Nonlinear Boundary Value Problems and a Global Inverse Function Theorem (*).

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Summary. – *Existence and uniqueness results are obtained for nonlinear periodic and Dirichlet boundary value problems by using results about the corresponding linearized problems and a global inverse function theorem.*

1. – In this paper we establish existence and uniqueness results about nonlinear boundary value problems using the following global inverse function theorem to be found in SCHWARTZ [11].

PROPOSITION 1.1. – Let X and Y be Banach spaces and $\Phi: X \to Y$ a continuously Fréchet differentiable function and suppose Φ' is invertible (as a linear operator) at every $x \in X$ and moreover $\|\Phi'(x)^{-1}\| \leqslant K < \infty$ uniformly in x. Then Φ is a homeomorphism of X onto Y.

We shall consider equations of the form

$$Lu + Nu = Fu \tag{1.1}$$

where L is a linear differential expression, N is a continuously Fréchet differentiable operator such that $L + N'(u)$ has a uniformly bounded inverse and F is an operator with bounded range. The plan of the paper is as follows. In section 2 we prove a version of proposition 1.1 where $\Phi = L + N$ and L is unbounded since in applications we find it simpler to prove the existence of a uniform bound for $L + N'(u)$ in spaces where L is unbounded. In section 4 we prove existence and uniqueness results for periodic solutions of ordinary differential systems using the abstract results of section 2 and some elementary results about matrices proved in section 3. Finally in section 5 we discuss the application of the results in section 2 to some Dirichlet boundary value problems.

Several other papers have been written about equation (1.1) and about the related equation

$$(Lu)(x) + g\big(x, u(x), u'(x)\big)\, u(x) = f\big(x, u(x), u'(x)\big) \tag{1.2}$$

where L is a linear second order differential expression, g is bounded away from the eigenvalues of L and f is bounded. DOLPH [3] studies equations analogous

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to (1.1), (1.2) in the form of Hammerstein integral equations. LEACH [9] obtains existence and uniqueness theorems for the periodic solutions of second order systems of ordinary differential equations which can be expressed in the form (1.1). LEACH and LAZER [8] and LANDESMAN and LAZER [7] study a Dirichlet boundary value problem of the form (1.2) when L is a self adjoint operator corresponding to an ordinary and partial differential expression respectively. WILLIAMS [13] generalises these results to the case where L is normal. In all these papers existence of solutions is a consequence of the Schauder fixed point theorem and the main technical effort of the papers is the establishing of a priori bounds to enable the Schauder theorem to be applied. Similarly, in the examples considered in the present paper the main difficulty is in proving that $[L + N'(u)]^{-1}$ is uniformly bounded.

2. – Throughout this section X and Y will be Banach spaces with norms $\| \|_X$ and $\| \|_Y$ and L will be a linear operator with domain $D(L)$ dense in X and range contained in Y. $\| \|$ will denote the operator norm of any bounded linear operator.

We recall that L is closed if $\{u_n\} \subset D(L)$, $u_n \to u$ in X and $Lu_n \to y$ in Y implies that $u \in D(L)$ and $Lu = y$. If L is closed, it is well known that $D(L)$ is a Banach space with respect to the graph norm $|||u||| = \|u\|_X + \|Lu\|_Y$.

THEOREM 2.1. – Let $N: X \to Y$ be continuously Fréchet differentiable and let there exist $K > 0$ such that $\|N'(u)\| \leqslant K$ for all $u \in X$. If $L + N'(u)$ has an everywhere defined inverse for all $u \in X$ and if $[L + N'(u)]^{-1}: Y \to X$ is uniformly bounded, i.e. there exists $C > 0$ such that $\|[L + N'(u)]^{-1}\| \leqslant C$ for all $u \in X$, then $L + N$ is a homeomorphism of the Banach space $D(L)$ onto Y.

PROOF. – Since $N: X \to Y$ is continuously Fréchet differentiable, N regarded as an operator from $D(L)$ to Y is also continuously Fréchet differentiable. Since the derivatives coincide on $D(L)$, we denote both by $N'(u)$.

