable for $0 \leq \rho < 1$, $\tau \geq 0$, with each derivative bounded in this range. Then for any integer $b \geq 1$ and $|w| \geq 1$,*

$$(4.11) \quad \left| \int_0^a \chi(\rho/a) \rho^{-b} [f(\rho, \rho y) e^{i w \rho}]_b^\rho d\rho \right| \leq C |w|^{b-1} (1 + \log |w|).$$

PROOF. - By Leibniz's rule, the stated properties of f , and the fact that $y = \text{Im } w$, we see

$$\left| \left(\frac{d}{d\rho} \right)^n (f(\rho, \rho y) e^{i w \rho}) \right| \leq C_n |w|^n$$

for $w \geq 1$, $\rho \geq 0$ and $y \geq 0$. Thus from the right side of (4.9) we see

$$|\rho^{-b} [f e^{i w \rho}]_b^\rho| \leq C_b |w|^b.$$

Hence

$$\int_0^{1/|w|} \rho^{-b} [f e^{i w \rho}]_b^\rho d\rho \leq C_b |w|^{b-1}.$$

Also from (4.9) we have

$$[f e^{i w \rho}]_b^\rho = f e^{i w \rho} - \sum_{n=0}^{b-1} \left(\frac{d}{d\rho} \right)^n (f e^{i w \rho}) \bigg|_{\rho=0} \frac{\rho^n}{n!}.$$

Thus

$$|[f e^{i w \rho}]_b^\rho| \leq C(1 + (\rho |w|)^{b-1}), \text{ and}$$

$$\int_{1/|w|}^a \chi(\rho/a) \rho^{-b} [f e^{i w \rho}]_b^\rho d\rho \leq C \int_{1/|w|}^a (\rho^{-b} + |w|^{b-1} \rho^{-1}) d\rho \leq C |w|^{b-1} (1 + \log |w|).$$

Combining this with the above estimate yields (4.11).

4.4. - Proof of Theorem 4.1.

LEMMA 4.6. - *The functions L_{j0} and L_{j1} are bounded for $|x|^2 + y^2$ in bounded sets. Furthermore the following holds for $|x|^2 + y^2 \geq 1$ and some posi-*

tive constants C_0, C_1, C_2 depending only on the operators L and B_j :

$$(4.12a) \quad |L_{j0}(x, y)| \leq C_2(|x|^2 + y^2)^{-(x_{j0}+n-1)/2}(1 + C_0 \log(|x|^2 + y^2))$$

$$(4.12b) \quad |L_{j1}(x, y)| \leq C_2 e^{-\beta y}(|x|^2 + y^2)^{-(x_{j1}+n-1)/2}(1 + C_1 \log(|x|^2 + y^2)),$$

with $C_i = 0$ if $x_{ji} + n - 2 \geq 0, i = 0, 1$.

PROOF. - The first assertion is immediate, since the integrands in (4.2a, b) have no singularity on the integration domain for bounded x and y .

First we take the case $x_{j0} + n - 2 \geq 0$. Then interchanging the orders of the integrations with respect to z and ρ in (4.6a), we obtain

$$L_{j0} = \int_{|\omega|=1} d\omega \int \rho^{x_{j0}+n-2} \varphi(\rho, \omega, \rho y) e^{i(x \cdot \omega + i\alpha y)\rho} d\rho,$$

where

$$\varphi(\rho, \omega, \tau) = (2\pi)^{-n+1} \chi(\rho/a) \Psi_{j0}(\rho, \omega, \tau).$$

We shall check that φ satisfies the hypotheses of Lemma 4.2. It is certainly infinitely differentiable, since the G_k and A_{jk} are analytic in a deleted neighborhood of the σ -origin. Furthermore the factor χ guarantees φ to vanish for ρ near a . Thus we may apply Lemma 4.2 to conclude (4.12a) with $C_0 = 0$.

Next suppose $x_{j0} + n - 2 < 0$. Then we represent L_{j0} by (4.10) and apply Lemma 4.5 to obtain (4.12a).

This proves (4.12a), but (4.12b) is proved in the same manner. The presence of the extra factor $e^{-\beta y}$ is of course due to its presence in (4.6b).

LEMMA 4.7. - *If $m + n - 2 - \mu_j \geq 0$, then there exist constants C_N such that the following estimate holds for all x and for $y \geq 0$, and any integer $N \geq m + n - 1 - \mu_j$:*

$$(4.13) \quad |L_{j2}(x, y)| \leq C_N(|x|^2 + y^2)^{-N/2}$$

PROOF. - We recall the expression (3.10) for $K_{j1}(x, y)$ and (4.2c) for L_{j2} . We bring the differential operator P inside the integral sign, where it applies only to the factor $e^{\rho(ix \cdot \omega + \zeta y)}$. We then find

$$(4.14) \quad L_{j2}(x, y) = \int_{\tilde{C}_a} d\rho \int_{|\omega|=1} d\omega \int_0^\infty \rho^{-\mu_j+n-2+m} \varphi(\rho, \omega, \zeta) e^{i\rho(x \cdot \omega - i\zeta y)} d\rho$$

where

$$\varphi(\rho, \omega, \zeta) = (1 - \chi(\rho/a)) N_j(\rho, \omega, \zeta) \left(\Pi \left(\zeta - \frac{\eta_i(\rho\omega)}{\rho} \right) \right)^{-1} \rho^{-m} P(\rho\omega, -\rho\zeta).$$

We shall check that for each $\zeta \in C_a$, φ satisfies the hypotheses of Lemma 4.3. For each fixed ω and ζ , φ is analytic in ρ for large ρ and approaches a limit

$$\varphi(\infty, \omega, \zeta) = N_j(\infty, \omega, \zeta) \prod_i (\zeta - \gamma_i(\omega))^{-1} P'(\omega, -\zeta).$$

(P' is the principal part of P) as $\rho \rightarrow \infty$. Setting $v = \frac{1}{\rho}$ and $\Phi(v, \omega, \zeta) \equiv \varphi(\rho, \omega, \zeta)$, we have that Φ is analytic in v in an interval centered at $v=0$. Part of Assumption III is that the η_i are analytic in ω also, for $|\omega|=1$. The same is true of the N_k by Lemma 3.4. The analytic dependence of Φ on ζ for $\zeta \in C_a$ is clear. In all, we find that Φ and all its derivatives will be analytic for v near 0, $|\omega|=1$ and $\zeta \in C_a$. In particular these derivatives will be bounded in this range. This implies that

$$|D_\rho^N \varphi| = |(-v^2 D_v)^N \varphi| = \left| \sum_{\nu=1}^N b_\nu v^{N+\nu} D_v^\nu \varphi \right| \leq C_N v^{N+1} = C_N \rho^{-N-1}.$$

for large ρ , uniformly in ω and ζ . This establishes part of the hypotheses. The same estimates are true of all derivatives of φ with respect to components of ω .

Thus the hypotheses of Lemma 4.3 are verified, and we may state the conclusion: if $\mu_j + n - 2 + m \geq 0$, then

$$|L_{j2}(x, y)| \leq C(|x|^2 + y^2)^{(\mu_j+1-n-m)/2}.$$

This proves (4.13) with $N = n + m - 1 - \mu_j$. To prove for $N > n + m - 1 - \mu_j$, simply define

$$\psi \equiv \rho^{-\mu_j+n-2+m-N} \varphi,$$

and replace the integrand in (4.14) by $\rho^N \psi e^{i\rho(x \cdot \omega - i\epsilon y)}$.

Lemmas 4.6 and 4.7 now yield Theorem 4.1.

4.5. - Extension to $y = 0$.

Although the Poisson kernels were originally defined only for $y > 0$, these estimates imply that they and their derivatives can be extended as continuous functions for $y = 0$ as well.

THEOREM 4.2. - *The kernels $K_j(x, y)$ and all their derivatives may be extended as continuous functions in the closed half space $y \leq 0$, exclusive of the origin. In particular,*

$$B_l \left(\frac{1}{i} D_x, -D_y \right) K_j(x, y) \Big|_{y=0} \equiv 0$$

for $x \neq 0$ or for $l \neq j$.

PROOF. - Let S be a bounded subset of the open half-space $\{y > 0\}$ such that S is adjacent to $\{y = 0\}$, and is bounded away from the origin. In S , each K_j is infinitely differentiable: furthermore Theorem 4.1 shows that each derivative of high enough order is bounded in S . By integration, this property is extended to all derivatives. Therefore all the derivatives have limits as $y \rightarrow 0$ which are continuous functions of x .

The second statement of the theorem now follows from Lemma 3.5. In fact, if, at some point $x_0 \neq 0$ the stated function were not zero, a contradiction to (3.12) could be deduced by choosing φ_i to have small support in a neighborhood of x_0 .

5. - Potential theory.