We shall show that $[L + N'(u)]^{-1}: Y \to D(L)$ is uniformly bounded in norm with respect to u. Let $u_0 \in D(L)$. Suppose that $[L + N'(u_0)]^{-1}y = x$. Then

$$\begin{aligned} |||x||| &= \|[L + N'(u_0)]^{-1}y\|_X + \|L[L + N'(u_0)]^{-1}y\|_Y \\ &\leqslant C\|y\|_Y + \|y\|_Y + \|N'(u_0)[L + N'(u_0)]^{-1}y\|_Y \leqslant (C+1)\|y\|_Y + KC\|y\|_Y = C_1\|y\|_Y . \end{aligned}$$

Hence $\|[L + N'(u)]^{-1}\| \leqslant C_1$ for all $u \in D(L)$ and so by proposition 1.1 $L + N$ is an homeomorphism of $D(L)$ onto Y.

The next theorem gives more information about $(L + N)^{-1}: Y \to D(L)$.

THEOREM 2.2. – Let L and N be as in theorem 2.1. $(L + N)^{-1}: Y \to D(L)$ is continuously Fréchet differentiable and Lipschitz.

PROOF. – Let $(L + N)^{-1} = G$. Let $y \in Y$ and let $Gy = x$. We shall prove that G is continuously Fréchet differentiable by showing that $G'(y) = [L + N'(x)]^{-1}$.

Let $h \in Y$ and let $G(y+h) = x + k$. Then

$$|||G(y+h) - G(y) - [L + N'(x)]^{-1}h|||$$
$$= |||[L+N'(x)]^{-1}\{h - [L+N'(x)][G(y+h) - G(y)]\}|||$$
$$\leqslant C_1 \|(L+N)(x+k) - (L+N)(x) - [L+N'(x)](k)\|_Y,$$

using the bound obtained for $[L+N'(x)]^{-1}\colon Y \to D(L)$ in theorem 2.1,

$$= C_1 \|N(x+k) - Nx - N'(x)k\|_Y$$

$\to 0$ as $\|h\|_Y \to 0$ since, by theorem 2.1, $\|h\|_Y \to 0$ implies that $|||k||| \to 0$.

Hence $G'(y) = [L+N'(x)]^{-1}$.

Let $y_1, y_2 \in Y$. Then

$$G(y_1) - G(y_2) = \int_0^1 G'\big(y_2 + t(y_1 - y_2)\big)(y_1 - y_2)\, dt.$$

Hence

$$|||G(y_1) - G(y_2)||| \leqslant \int_0^1 \|G'\big(y_2 + t(y_1-y_2)\big)\|\, dt \|y_1 - y_2\|$$
$$= \int_0^1 \|\big[L + N'\big(x(t)\big)\big]^{-1}\|\, dt \|y_1 - y_2\| \qquad \text{where } x(t) = G\big(y_2 + t(y_1 - y_2)\big)$$
$$\leqslant C_1 \|y_1 - y_2\|$$

i.e. G is Lipschitz.

We can now prove another existence theorem.

THEOREM 2.3. Let L and N be as in theorem 2.1. If $F\colon D(L) \to Y$ is continuous, compact and has bounded range, then there exists at least one solution u of $L(u) + N(u) = F(u)$.

PROOF. – Choose $K_1 > 0$ such that $\|Fu\|_Y \leqslant K_1$ for all $u \in D(L)$. Hence if $u \in D(L)$,

$$|||(L+N)^{-1}F(u)||| \leqslant |||(L+N)^{-1}F(u) - (L+N)^{-1}(0)||| + |||(L+N)^{-1}(0)|||$$
$$\leqslant C_1 \|Fu\|_Y + K_2 \text{ by theorem 2.2}$$
$$\leqslant C_1 K_1 + K_2 = R.$$

Let $B_R = \{x \in D(L)\colon |||x||| \leqslant R\}$. If $T = (L+N)^{-1}F$, it is clear that T maps B_R into B_R. Since $F\colon D(L) \to Y$ is compact and continuous and $(L+N)^{-1}\colon Y \to D(L)$

is continuous, T is compact and continuous. By the Schauder fixed point theorem T has a fixed point u in B_R and clearly $L(u) + N(u) = F(u)$.

3. – In this section we introduce some notation and prove some simple results about matrices which we shall use in section 4.

Let C^n denote the set of all n-tuples of complex numbers. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we define an inner product on C^n by $(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$ and the corresponding norm by $\|x\| = \left\{\sum_{i=1}^{n} |x_i|^2\right\}^{\frac{1}{2}}$. Let M_n denote the set of all complex $n \times n$ matrices and let $\|\,\|$ denote the natural norm on M_n corresponding to the norm $\|\,\|$ on C^n, i.e. if $A \in M_n$; $\|A\| = \sup\{\|Ax\| : x \in C^n, \|x\| = 1\}$. It can be shown (see JOHN [5]) that $\|A\| = \max \lambda_k^{\frac{1}{2}}$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the positive symmetric matrix A^*A (A^* denotes the adjoint of A i.e. if $A = (a_{ij})$, $A^* = (\bar{a}_{ji})$.)

LEMMA 3.1. – *If $A \in M_n$ and there exists $\delta > 0$ such that $|\lambda| > \delta$ for all eigenvalues λ of A, then $\|A^{-1}\| \leqslant \delta^{-n} \|A\|^{n-1}$.*

PROOF. – Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of A^*A and let e_i, $i = 1, \ldots, n$ be any eigenvector corresponding to μ_i such that $\|e_i\| = 1$. Since $\|A^*Ae_i\| = \mu_i$ we have

$$\|A^*A\| \geqslant \mu_i, \qquad i = 1, 2, \ldots, n. \tag{3.1}$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. By considering the characteristic polynomials of A^*A and A we have

$$\mu_1 \mu_2 \ldots \mu_n = \det A^*A = |\det A|^2 = |\lambda_1|^2 |\lambda_2|^2 \ldots |\lambda_n|^2. \tag{3.2}$$

Now $\|A^{-1}\| = [\min_{1 \leqslant k \leqslant n} \mu_k^{\frac{1}{2}}]^{-1}$ (JOHN [5], p. 14). Suppose that $\mu_{k_0} = \min_{1 \leqslant k \leqslant n} \mu_k$. Then, by (3.1) and (3.2),

$$\mu_k \|A^*A\|^{n-1} \geqslant |\lambda_1|^2 |\lambda_2|^2 \ldots |\lambda_n|^2 \geqslant \delta^{2n}.$$

Since $\|A^*\| = \|A\|$, it follows that $\mu_{k_0} \geqslant \delta^{2n} \|A\|^{-2n+2}$ and so $\|A^{-1}\| \leqslant \delta^{-n} \|A\|^{n-1}$.

If $A \in M_n$ define $\exp A = \sum_{n=0}^{\infty} A^n/n!$.

LEMMA 3.2. – *If $A \in M_n$, λ is an eigenvalue of A if and only if $\exp \lambda$ is an eigenvalue of $\exp A$.*

PROOF. – Let the eigenvalues of A be $\lambda_1, \ldots, \lambda_n$. There exists an invertible matrix P such that $PAP^{-1} = J$ where J is a Jordan canonical matrix i.e. J is upper triangular and $\operatorname{diag} J = \{\lambda_1, \ldots, \lambda_n\}$. Clearly $\exp J$ is upper triangular with $\operatorname{diag}(\exp J) = \{\exp J, \ldots, \exp \lambda_n\}$ and so $\exp J$ has eigenvalues $\exp \lambda_1, \ldots, \exp \lambda_n$. Now $\exp J = P \exp AP^{-1}$. Let μ be an eigenvalue of $\exp A$ and u a corresponding eigenfunction. Then $(\exp J)Pu = P(\exp A)u = \mu Pu$ and so μ is also an eigenvalue of $\exp A$. Hence every eigenvalue of $\exp A$ is an eigenvalue of $\exp J$ and similarly, since $\exp A = P^{-1}\exp JP$, every eigenvalue of $\exp J$ is an eigenvalue of $\exp A$. Hence the eigenvalues of $\exp A$ are $\exp \lambda_1, \ldots, \exp \lambda_n$.

LEMMA 3.3. – Let $S = \{2\pi ni : i^2 = -1$ and n is an integer$\}$. If $A \in M_n$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and there exists $\delta > 0$ such that $\operatorname{dist}(\lambda_i, S) \geqslant \delta$ for $i = 1, \ldots, n$, then there exists $C > 0$, where C depends only on δ and $\|A\|$, such that $|\mu| \geqslant C$ where μ is any eigenvalue of $I - \exp A$.