In this section, which is independent of the preceding material, we develop estimates for functions u given by integrals of the form

$$(5.1) \quad u(x, y) = \int_S K(kx, ky; k\xi) f(\xi) d\xi$$

where k is a large positive parameter, and the behavior of K is known both for large and for small values of its arguments. More specifically, pointwise bounds on the magnitude of u and its HÖLDER difference quotients with respect to x will be obtained. We use the notation $x = (x_1, \dots, x_{n-1})$ and $\xi = (\xi_1, \dots, \xi_{n-1})$; also (x, y) will denote a point in the n -dimensional half-space $y > 0$. The symbol $S_a(x_0)$ for any $a > 0$ will denote the $(n-1)$ -ball $\{\xi : |\xi - x_0| < a\}$; the region of integration in (5.1) is specified as $S = S_d(0)$, where d is a fixed number greater than $2k^{-1}$. Finally, we define $\varepsilon = 1/k$, $r^2 = |x - \xi|^2 + y^2$ and $\rho^2 = |x - \xi|^2$.

5.1. - The estimates.

It will always be assumed that $y > 0$.

The following assumptions are imposed on K . There are real numbers $t \geq 0$ and s , and constants A and B (more generally, we allow A and B to be functions of y and k), such that

$$(5.2) \quad |K(x, y; \xi)| \leq \begin{cases} Ar^{1-n+t} & \text{for } r < 2, \\ B\rho^{1-n+s} & \text{for } r > \frac{1}{2}, \end{cases}$$

In the case $t = 0$ only, we further assume the following estimates (5.3-5.5)

$$(5.3) \quad |D_x K|, |D_\xi K| \leq \begin{cases} Ar^{-n} & \text{for } r < 2 \\ Br^{-n+s} & \text{for } r > \frac{1}{2}. \end{cases}$$

It is assumed also that K may be represented as $K = \sum_{i=1}^{n-1} \frac{\partial}{\partial \xi_i} L_i(x, y; \xi)$, where

$$(5.4) \quad |L_i(x, y; \xi)| \leq \begin{cases} Ar^{2-n} & \text{for } r < 2 \\ Br^{2-n+s} & \text{for } r > \frac{1}{2}, \end{cases}$$

$$(5.5) \quad |D_x L_i|, |D_\xi L_i| \leq \begin{cases} Ar^{1-n} & \text{for } r < 2 \\ Br^{1-n+s} & \text{for } r > \frac{1}{2}. \end{cases}$$

The following notation will be used:

$$|f|_0 = \sup_S |f(x)|,$$

$$[f]_\alpha = \sup_{x, x' \in S} \frac{|f(x) - f(x')|}{|x - x'|^\alpha},$$

where $0 < \alpha < 1$.

THEOREM 5.1. - *Under the hypotheses (5.2-5.5) with $t = 0$, the following estimates hold for $|x| < d/2$, $0 < y < d/2$ (here we set $\lambda = \max(1, ky)$):*

$$(5.6) \quad |u(x, y)| \leq Ck^{1-n} \{ B(kd)^s |f|_0 + [A\epsilon^\alpha + Bd^\alpha(\lambda^s + (kd)^s)][f]_\alpha \}.$$

$$(5.7) \quad \frac{|u(x, y) - u(0, y)|}{|x|^\alpha} \leq Ck^{1-n} \{ B(kd)^s d^{-\alpha} |f|_0 + [A + B(\lambda^s + (kd)^s)][f]_\alpha \}$$

where C depends only on n, s , and α . When $y > \epsilon$ we may set $A = 0$ in (5.6, 7).

THEOREM 5.2. - *Let K satisfy (5.2-5) with $t = 0$ and $s < 0$, and let $d = \infty$, so the integration in (5.1) is over all space. Then*

$$(5.8a) \quad |u(x, y)| \leq Ck^{1-n} B(ky)^s [|f|_0 + y^\alpha [f]_\alpha], \quad y \geq \epsilon;$$

$$(5.8b) \quad |u(x, y)| \leq Ck^{1-n} [B |f|_0 + A\epsilon^\alpha [f]_\alpha], \quad y \leq \epsilon.$$

THEOREM 5.3. - Let $K(x, \xi; y)$ satisfy (5.2) with $s \neq 0$. Let $|x| \leq d/2$. If $t > 0$, then

$$(5.9) \quad |u(x, y)| \leq Ck^{1-n} \{ A + B[\lambda^s + (kd)^s] \} \|f\|_0.$$

Furthermore if $y > \varepsilon$, and $t \geq 0$, (5.9) holds with the term A missing.

5.2. - Preliminary inequalities.

Most of the estimates involved in Theorems 5.1-5.3 will be obtained by comparing various integrals with one of the specific integrals treated in the following lemma and its corollary.

LEMMA 5.4. - Let a, b, y, γ , and β be real numbers satisfying $0 \leq a \leq b$, $0 < y < b$, and $\gamma > -1$. Define $\delta = \gamma - \beta + 1$ and assume $|\delta| > 0$. Then there is a C depending only on lower bounds for $|\delta|$ and $\gamma + 1$ such that

$$\int_a^b \rho^\gamma (\rho^2 + y^2)^{-\beta/2} d\rho \leq C[(\text{Max } [a, y])^\delta + b^\delta].$$

PROOF. - For $y \leq \rho$ we have $\rho^2 \leq \rho^2 + y^2 \leq 2\rho^2$, and for $y \geq \rho$ we have $y^2 \leq \rho^2 + y^2 \leq 2y^2$. Use these estimates for $\rho^2 + y^2$ in the integrand, and integrate the resulting expression exactly, splitting the interval of integration into the segments (a, y) and (y, b) if $y > a$.

COROLLARY 5.5. - Let $a, y, k = \varepsilon^{-1}$, n, q, α , and b be numbers (n and q integers), all nonnegative except possibly q , such that $\mu \equiv \text{Max } [a, y] \leq b$, $n \geq 2$, $0 < \alpha < 1$. Set $r^2 = \rho^2 + y^2$. There is a C depending only on α such that

$$(5.10) \quad k^{1-n+q} \int_a^b r^{1-n+q} \rho^{\alpha+n-2} d\rho \leq Ck^{1-n} [(k\mu)^q + (kb)^q] b^\alpha.$$

If, furthermore, σ is a number satisfying $0 < \sigma \leq \mu \leq b$, then

$$(5.11) \quad \sigma k^{1-n+q} \int_a^b r^{1-n+q} \rho^{\alpha+n-2} d\rho \leq Ck^{1-n} [(k\mu)^q + (kb)^q] \sigma^\alpha.$$

PROOF. - Set $\gamma = \alpha + n - 2$, $\beta = n - 1 - q$. The lemma yields that the integral in (5.10) is bounded by

$$\begin{aligned} Ck^{1-n+q} [\mu^{\alpha+q} + b^{\alpha+q}] &= Ck^{1-n} [\mu^\alpha (k\mu)^q + b^\alpha (kb)^q] \\ &\leq Ck^{1-n} b^\alpha [(k\mu)^q + (kb)^q]. \end{aligned}$$

Similarly the integral in (5.11) is bounded by

$$\begin{aligned} C\sigma k^{1-n+q}[\mu^{\alpha+q-1} + b^{\alpha+q-1}] \\ = Ck^{1-n}[(\sigma/\mu)^{1-\alpha}(k\mu)^q + (\sigma/b)^{1-\alpha}(kb)^q]\sigma^\alpha \\ \leq Ck^{1-n}[(k\mu)^q + (kb)^q]\sigma^\alpha. \end{aligned}$$

In the first case the constant C depends only on lower bounds for $|\delta| = |\alpha + q|$ and $\gamma + 1 = \alpha + n - 1$. Since q is an integer and $n \geq 2$, C really depends only on α . The same argument holds in the second case.

5.3. - Proof of Theorem 5.1.

In the following the letter C will be used to denote various different constants, each depending only on n, s, t , and α .

The following representation is immediate:

$$\begin{aligned} u(x, y) &= \int_S K(kx, ky; k\xi)[f(\xi) - f(x)]d\xi \\ &+ f(x) \int_S \sum_{i=1}^{n-1} D_{\xi_i} L_i(kx, ky; k\xi) d\xi \equiv U_1(x, y) + U_2(x, y); \end{aligned}$$

furthermore U_2 may be transformed into the form

$$U_2(x, y) = \varepsilon f(x) \sum_{i=1}^{n-1} \int_{\partial S} L_i(kx, ky; k\xi) \gamma_i(\xi) d\xi,$$

where γ_i is the i -th direction cosine of the outward normal to the boundary ∂S

For purposes of deriving (5.7) we define $S_0 = S_{2|x|}(0)$ if $y \leq 2|x|$, S_0 void if $y > 2|x|$, and write

$$u(x, y) - u(0, y) = J_1 + \dots + J_5.$$

where

$$\begin{aligned} J_1 &= \int_{S-S_0} [K(kx, ky; k\xi) - K(0, ky; k\xi)][f(\xi) - f(x)]d\xi, \\ J_2 &= [-f(x) + f(0)] \int_{S-S_0} K(0, ky; k\xi) d\xi \\ &= (f(0) - f(x)) \varepsilon \int_{\partial(S-S_0)} \sum L_i(0, ky; k\xi) \gamma_i(\xi) d\xi, \end{aligned}$$

$$J_3 = \int_{S_0} K(kx, ky; k\xi) [f(\xi) - f(x)] d\xi,$$

$$J_4 = - \int_{S_0} K(0, ky; k\xi) [f(\xi) - f(0)] d\xi,$$

$$J_5 = U_2(x, y) - U_2(0, y).$$

Estimate (5.6) will be established by proving it separately for U_1 and U_2 . In like manner (5.7) will be proved separately for each of the J_i .

PROOF OF (5.6) FOR U_1 . - Consider two cases:

CASE 1. $y \leq \varepsilon$. - Split the domain S of integration into $S_\varepsilon(x)$ and $S - S_\varepsilon(x)$, and use (5.2, 5.3). (Since $d > 2\varepsilon$ and $|x| < d/2$, $S_\varepsilon(x) \subset S_0$.) In the first domain $|K| \leq A(kr)^{1-n}$, and in the second, $|K| \leq B(kr)^{1-n+s}$. In both cases $|f(\xi) - f(x)| \leq \rho^\alpha [f]_\alpha$.

Thus

$$\begin{aligned} |U_1(x, y)| &\leq \int_{S_\varepsilon(x)} A(kr)^{1-n} \rho^{n-2+\alpha} [f]_\alpha d\xi \\ &\quad + \int_{S-S_\varepsilon(x)} B(kr)^{1-n+s} \rho^{n-2+\alpha} [f]_\alpha d\xi \\ &\leq CAk^{1-n} [f]_\alpha \int_0^\varepsilon r^{1-n} \rho^{\alpha+n-2} d\rho \\ &\quad + CBk^{1-n+s} [f]_\alpha \int_\varepsilon^d r^{1-n+s} \rho^{\alpha+n-2} d\rho. \end{aligned}$$

To the first integral apply (5.10) with $a = 0$, $b = \varepsilon$, $q = 0$, and to the second apply (5.10) with $a = \varepsilon$, $b = d$, $q = s$; (5.6) is seen to be verified.

CASE 2. $y > \varepsilon$. - Then no splitting is needed: $|K| \leq B(kr)^{1-n+s}$ for the entire range:

$$|U_1(x, y)| \leq CB[f]_\alpha k^{1-n+s} \int_0^d r^{1-n+s} \rho^{\alpha+n-2} d\rho.$$

Apply (5.10) with $a = 0$, $b = d$ to verify (5.6).

PROOF OF (5.6) FOR U_2 . - From (5.4) and $\int_S d\xi \leq Cd^{n-2}$ we see that

$$|U_2(x, y)| \leq C\varepsilon |f(x)| B(kd)^{2-n+s} d^{n-2} \leq CBk^{1-n}(kd)^s |f|_0,$$

which inequality is included in (5.6).

PROOF OF (5.7) FOR J_1 . - Throughout the proof we use the symbol $\sigma = |x|$.

CASE 1. $y \geq 2\sigma$, $y \geq \varepsilon$. - Using the shorthand notation $J_1 = \int_{S-S_0} \Delta K \Delta f$, we estimate, by the mean value theorem and (5.3),

$$(5.12) \quad |\Delta K| = |k\sigma D_x K(k\hat{x}, ky; k\xi)| \leq k\sigma B(k\hat{r})^{-n+s} \leq CB\sigma k^{1-n+s} r^{-n+s}$$

(here $\hat{r}^2 = |\hat{x} - \xi|^2 + y^2$ and the last inequality follows from $y \geq 2\sigma$, since $r \leq |x - \hat{x}| + \hat{r} \leq \sigma + \hat{r} \leq y + \hat{r} \leq 2\hat{r}$). Also

$$|\Delta f| \leq [f]_\alpha \rho^\alpha.$$

Thus $|J_1| \leq CB[f]_\alpha \sigma k^{1-n+s} \int_0^d r^{-n+s} \rho^{\alpha+n-2} d\rho$. Apply (5.11) with $a=0$, $b=d$, $q=s$ to verify (5.7).

CASE 2. $2\sigma \leq y \leq \varepsilon$. - Write $J_1 = J_{11} + J_{12}$, where $J_{11} = \int_{S_\varepsilon(0)}$, $J_{12} = \int_{S-S_\varepsilon(0)}$. For J_{11} , the inequality analogous to (5.12) is

$$|\Delta K| \leq CA\sigma k^{1-n} r^{-n}.$$

Hence ((5.11) with $b = \varepsilon$, $q = 0$)

$$|J_{11}| \leq CA[f]_\alpha \sigma k^{1-n} \int_0^\varepsilon r^{-n} \rho^{\alpha+n-2} d\rho \leq CAk^{1-n} [f]_\alpha \sigma^\alpha.$$

For J_{12} , (5.12) holds and we use (5.11) with $a = \varepsilon/2$, $b = 2d$ to attain the desired estimate.

CASE 3. $y \leq 2\sigma$, $\varepsilon \leq 2\sigma$. - In this case $S_0 = S_{2\sigma}(0)$ and $\xi \in S - S_0$, $\hat{x} \in S_\sigma(0)$ imply $\hat{r} \geq |\xi - \hat{x}| \geq \sigma$, so $r \leq |x - \hat{x}| + \hat{r} \leq \sigma + \hat{r} \leq 2\hat{r}$. Similarly $\hat{r} \leq 2r$. Hence (5.12) is valid, and we have

$$|J_1| \leq CBk^{1-n+s} \sigma \int_\sigma^{2d} r^{-n+s} \rho^{\alpha+n-2} d\rho.$$

The use of (5.11) with $a = \sigma$, $b = 2d$ yields (5.7).

CASE 4. $y \leq 2\sigma \leq \varepsilon$. - Write $J_1 = J_{13} + J_{14}$, where

$$J_{13} = \int_{\bar{S} - S_\varepsilon(0)}, \quad J_{14} = \int_{S_\varepsilon(0) - S_{2\sigma}(0)}.$$

For J_{13} , use (5.12):

$$|J_{13}| \leq CB\sigma k^{1-n+s} \int_{\varepsilon/2}^{2d} r^{-n+s} \rho^{z+n-2} d\rho.$$

Again (5.11) with $a = \varepsilon/2$, $b = 2d$, $q = s$ yields the result. Similarly

$$|J_{14}| \leq CA\sigma k^{1-n} \int_{\sigma}^{2\varepsilon} r^{-n} \rho^{z+n-2} d\rho$$

and (5.11) again applies.

PROOF OF (5.7) FOR J_2 . - Write $J_2 = J_{21} - J_{22}$, where $J_{21} = \int_{\partial \bar{S}}$ and $J_{22} = \int_{\partial S_0}$. For J_{21} we use $|L_i| \leq B(kd)^{2-n+s}$ and the fact that $\int_{\partial \bar{S}} d\xi \leq Cd^{n-2}$ to obtain

$$|J_{21}| \leq C[f]_{\alpha} \sigma^{\alpha} \varepsilon B(kd)^{2-n+s} d^{n-2} \leq CB[f]_{\alpha} k^{1-n} (kd)^s \sigma^{\alpha}.$$

$J_{22} = 0$ if $y \geq 2\sigma$; for $y \leq 2\sigma$ we consider two cases. When $\varepsilon \leq 2\sigma$, the estimate for J_{21} is repeated with d replaced by σ . We then obtain the estimate $CB[f]_{\alpha} k^{1-n} (k\sigma)^s \sigma^{\alpha}$, which can be seen to be bounded by the right hand side of (5.7), when we observe that in this case $m \leq 2k\sigma \leq 2kd$.

When $\varepsilon \geq 2\sigma$, the same is done with B furthermore replaced by A and s replaced by 0. Then

$$|J_{22}| \leq CA[f]_{\alpha} k^{1-n} \sigma^{\alpha}.$$

Thus in all cases (5.7) is verified.

PROOF OF (5.7) FOR J_3 AND J_4 . - Since J_4 is a special case of J_3 we need only estimate the latter. We may assume $y \leq 2\sigma$, since otherwise S_0 is empty and $J_3 = J_4 = 0$.

CASE 1. - $\varepsilon \geq \sigma$. Here we use (5.2) to obtain

$$|J_3| \leq \int_{S_0} A(kr)^{1-n} [f]_{\alpha} \rho^{\alpha} d\xi.$$

For ξ in S_0 , ρ cannot surpass 3σ , so

$$|J_3| \leq CA[f]_\alpha k^{1-n} \int_0^{3\sigma} r^{1-n} \rho^{\alpha+n-2} d\rho.$$

By applying (5.10) with $a = 0$ and $b = 3\sigma$, (5.7) is verified.

CASE 2. $\varepsilon < \sigma$. - Here we set $J_3 = J_{31} + J_{32}$, where $J_{31} = \int_{S_\varepsilon(x)}$, $J_{32} = \int_{S_0 - S_\varepsilon(x)}$. The above estimate is repeated for J_{31} with σ replaced by ε ; the same result is obtained. In a similar way we have

$$|J_{32}| \leq CB[f]_\alpha k^{1-n+s} \int_{\varepsilon/2}^{3\sigma} r^{1-n+s} \rho^{\alpha+n-2} d\rho.$$

Again (5.10) applies and (5.7) is verified.