PROOF. – Let $D = \{\lambda \in C : \operatorname{dist}(S, \lambda) \geqslant \delta$ and $|\lambda| \leqslant \|A\|\}$. D is a compact subset of C which contains all the eigenvalues of A. If $f(\lambda) = 1 - \exp \lambda$, $|f|$ is continuous on D, has no zeros on D and so attains a positive minimum on D i.e. there exists $C > 0$, depending only on δ and $\|A\|$, such that $|f(\lambda)| > C$ if $\lambda \in D$. If μ is an eigenvalue of $I - \exp A$, by lemma 3.2 there exists on eigenvalue λ of A such that $\mu = 1 - \exp \lambda = f(\lambda)$. Hence $|\mu| \geqslant C$.

4. – In this section we shall study the existence and uniqueness of solutions of the systems of eqs. (4.1), (4.3) and (4.2), (4.3) where

$$u'(t) = A(t, u(t)) + f(t) \tag{4.1}$$

$$u'(t) = A(t, u(t)) + h(t, u(t)) \tag{4.2}$$

$$u(0) = u(2\pi) \tag{4.3}$$

and $u: [0, 2\pi] \to C^n$, $A: [0, 2\pi] \times C^n \to C^n$, $f: [0, 2\pi] \to C^n$ and $h: [0, 2\pi] \times C^n \to C^n$ and A, f and h satisfy the conditions of theorem 4.1.

We shall use known results about the following linear periodic systems (4.4), (4.3) and (4.5), (4.3).

$$u'(t) = B(t)u(t) \tag{4.4}$$

$$u'(t) = B(t)u(t) + g(t) \tag{4.5}$$

where $B: [0, 2\pi] \to M_n$ and $g: [0, 2\pi] \to C^n$. If $Y(t) = \exp\left[\int_0^t B(s)\, ds\right]$, then $Y(t)$ is a fundamental solution for (4.4) with $Y(0) = I$ (the identity matrix). HARTMAN [4] p. 407-8 proves the following:

PROPOSITION 4.1. Let $B: [0, 2\pi] \to M_n$ be continuous. (4.5), (4.3) has a unique solution for every continuous g if and only if $I - Y(2\pi)$ is invertible. This solution is given by

$$y(t) = Y(t)\left\{[I - Y(2\pi)]^{-1}\int_0^{2\pi} Y^{-1}(s)g(s)\, ds + \int_0^t Y^{-1}(s)g(s)\, ds\right\}. \tag{4.6}$$

Let $C[0, 2\pi] = \{u\colon [0, 2\pi] \to C^n$ and u is continuous$\}$. $C[0, 2\pi]$ is a Banach space with respect to the norm $\|u\| = \sup\{\|u(x)\|\colon x \in [0, 2\pi]\}$. If $C_p[0, 2\pi] = \{u \in C[0, 2\pi]\colon u(0) = u(2\pi)\}$, $C_p[0, 2\pi]$ is a closed subspace of $C[0, 2\pi]$ and so is also a Banach space.

We can now state and prove our main theorem.

THEOREM 4.2. – Let the function $(t, x) \to A'(t, x)$ be continuous on $[0, 2\pi] \times C^n$ where $A'(t, x_0)$ denotes the Fréchet derivative of $x \to A(t, x)$ at x_0. If

(1) there exists $K > 0$ such that $\|A'(t, x)\| \leqslant K$ for all $x \in C^n$ and all $t \in [0, 2\pi]$;

(2) there exists $\delta > 0$ such that $\operatorname{dist}\left[\left\{2\pi n i\colon n \text{ is an integer}\right\}, \left\{\lambda\colon \lambda \text{ is an eigenvalue of } \int_0^{2\pi} A'(s, u(s))\, ds\right\}\right] > \delta$ for all $u \in C_p[0, 2\pi]$;

(3) h is continuous with bounded range;

then

(*a*) (4.1), (4.3) has a unique solution for all $f \in C[0, 2\pi]$;

(*b*) (4.2), (4.3) has at least one solution.