PROOF OF (5.7) FOR J_5 . - Referring to the definition, we write $J_5 = J_{51} + J_{52}$, where

$$J_{51} = \varepsilon[f(x) - f(0)] \sum_i \int_{\partial S} L_i(kx, ky; k\xi) \gamma_i(\xi) d\xi,$$

$$J_{52} = \varepsilon f(0) \sum_i \int_{\partial S} [L_i(kx, ky; k\xi) - L_i(0, ky; k\xi)] \gamma_i d\xi.$$

The estimation of J_{51} goes exactly like that of J_2 , so will not be given. For J_{52} we use

$$|\Delta L_i| \leq |D_x L_i(k\hat{x}, k\xi)| k\sigma.$$

Remembering that for $|\xi| = d$ and $|\hat{x}| < d/2$, we have $(|\hat{x} - \xi|^2 + y^2) \geq C^{-1}d^2$ and using (5.5), we obtain

$$|J_{52}| \leq C\varepsilon |f|_0 B(kd)^{1-n+s} (k\sigma) d^{n-2} = CB |f|_0 k^{1-n+s} d^{s-1} \sigma$$

$$\leq CB |f|_0 k^{1-n} (kd)^s d^{-\alpha} \sigma^\alpha,$$

verifying (5.7).

This completes the proof of (5.6), (5.7). As for the last statement in the theorem, we note that when $y > \varepsilon$, $kr \geq 1$ for all ξ in the domain of integration; hence the value of A in (5.2, 3.4) is irrelevant; we may as well replace it by 0. This finishes the proof of Theorem 5.1.

5.4. - Proof of theorem 5.2.

With no loss of generality we may suppose $\alpha = 0$. First consider the case $y > \varepsilon$. Then write

$$(5.13) \quad u = u_1 + u_2$$

where $u_1 = \int_{S_{2y}(0)} K f d\xi$, $u_2 = \int_{\rho > 2y} K f d\xi$. For u_1 we have the estimate (5.6) with $A = 0$ and $d = 2y$:

$$|u_1(0, y)| \leq Ck^{1-n}B(ky)^s(|f|_0 + y^2[f]_\alpha).$$

For u_2 we use the fact that for $\rho > 2y$, $\rho \leq r \leq 2\rho$; hence

$$|u_2(0, y)| \leq C \int_{2y}^{\infty} B(k\rho)^{1-n+s} |f|_0 \rho^{n-2} d\rho \leq CBk^{1-n}(ky)^s |f|_0.$$

Thus the estimate (5.10a) holds for u_2 as well.

Next consider the case $y < \varepsilon$ and use a decomposition (5.13) where this time $u_1 = \int_{S_{2\varepsilon}(0)}$ and $u_2 = \int_{\rho > 2\varepsilon}$. For u_1 we have (5.6) with $B = 0$:

$$|u_1(0, y)| \leq CAk^{1-n}\varepsilon^\alpha[f]_\alpha,$$

For u_2 we use the same reasoning as above to obtain

$$|u_2(0, y)| \leq C \int_{2\varepsilon}^{\infty} B(k\rho)^{1-n+s} |f|_0 \rho^{n-2} d\rho \leq CBk^{1-n} |f|_0.$$

Altogether we obtain (5.8) and the theorem is proved.

5.5. - Proof of theorem 5.3.

Suppose first that $y \geq \varepsilon$. Then from (5.1) we have

$$\begin{aligned} |u(x, y)| &\leq C \int_0^d B(kr)^{1-n+s} \rho^{n-2} |f|_0 d\rho \\ &= CBk^{1-n+s} |f|_0 \int_0^d r^{1-n+s} \rho^{n-2} d\rho. \end{aligned}$$

Applying Lemma 5.4 yields (5.9) with A missing.

Now suppose $y \leq \varepsilon$. Then we have

$$|u| \leq C \left[\int_0^\infty A(kr)^{1-n+\varepsilon} |f|_0 \rho^{n-2} d\rho + \int_\varepsilon^d B(kr)^{1-n+s} |f|_0 \rho^{n-2} d\rho \right].$$

Applying Lemma 5.4, we obtain (5.9). This completes the proof.

6. - Estimation of solutions in the half-space.

6.1. - Construction of solutions.

For functions $\varphi_j(x)$, $j = 1, 2, \dots, m_1$, we define

$$(6.1) \quad u_j(x, y) = k^{n+1} \int_{R^n} K_j(k(x - \xi), ky) \varphi_j(\xi) d\xi \quad (\text{where } k = \varepsilon^{-1}).$$

In the theorems to follow, the degree of smoothness of the φ_j will either be explicitly stated, or assumed to be sufficient for the conclusions to be meaningful.

LEMMA 6.1. - *Let the φ_j be continuous with compact support. Then the functions $u_j(x)$ satisfy*

$$(6.2a) \quad L\left(\frac{\varepsilon}{i} D_x^-\right) u_j(\bar{x}) = 0, \quad x_n = y > 0,$$

$$(6.2b) \quad B\left(\frac{\varepsilon}{i} D_x, -\varepsilon D_y\right) u_j \Big|_{y=0} = \delta_{ji} k^{n_j} \varphi_j(x).$$

The same is true if the φ_j are not of compact support, but decay sufficiently rapidly at ∞ .

PROOF. - The Poisson kernels were constructed to satisfy

$$L\left(\frac{1}{i} D_x^-\right) K_j(x - \xi; y) = 0 \quad \text{for } y > 0;$$

it follows immediately that (6.2a) holds. Furthermore defining $x' = kx$, $\xi' = ky$,

$y' = kx$, we have

$$\begin{aligned} & k^{\gamma_j + n - 1} B_i \left(\frac{\varepsilon}{i} D_x, -\varepsilon D_y \right) \int K_j(k(x - \xi), ky) \varphi_j d\xi \\ &= k^{\gamma_j + n - 1} B_i \left(\frac{1}{i} D_{x'}, -D_{y'} \right) \int K_j(x' - \xi', y') \varphi_j(\varepsilon \xi') \varepsilon^{n-1} d\xi, \end{aligned}$$

which by Lemma 3.5 approaches $k^{\gamma_j} \delta_{ji} \varphi_j(\varepsilon x') = \delta_{ji} k^{\gamma_j} \varphi_j(x)$ as $y \rightarrow 0$. This establishes the lemma for the case of compact support.

We now define κ_j to be the minimum of the numbers κ_{j0} and κ_{j1} occurring in Theorem 4.1, for the case when the polynomial $P \equiv 1$. According to Theorem 4.1, K_j grows or decays like $r^{1-n-\kappa_j}$ or $r^{1-n-\kappa_j} \log r$ as $r \rightarrow \infty$. It follows that if φ_j has order of magnitude r^{-q} as $r \rightarrow \infty$, where $q + \kappa_j > 0$, the integral in (6.1) will converge, and in fact uniformly for x and y in bounded sets. Thus u_j will be defined.

For any $r_0 > 0$, we can separate the region of integration into $S_{r_0}(0)$ and the complement, inducing a decomposition

$$u_j = u_{jr_0} + v_{jr_0},$$

with u_{jr_0} obtained as in (6.1), except that the integral is only over $S_{r_0}(0)$. Suppose that x is a fixed point in $S_1(0)$, and that for every derivative D^ν occurring in the operators L and B_i , $D^\nu K(x, y) \leq O(r^{1-n-\kappa} \log r)$ ($r \rightarrow \infty$), where $\kappa + q > 0$. Thus u_j may be differentiated to obtain $Lu_j = 0$. Furthermore $B_i v_{jr_0} \rightarrow 0$ as $r_0 \rightarrow \infty$, uniformly for small y , and of course $\lim_{y \rightarrow 0} B_i u_{jr_0} = \delta_{ji} k^{\gamma_j} \varphi_j(x)$ for each $r_0 > 1$. Given any $\varepsilon' > 0$, we may choose r_0 so large that $|B_i v_{jr_0}| < \varepsilon'/2$, then y so small that $B_i u_{jr_0}(x, y)$ is closer than $\varepsilon'/2$ to its limit. Then $|B_i u_j - \delta_{ji} k^{\gamma_j} \varphi_j| < \varepsilon'$. This proves that (6.2) holds, completing the proof of the lemma.

Hence the function $u(x, y) \equiv \sum_{j=1}^{m_1} u_j(x, y)$ is a solution of our original problem (1.1), (1.8).

We shall be interested in estimating

$$P \left(\frac{\varepsilon}{i} D_x, -\varepsilon D_y \right) u_j(x, y),$$

where P is an arbitrary polynomial (to be identified with that in Sec. 4). Corresponding to (4.1) we make the decomposition

$$(6.3) \quad Pu_j = Pu_{j0} + Pu_{j1}$$

where

$$(6.4a) \quad Pu_{j0}(x, y) = k^{r_j+n-1} \int_{R^n} (L_{j0}(kx - k\xi, ky) + L_{j2}(kx - k\xi, ky)) \varphi_j(\xi) d\xi,$$

$$(6.4b) \quad Pu_{j1}(x, y) = k^{r_j+n-1} \int_{R^n} L_{j1}(kx - k\xi, ky) \varphi_j(\xi) d\xi.$$

6.2. - Estimation of Pu_{j0} .

In the following the single symbol C will be used to denote many different constants independent of u_j , φ_j , x , y , d , or ε , but possibly depending on the other parameters of the problem. The integers μ_j and γ_j are those introduced in Sec. 1.2, and κ_{j0} , κ_{j1} are those defined in Sec. 4.1. The degree of the polynomial P is denoted by m , and we define $\lambda = \text{Max}[1, ky]$. It will be assumed that φ_j has support in $S_d(0)$, where d is a finite number or ∞ . If $d = \infty$, we assume (1) that φ_j is bounded and (2) that φ_j decays rapidly enough at infinity that the integral in (6.1) converges, as well as the integrals in (6.4). Finally, we set $\sigma = \text{Max}[d, 2r]$, where $r^2 = |x|^2 + y^2$.

THEOREM 6.2. (*The Case $m < \mu_j$*). - Assume $\kappa_{j0} + n - 2 \geq 0$, $m < \mu_j$, and either σ is finite or $\kappa_{j0} > 0$. Then

$$(6.5) \quad \left| P\left(\frac{\varepsilon}{i} D_x, -\varepsilon D_y\right) u_{j0}(x, y) \right| \leq C k^{r_j} [\lambda^{-\kappa_{j0}} + (k\sigma)^{-\kappa_{j0}}] \|f\|_0$$

(if $\kappa_{j0} > 0$ and $\sigma = \infty$, the last term in brackets is understood to vanish). If $\kappa_{j0} + n - 2 < 0$, then (6.5) holds with κ_{j0} replaced by $\kappa_{j0} - \varepsilon'$ for any $\varepsilon' > 0$. Then C depends on ε' .

PROOF. - Let $t = \mu_j - m > 0$. We first establish that the kernel in (6.4a) satisfies (5.2) with $s = -\kappa_{j0}$ and $A = B = C$. The first inequality in (5.2) follows from the first statement of Lemma 4.6 together with (4.13) with $N = n - 1 - t$. The second follows from (4.12a) and (4.13) with large N . Also observe that by definition of σ we always have $|x| < \sigma/2$, $0 < y < \sigma/2$, and the support of φ_j will certainly lie in $S_\sigma(0)$. The result (6.5) then follows immediately from Theorem 5.3, with d replaced by σ . Now consider the case $\kappa_{j0} + n - 2 < 0$. The logarithm term which then appears in (4.12a) may be eliminated by subtracting ε' from the κ_{j0} in (4.12a). Then (6.5) follows with the same replacement made.

THEOREM 6.3. (*The Case $m = \mu_j$*). - Assume $\kappa_{j0} + n - 2 \geq 0$, $m = \mu_j$, and σ is finite. Then

$$(6.6) \quad |Pu_{j0}(x, y)| + \sigma^x \frac{|Pu_{j0}(x, y) - Pu_{j0}(0, y)|}{|x|^x} \\ \leq Ck^{y_j} \{ (k\sigma)^{-x_{j0}} |\varphi_j|_0 + \sigma^x (\lambda^{-x_{j0}} + (k\sigma)^{-x_{j0}}) [\varphi_j]_x \}.$$

If $x_{j0} \neq 0$ and $y \geq \varepsilon$, then

$$(6.7) \quad |Pu_{j0}(x, y)| \leq Ck^{y_j} [\lambda^{-x_{j0}} + (k\sigma)^{-x_{j0}}] |\varphi_j|_0.$$

If $\sigma = \infty$, $x_{j0} > 0$, and $y \leq \varepsilon$, then

$$(6.8) \quad |Pu_{j0}(x, y)| \leq Ck^{y_j} (|\varphi_j|_0 + \varepsilon^x [\varphi_j]_x).$$

If $x_{j0} + n - 2 < 0$, the above estimates hold with x_{j0} replaced by $x_{j0} - \varepsilon'$.

PROOF. - We must verify that the kernel in (6.4a) satisfies (5.2-5) with $t = 0$ and $s = -x_{j0}$ and $A = B = C$. The first statement of Lemma 4.6, and (4.13) with $N = n - 1$, establishes the first inequality in (5.2). The second derives from (4.12a) and (4.13) with large N .

Differentiating L_{j0} in (4.2a) with respect to a component of x amounts to placing an extra factor $\xi_i + z\omega_i = (\rho + z)\omega_i$ in the integrand. The result can be thought of as replacing $A_{jk}(\rho\omega)$ by $\rho\omega_i A_{jk}(\rho\omega)$. If a new integer x'_{j0} is computed for the new integrand, it is therefore seen to be one greater than the old one: $x'_{j0} = x_{j0} + 1$. Hence the second part of (5.3) holds with $s = -x_{j0}$. Similarly the first part of (5.3) can be seen to hold by examining the result of differentiating the expression (3.10) for $K_{j1}(x, y)$ with respect to a component of x .

We must now show that the kernel

$$K(x, y; \xi) = L_{j0}(x - \xi, y) + L_{j2}(x - \xi, y)$$

may be expressed in the form

$$(6.9) \quad K(x, y; \xi) = \sum_{i=1}^{n-1} \frac{\partial}{\partial \xi_i} L_j^*(x, y; \xi)$$

with L_i^* satisfying (5.4, 5). Such L_i^* are constructed from the L_{j0} and L_{j1} by (1) inserting the factor

$$\frac{\xi_i + z\omega_i}{(\rho + z)^2} = \frac{\omega_i}{\rho + z}$$

into the integrand of (4.2a), thus obtaining a new function L_{j0}^i ; (2) inserting the factor $(\xi_i/\rho^2) = (\omega_i/\rho)$ into the integrand of K_{j1} in (3.10), thus through

(4.2c) obtaining a new function L_{j2}^i ; and (3) setting $L_i^* = L_{j0}^i + L_{j2}^i$. It is immediate that (6.9) holds. Finally the argument used to verify (5.3) above can be used to verify (5.4, 5).

Now that all the hypotheses of Theorem 5.1 (with d replaced by σ) are verified with $s = -\kappa_{j0}$, we may apply that theorem to obtain (6.6). In fact, (6.6) is obtained by adding the estimate for $|Pu_{j0}|$ obtained from (5.6) to σ^α times that obtained from (5.7). In a similar manner, (6.7) is obtained from the last statement of Theorem 5.3, and (6.8) from (5.8b). The last assertion of the theorem is proved as in Theorem 6.2. This completes the proof of Theorem 6.3.

THEOREM 6.4. (*The Case $m > \mu_j$*). - Assume $\kappa_{j0} + n - 2 \geq 0$, $m > \mu_j$. Then

$$(6.10) \quad |Pu_{j0}(x, y)| + \sigma^\alpha \frac{|Pu_{j0}(x, y) - Pu_{j0}(0, y)|}{|x|^\alpha} \\ \leq Ck^{\gamma_j - m + \mu_j} \{ (k\sigma)^{-\kappa_{j0} - m + \mu_j} |\varphi_j|_{m - \mu_j} + \sigma^\alpha (\lambda^{-\kappa_{j0} - m + \mu_j} + (k\sigma)^{-\kappa_{j0} - m + \mu_j}) [\varphi_j]_{m - \mu_j + a} \}.$$

Furthermore the conclusions analogous to (6.7) and (6.8) are true if, in these inequalities and their respective premises, the numbers γ_j and κ_{j0} are decreased by the amount $m - \mu_j$ and the norm indices are increased by the same amount. If $\kappa_{j0} + n - 2 < 0$, the same estimates hold with κ_{j0} reduced by ϵ .

PROOF. - The proof is based on a representation of Pu_{j0} by modified kernels $L_{j0}^{(q)}$ constructed as follows (here q is any positive integer):

$L_{j0}^{(q)}$ and $L_{j1}^{(q)}$ are given by (4.2a) and (4.2b) respectively, with an extra factor $(\rho + z)^{-2q}$ inserted into the integrands.

$L_{j2}^{(q)} \equiv P\left(\frac{1}{i}D, -D_y\right)K_{j1}^{(q)}$ where K_{j1} is given by the second integrals in (3.10), with $\rho^{-\mu_j}$ replaced by $\rho^{-\mu_j - 2q}$.

The following properties are immediate:

- (6.11) 1. $(-\Delta_x)^q L_{ji}^{(q)}(x, y) = L_{ji}(x, y)$;
 2. Estimate (4.12) holds with κ_{ji} replaced by $\kappa_{ji} - 2q$;
 3. Estimate (4.13) holds for any $N \geq m + n - 1 - \mu_j - 2q$.

Suppose now that $m - \mu_j$ is even; we set it equal to $2q$. The new representation is the following, obtained by noticing $D_x L_{ji} = -D_\xi L_{ji}$ and integrating by parts:

$$Pu_{j0} = k^{\gamma_j + n - 1} \int (-k^{-2} \Delta_x)^q (L_{j0}^{(q)} + L_{j2}^{(q)}) \varphi_j d\xi \\ = k^{\gamma_j - m + \mu_j + n - 1} \int (L_{j0}^{(q)} + L_{j2}^{(q)}) (-\Delta_\xi)^q \varphi_j(\xi) d\xi.$$

With this representation at hand, the proof of Theorem 6.3 may be repeated; the only changes are those mentioned in the statement of the theorem. For example, it is clear that estimates will be in terms of $|\Delta^q \varphi_j|_0$ and $[\Delta^q \varphi_j]_\alpha$, which are in turn estimable by $|\varphi_j|_{m-\mu_j}$ and $[\varphi_j]_{m-\mu_j+\alpha}$.

This proves the theorem for the case that $m - \mu_j$ is even; next suppose it is odd, and set $m - \mu_j = 2q - 1 = 2(q - 1) + 1$. This time the representation is

$$\begin{aligned} Pu_{j_0} &= k^{\gamma_j+n-1} \int (-k^{-2} \Delta_x)^q (L_{j_0}^{(q)} + L_{j_2}^{(q)}) \varphi_j d\xi \\ &= k^{\gamma_j+n-1-2(q-1)} \int (-k^{-2} \Delta_x) (L_{j_0}^{(q)} + L_{j_2}^{(q)}) (-\Delta_\xi)^{q-1} \varphi_j d\xi \\ &= k^{\gamma_j+n-1-m+\mu_j} \sum_{i=1}^n -k^{-1} D_{x_i} (L_{j_0}^{(q)} + L_{j_2}^{(q)}) D_{\xi_i} (-\Delta_\xi)^{q-1} \varphi_j d\xi. \end{aligned}$$

As before, one has only to verify that the integrand $k^{-1} D_{x_i} (L_{j_0}^{(q)} + L_{j_2}^{(q)})$ satisfies the hypotheses of Theorem 5.1, with $s = \alpha_{j_0} - m + \mu_j$. This completes the proof of Theorem 6.4.

6.3. - Estimation of Pu_{j_1} .

THEOREM 6.5. - Assume $\alpha_{j_1} + n - 2 \geq 0$. Then

$$|Pu_{j_1}(x, y)| \leq Ck^{1-n}[1 + (k\sigma)^{-\alpha_{j_1}}] |\varphi_j|_0 e^{-\beta_k y}.$$

If $\alpha_{j_1} + n - 2 < 0$, the same holds with α_{j_1} reduced by $\epsilon' > 0$.

PROOF. - Lemma 4.6 assures us that the integrand of (6.4b) satisfies the hypotheses of Theorem 5.3 with $t = n - 1 > 0$, $s = -\alpha_{j_1}$, and $A = B = Ce^{-\beta y}$. The theorem follows.

7. - The interior limit.

We study the solutions u_j ($j = 1, \dots, m_1$) given by (6.1) and, to emphasize their dependence on ϵ , write $u_j(x, y; \epsilon)$. Specifically, we shall be interested in the function

$$(7.1) \quad u_j^0(x, y) \equiv \lim_{\epsilon \rightarrow 0} u_j(x, y; \epsilon) \quad (y > 0),$$

if such a limit exists. It will be called the interior limit solution.

In fact, more generally we shall study

$$\lim_{\varepsilon \rightarrow 0} P\left(\frac{1}{\varepsilon} D_x, -D_y\right) u_j(x, y; \varepsilon),$$

where P is a given polynomial, as in Secs. 4 and 6. However this time we restrict attention to the case P is homogeneous (of degree m). We use decomposition (6.3) and study separately $\lim_{\varepsilon \rightarrow 0} P\left(\frac{1}{\varepsilon} D_x, -D_y\right) u_{j0} = \lim_{\varepsilon \rightarrow 0} k^m P\left(\frac{\varepsilon}{\varepsilon} D_x, -\varepsilon D_y\right) u_{j0}$ and $\lim_{\varepsilon \rightarrow 0} P u_{j1}$. These functions are given by (6.4) with the exponent of k increased by the amount m , and L_{ji} given by (4.2). The integers α_{ji} are those defined in Sec. 4. It is seen from (4.12b) that for fixed $y > 0$, $L_n(kx, ky)$ vanishes exponentially as $k \rightarrow \infty$. Hence we immediately obtain

$$(7.2) \quad \lim_{\varepsilon \rightarrow 0} P u_{j1} = 0 \quad (y > 0),$$

which is reasonable since u_{j1} represents the «boundary layer effect» which should vanish in the interior.

We now consider $P u_{j0}$. From (4.13) we see that for fixed $y > 0$,

$$L_{j2}(k(x - \xi), ky) \leq C_N k^{-N} = C_N \varepsilon^N$$

for any integer N . Hence

$$(7.3) \quad \lim_{\varepsilon \rightarrow 0} k^{\gamma_j + m + n - 1} \int L_{j2}(kx - k\xi, ky) \varphi_j d\xi = 0,$$

and so the desired limit reduces to

$$(7.4) \quad \lim_{\varepsilon \rightarrow 0} P u_j = \lim_{\varepsilon \rightarrow 0} k^{\gamma_j + m + n - 1} \int L_{j0}(kx - k\xi, ky) \varphi_j(\xi) d\xi,$$

with L_{j0} given by (4.2a).

For the sake of definiteness in the following analysis, we suppose that the functions $G_j(\sigma, \omega, y)$ are constructed as suggested in the proof of Lemma 3.3, namely

$$G_j(\sigma, \omega, y) = \int_C \frac{z^{j-1} e^{\sigma y z} dz}{\bar{L}^1(\sigma, \omega, z)},$$

$$\widehat{L}^1 = \prod_{i=1}^{m_0} \left(z - \frac{\eta_i(\sigma \omega)}{\rho} \right).$$

Then a portion of the integrand of (4.2a) can be expressed in the form

$$(7.5) \quad \sum_{i=1}^{m_0} A_{ji}(\sigma\omega) P(\sigma\omega, -D_\gamma) G_i(\sigma, \omega, y) = \int_{\bar{C}} \frac{(\sum_{i=1}^{m_0} z^{i-1} A_{ji}(\sigma\omega)) P(\sigma\omega, -\sigma z) e^{\sigma y z} dz}{L^1(\sigma, \omega, z)}$$

$$= \sigma^m \int_{\bar{C}} (\hat{L}^1(\sigma, \omega, z))^{-1} P(\omega, -z) \sum_{i=1}^{m_0} z^{i-1} A_{ji}(\sigma\omega) e^{\sigma y z} dz.$$

We expand the summation in the integrand in powers of σ (recall the A_{jk} are analytic in σ except for a possible pole at the origin):

$$(7.6) \quad \sum z^{i-1} A_{ji}(\sigma\omega) = \sigma^{\nu_j} (f_{j1}(\omega, z) + \sigma f_{j2}(\omega, z) + \dots) \quad (f_{j1} \not\equiv 0).$$

This defines the integers ν_j .

We are now ready for the main theorem:

THEOREM 7.1. - For fixed x and $y > 0$,

$$(7.7) \quad \lim_{\varepsilon \rightarrow 0} P\left(\frac{1}{\varepsilon} D_x, -D_\gamma\right) u_j(x, y; \xi) = \begin{cases} 0 & \text{if } \gamma_j < \nu_j \\ v_j(x, y) & \text{if } \gamma_j = \nu_j \end{cases}$$

where ν_j is the lowest power of σ occurring in the expansion (7.6),

$$(7.8) \quad v_j(x, y) = \int_{R^n} \Lambda_j(x - \xi, y) \varphi_j(\xi) d\xi,$$

$$(7.9) \quad \Lambda_j(x, y) = (2\pi)^{n-1} \int_{|z|=1} \frac{dz}{2\pi i z} \int_{R^n} \rho^{2-n} (\rho + z)^{\nu_j + m + n - 2} H(\omega, (\rho + z)y) e^{i(\rho+z)x \cdot \omega} d\bar{\xi}$$

(in the latter integral we use polar coordinates $\bar{\xi} = \rho\omega$), and

$$(7.10) \quad H(\omega, \sigma y) \equiv \int_{\bar{C}} f_{j1}(\omega, z) P(\omega, -z) \prod_{i=1}^{m_0} (z - \alpha_i(\omega))^{-1} e^{\sigma y z} dz,$$

where f_{ji} is given by (7.6).

PROOF. - From (4.3), (7.5), and (7.6) one sees that

$$(7.11) \quad m + \nu_j = \kappa_{j0}.$$

In fact, we have

$$(7.12) \quad \begin{aligned} \sigma_{j0} \Psi_{j0}(\sigma, \omega, \tau) e^{-\sigma\tau} &= \sigma_{j0} \Psi_{j0}(0, \omega, \tau) e^{-\sigma\tau} \\ &+ \sigma_{j0}^{n+1} \widehat{\Psi}'_{j0}(\sigma, \omega, \tau) e^{-\sigma\tau}, \end{aligned}$$

where

$$\Psi_{j0}(0, \omega, \tau) e^{-\sigma\tau} = \int_{\mathcal{C}} (L^1(0, \omega, z))^{-1} P(\omega, z) f_{j1}(\omega, z) e^{\tau z} dz$$

and

$$(7.13) \quad \widehat{\Psi}'_{j0}(\rho, \omega, \tau) = \rho^{-1} [\Psi_{j0}(\rho, \omega, \tau) - \Psi_{j0}(0, \omega, \tau)]$$

Equation (7.12) yields a decomposition of the left hand side of (4.3), which in turn induces a decomposition of (4.2a):

$$(7.14) \quad L_{j0} = L_{j01} + L_{j02},$$

where

$$(7.15) \quad \begin{aligned} L_{j01}(x, y) &= (2\pi)^{-n+1} \int_{|z|=1} \frac{dz}{2\pi iz} \int \chi(\rho/\omega) \left(\frac{\rho+z}{\rho} \right)^{n-2} (\rho+z)^{n_0} H(\omega, (\rho+z)y) e^{ix(\rho+z)\omega} d\xi, \end{aligned}$$

in which

$$(7.16) \quad \begin{aligned} H(\omega, \tau) &= \int_{\mathcal{C}} f_{j1}(\omega, z) P(\omega, -z) (\widehat{L}^1(0, \omega, z))^{-1} e^{\tau z} dz \\ &= \Psi_{j0}(0, \omega, \tau) e^{-\sigma\tau}; \end{aligned}$$

and L_{j02} is the same expression (7.15) with H replaced by

$$(\rho+z) \widehat{\Psi}'(\rho, \omega, \tau) e^{-\tau z}.$$

From (7.4) and (7.14) we have

$$(7.17) \quad \begin{aligned} \lim_{\epsilon} P\left(\frac{1}{\epsilon} D, -D_y\right) u_f(x, y; \epsilon) \\ = \lim_{\epsilon} k^{\gamma_j+n-1+m} \int (L_{j01} + L_{j02}) \varphi_j(\xi) d\xi. \end{aligned}$$

LEMMA 7.2. - The kernel $L_{j01}(x, y)$ is subject to the same estimate (4.12a) as L_{j0} . Also L_{j02} satisfies (4.12a) with κ_{j0} replaced by $\kappa_{j0} + 1$.

PROOF. - Our auxiliary function $H(\omega, \tau)$ defined by (7.16) is related to the function φ used in the proof of Lemma 4.6 by

$$H(\omega, \tau) = \varphi(0, \omega, \tau).$$

It therefore has all the properties needed for the same proof to be effective. This establishes the estimate for L_{j01} . In a similar manner, one sees that the function

$$(2\pi)^{-n+1} \chi(\rho/a) \hat{\Psi}_{j0}(\rho, \omega, \tau)$$

given by (7.13) has the same properties. But if one uses this function for φ in the proof of Lemma 4.6, and replaces κ_{j0} by $\kappa_{j0} + 1$, then one obtains the proper estimate for L_{j02} . This completes the proof of Lemma 7.2.

CONTINUATION OF THE PROOF OF THEOREM 7.1. - Lemma 7.2 establishes that the kernel L_{j01} satisfies (5.2) with $t > 0$ and $s = -\kappa_{j0}$ when $\kappa_{j0} + n - 2 \geq 0$, and $s = -\kappa_{j0} + \epsilon'$ for any $\epsilon' > 0$ when $\kappa_{j0} + n - 2 < 0$. In either case, (5.2) will be satisfied if we set $s = -\kappa_{j0} + \frac{1}{2}$. Therefore Theorem 5.3 may be applied with this value of s , yielding

$$(7.18) \quad \left| k^{\gamma_j + n - 1 + m} \int L_{j01} \varphi_j d\xi \right| \leq C k^{\gamma_j + m} [(ky)^s + (kd)^s] |\varphi_j|_0$$

for $y > \epsilon$. If we fix $y > 0$ and let $\epsilon \rightarrow 0$, we certainly have $y > \epsilon$, and the right side of (7.18) takes the form $k^{\gamma_j + m + s} C(y, d, \varphi_j)$. But recalling (7.11), we see this exponent is $\gamma_j + m - (\nu_j + m) + \frac{1}{2} = \gamma_j - \nu_j + \frac{1}{2} < 0$. Hence the left side of (7.18) approaches 0 as $k \rightarrow \infty$.

The kernel L_{j02} satisfies (5.2) with an even smaller value of s , so the same is true of its corresponding integral. This proves the assertion of Theorem 7.1 for the case $\gamma_j < \nu_j$.

Now consider the case $\gamma_j = \nu_j$. The same reasoning yields that

$$\lim_{\epsilon \rightarrow 0} k^{\gamma_j + n - 1 + m} \int L_{j02} \varphi_j d\xi \rightarrow 0,$$

although this assertion is not necessarily true for L_{j01} . If, in the integrand of (7.15), we write $\chi(\rho/a)$ in the form $1 + (\chi(\rho/a) - 1)$, we induce a decompo-

sition $L_{j01} + L_{j011} + L_{j012}$, where L_{j011} appears as (7.15) with χ missing. The function L_{j012} , whose integrand vanishes near $\rho = 0$, satisfies (4.13) for large enough N ; the proof is the same as that of Lemma 4.7. Therefore from Theorem 5.3 we obtain that

$$\int L_{j012}(kx - k\xi, ky)\varphi_j d\xi = O(\varepsilon^N)$$

for arbitrarily large N ($y > 0$ fixed). This term vanishes in the limit as $\varepsilon \rightarrow 0$. We are left with the relation

$$(7.19) \quad \lim_{\varepsilon \rightarrow 0} P\left(\frac{1}{\varepsilon} D_x, -D_y\right) u_j(x, y; \varepsilon) \\ = \lim_{\varepsilon \rightarrow 0} k^{\gamma_j + m + n - 1} \int L_{j011}(kx - k\xi; ky)\varphi_j(\xi) d\xi.$$

From (7.15, 7.16) and the fact (see (1.5)) that $\widehat{L}^1(0, \omega, z) = \prod_{i=1}^{m_0} (z - \alpha_i(\omega))$, we see that $L_{j011}(x, y)$ is precisely the function Λ_j given by (7.8). Thus defining $\rho' = k\rho$, $\xi' = k\xi$, $z' = kz$, we have from (7.9)

$$\Lambda_j(kx, ky) \\ = (2\pi)^{n-1} \int_{|z'|=k} \frac{dz'}{2\pi i z'} \int_{R^n} (\rho')^{2-n} (\rho' + z')^{\gamma_j + m + n - 2} \\ \cdot H(\omega, (\rho' + z')y) e^{i(\rho' + z')x \cdot \omega} d\xi' k^{-\gamma_j - m - n + 1}.$$

Shrinking the contour of integration $|z'| = k$ to $|z'| = 1$, we see that

$$\Lambda_j(kx, ky) = k^{-\gamma_j - m - n + 1} \Lambda_j(x, y).$$

Recalling that $\gamma_j = \nu_j$, we find from this and (7.19) that

$$\lim_{\varepsilon \rightarrow 0} P u_j(x, y; \varepsilon) = \int \Lambda_j(x - \xi; y) \varphi_j(\xi) d\xi,$$

which completes the proof of Theorem 7.1.

REMARK. - Assuming $\gamma_j = \nu_j$, it is clear from the above representation (7.8) that $u_j^0(x, y)$ is infinitely differentiable for $y > 0$ and satisfies

$$(7.20) \quad \lim_{\varepsilon \rightarrow 0} P\left(\frac{1}{\varepsilon} D_x, -D_y\right) u_j(x, y; \varepsilon) = P\left(\frac{1}{\varepsilon} D_x, -D_y\right) u_j^0(x, y)$$

for any operator P .

THEOREM 7.3. - *The interior limit satisfies*

$$L_0\left(\frac{1}{i}D_x, -D_y\right)u_j^0 = 0$$

for all j .

PROOF. - Setting $P = L_0$ and observing that $\widehat{L}^1$ divides $\widehat{L}_0$, we see that the contour integral (7.10) vanishes.

THEOREM 7.4. - *Assume $\gamma_j = \nu_j$. The limit $u_j^0(x, y)$ satisfies the boundary condition*

$$(7.21) \quad \lim_{y \rightarrow 0} P\left(\frac{1}{i}D_x, -D_y\right)u_j^0(x, y) = 0$$

if P is such that

$$(7.22) \quad \int_C f_{j1}(\omega, z)P(\omega, -D) \prod_{i=1}^{m_0} (z - \alpha_i(\omega))^{-1} dz = 0.$$

It satisfies

$$(7.23) \quad \lim_{y \rightarrow 0} Pu^0(x, y) = \varphi_j(x)$$

if $m = -\nu_j$ and the above integral is 1, identically in ω .

PROOF. - Suppose (7.22) is satisfied. Then, referring to (7.10), we have

$$\lim_{y \rightarrow 0} H(\omega, (\rho + z)y) = 0$$

uniformly for $\rho > 0$, $|z| = 1$, $|\omega| = 1$. (The uniformity in ρ follows from the fact that the contour integral in (7.10) lies in the left half plane). The result (7.20) then follows by interchanging integrations as in the proof of Lemma 3.5.

Next suppose the integral in (7.22) is one, and $m = -\nu$, i. e., $\kappa_0 = 0$. Then $H \rightarrow 1$ as $y \rightarrow 0$, uniformly in ω, ρ, z , and the proof again follows the lines of that of Lemma 3.5.

Appendix.

PROOF OF LEMMA 4.2. - Denoting the integral on the left of (4.7) by I , we have the representation

$$I = \int_0^1 J(x, \rho, \rho y) e^{-\alpha \rho y} d\rho,$$

where

$$(A.1) \quad J(x, \rho, \tau) = \int_{|\omega|=1} \rho^k \varphi(\rho, \omega, \tau) e^{i\rho\omega \cdot x} d\omega.$$

With no loss of generality, we may assume $x = (x_1, 0, 0, \dots)$, so $x_1 = |x|$. For simplicity we write $x = x_1$. We also denote the first component of ω by t , the others by $\hat{\omega} = (\omega_2, \dots, \omega_n)$, and employ the cut-off function χ . Thus

$$(A.2) \quad J = J_1 + J_2,$$

where

$$J_1(x, \rho, \tau) = \int_{|z|=1} \chi(t) \rho^k \varphi(\rho, \omega, \tau) e^{i\rho t x} d\omega$$

and

$$J_2(x, \rho, t) = \int_{|\omega|=1} [1 - \chi(t)] \rho^k \varphi e^{i\rho t x} d\omega.$$

This induces a corresponding decomposition

$$(A.3) \quad I = I_1 + I_2.$$

LEMMA A.1. - *Under the hypotheses of Lemma 4.2, for $y > 0$,*

$$|I| \leq C_k y^{-k-1}.$$

PROOF. - It is immediate that

$$|I| \leq C \int_0^1 \rho^k \chi_1(\rho) e^{-\alpha \rho y} d\rho,$$

where χ_1 is some nonnegative infinitely differentiable function vanishing near $\rho = 1$. Integrating by parts $k+1$ times, we find the integral on the right equal to

$$\frac{(k+1)!}{(-\alpha y)^{k+1}} \left\{ e^{-\alpha \rho y} \partial_\rho^k [\rho^k \chi_1(\rho)] \Big|_{\rho=0}^{\rho=1} - \int_0^1 e^{-\alpha \rho y} \partial_\rho^{k+1} [\rho^k \chi_1(\rho)] d\rho \right\}.$$

The quantity in brackets is clearly bounded; this proves the lemma.

LEMMA A.2. - Under the hypotheses of Lemma 4.2,

$$(A.4) \quad |I_2| = \left| \int_0^1 J_2(x, \rho, \rho y) e^{-\alpha \rho y} d\rho \right| \leq C_k r^{k-1},$$

where $r^2 = |x|^2 + y^2$.

PROOF. - Interchanging the integrations, we have

$$(A.5) \quad I_2 = \int_{|\omega|=1} d\omega (1 - \chi(t)) \int_0^1 \rho^k \varphi(\rho, \omega, \rho y) e^{i t x - \alpha \rho y} d\rho.$$

We shall integrate with respect to ρ by parts, and as such shall need the estimate

$$(A.6) \quad \left| \frac{\partial^{k+1}}{\partial \rho^{k+1}} [\rho^k \varphi(\rho, \omega, \rho y) e^{-\alpha \rho y}] \right| \leq C_k (1 + y) e^{-\alpha \rho y^2},$$

obtained by use of Leibniz's rule and the given properties of φ .

Integrating (A.5) by parts $k+1$ times, we obtain, for some constants e^1 and e_2 ,

$$I_2 = \int_{|\omega|=1} d\omega (1 - \chi) \left[\frac{e_1}{(i t x - \alpha y)^{k+1}} + \frac{e_2}{(i t x - \alpha y)^{k+1}} \int_0^1 e^{i t x \rho} \partial_\rho^{k+1} [\rho^k \varphi e^{-\alpha \rho y}] d\rho \right].$$

This, together with (A.6) and the fact that $t \geq \frac{1}{2}$ for $1 - \chi(t) \neq 0$, yields (A.4), and Lemma A.2 is proved.

It will be convenient to use other representations for J . Integration over the unit sphere $|\omega| = 1$ may be expressed in the form

$$\int_{|\omega|=1} f(\omega) d\omega = \int_{-1}^1 \widehat{f}(t) dt,$$

where

$$\widehat{f}(t) = \int_{|\omega|=1} f(t, \widehat{\omega} (1 - t^2)^{1/2}) (1 - t^2)^{(n-3)/2} d\widehat{\omega}.$$

In this manner we have

$$(A.7) \quad J_1(x, \rho, \tau) = \int_{-1}^1 \chi(t) \rho^k \widehat{\varphi}(\rho, t, \tau) e^{i \rho t x} dt.$$

For $|t| < 1$, $\widehat{\varphi}$ shares with φ the properties stated as hypotheses in Lemma 4.2.

Another representation is obtained by integrating J_1 with respect to t by parts k times:

$$J_1(x, \rho, \tau) = \left(\frac{-1}{ix} \right) \int_{-1}^1 e^{i\rho t x} \partial_t^k [\chi(t) \widehat{\varphi}(\rho, t, \tau)] dt.$$

The derivative occurring in the integrand, being a differentiable function of t , can be written as $a(\rho, \tau) + tb(\rho, t, \tau)$ with a and b bounded. Hence

$$\begin{aligned} \text{(A.8)} \quad J_1(x, \rho, \tau) &= 2 \left(\frac{-1}{ix} \right)^k a(\rho, \tau) \frac{\sin x\rho}{x\rho} + \left(\frac{-1}{ix} \right)^k \int_{-1}^1 b(\rho, t, \tau) t e^{ix\rho} dt \\ &\equiv J_{11} + J_{12}. \end{aligned}$$

LEMMA A.3. - *Under the hypotheses of Lemma 4.2.*

$$\text{(A.9)} \quad |I_1| = \left| \int_0^1 J_1(x, \rho, \rho y) e^{-\alpha \rho y} d\rho \right| \leq C_k |x|^{-k-1}.$$

PROOF. - We examine the integral $\int_0^1 J_{11} e^{-\alpha \rho y} d\rho$. Denoting

$$S(\rho) = \int_0^\rho \frac{\sin \xi}{\xi} d\xi,$$

we integrate by parts to obtain

$$\left(\frac{-1}{ix} \right)^k \left[2x^{-1} S(x\rho) a(\rho, \rho y) e^{-\alpha \rho y} \Big|_{\rho=0}^{\rho=1} - x^{-1} \int_0^1 S(x\rho) \partial_\rho (a e^{-\alpha \rho y}) d\rho \right].$$

As above, we estimate

$$|\partial_\rho (a e^{-\alpha \rho y})| \leq C(1 + y) e^{-\alpha \rho y}.$$

This estimate, together with the boundedness of $S(x\rho)$ and of a indicate that (A.9) holds with J_1 replaced by J_{11} . A similar integration by parts with respect to ρ performed on $\int J_{12} e^{-\alpha \rho y} d\rho$ establishes the same estimate. Lemma A.3 is proved by combining these two estimates.

COMPLETION OF THE PROOF OF LEMMA 4.2. - Clearly either $|x| \geq r/2$ or $y \geq r/2$. In the former case, (A.9) implies that

$$|I_1| \leq C_k r^{-k-1}.$$

This combined with (A.4) yields (4.7). In the latter case, Lemma A.1 yields the same result.

PROOF OF LEMMA 4.3. - Again set

$$I = \int_1^\infty J(x, \rho) e^{-x\rho^y} d\rho,$$

where $J(x, \rho)$ is given by (A.1),

The proof of Lemma A.1 can be easily adapted to yield

$$|I| \leq C_k y^{-k-1}.$$

We again decompose J and I according to (A.2), (A.3); thus

$$I_2 = \int_{|\omega|=1} d\omega (1 - \chi(t)) \int_1^\infty \rho^k \varphi(\rho, \omega) e^{i\rho t - x\rho^y} d\rho.$$

Integrating with respect to ρ by parts $k+1$ times and using the estimates given for derivatives of φ , we obtain as before,

$$|I_2| \leq C_k r^{-k-1}.$$

Finally, the estimate

$$|I_1| \leq C_k |x|^{-k-1}$$

follows as in the proof of Lemma A.3 from the representation (A.8). With these estimates at hand, the proof of Lemma 4.3 is completed in the same manner as was that of Lemma 4.2.

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