PROOF. – Define $L\colon D(L) \to C[0, 2\pi]$ by $Lu = -u'$ where $u \in D(L)$ if and only if $u \in C_p[0, 2\pi]$ and u' is continuous. L is a densely defined operator in $C_p[0, 2\pi]$. If $\{u_n\} \subset D(L)$ such that $u_n \to u$ in $C_p[0, 2\pi]$ and $Lu_n = -u_n' \to v$ in $C[0, 2\pi]$, then by RUDIN [10] u is differentiable, i.e. $u \in D(L)$, and $u' = -v$, i.e. $Lu_n \to Lu$. Hence L is closed.

Define $N\colon C[0, 2\pi] \to C[0, 2\pi]$ by $(Nu)(t) = A(t, u(t))$. A simple computation shows that, if $u, h \in C[0, 2\pi]$, $(N'(u)h)(t) = A'(t, u(t))h(t)$. Hence N is continuously Fréchet differentiable and it follows easily from (1) that $\|N'(u)\| \leqslant K$ for all $u \in C[0, 2\pi]$.

We now prove that $L + N'(u)$ is invertible and that $\|[L + N'(u)]^{-1}\|$ is uniformly bounded for $u \in C[0, 2\pi]$. Let $u_0 \in C[0, 2\pi]$ and consider the linear problem (4.7), (4.3).

$$u'(t) = A'(t, u_0(t))\, u(t) - f(t) \tag{4.7}$$

where $f \in C[0, 2\pi]$. Clearly u satisfies (4.7), (4.3) if and only if $[L + N'(u_0)]u = f$.

$Y(t) = \exp\left[\int_0^t A'(s, u_0(s))\, ds\right]$ is a fundamental matrix for (4.7). Since $\|\exp A\| \leqslant \exp\|A\|$ for all $A \in M_n$, $\|Y(t)\| \leqslant \exp 2K\pi$ and $\|Y^{-1}(t)\| = \left\|\exp\left[-\int_0^t A'(s, u_0(s))\, ds\right]\right\| \leqslant \exp 2K\pi$ for $0 \leqslant t \leqslant 2\pi$. If $V = I - Y(2\pi)$, $\|V\| \leqslant 1 + \exp 2K\pi$. By hypothesis (2) and lemma 3.3 there exists $C > 0$, independent of u_0, such that $|\mu| \geqslant C$ for all eigenvalues μ of V. Hence V is invertible and by lemma 3.1. $\|V^{-1}\| < C^{-n}[1 + \exp 2K\pi]^{n-1}$. Since V is invertible, (4.7), (4.3) has a unique solution by proposition 4.1 i.e. $L + N'(u_0)$ has an everywhere defined inverse. Moreover, if $[L + N'(u_0)]u = f$, by (4.6) we have

$$u(t) = Y(t)\left[V^{-1}\int_0^{2\pi} Y^{-1}(s)f(s)\,ds - \int_0^t Y^{-1}(s)f(s)\,ds\right].$$

Hence

$$\|u(t)\| \leqslant \exp 2K\pi[C^{-n}(1+\exp 2K\pi)^{n-1}2\pi\exp 2K\pi]\|f\| + 2\pi\exp 2K\pi\|f\|$$

and so $\|u\| < K_1\|f\|$ where K_1 depends only on K and C. Therefore $\|[L+N'(u)]^{-1}\| \leqslant K_1$ for all $u \in C[0, 2\pi]$. (*a*) now follows from theorem 2.1.

Define F: $D(L) \to C[0, 2\pi]$ by $(Fu)(t) = h(t, u(t))$. Clearly F has bounded range. Let $\{u_n\}$ be any bounded sequence in $D(L)$. Since $\{u_n'\}$ is bounded in $C[0, 2\pi]$, $\{u_n\}$ is equicontinuous and so by Ascoli's theorem there exists a subsequence $\{u_m\}$ of $\{u_n\}$ such that $u_m \to u$ in $C[0, 2\pi]$. Choose $M > 0$ such that $\|u_m\| < M$ for all m. Let $\varepsilon > 0$. Since h is uniformly continuous on the compact set $[0, 2\pi]\times\{x \in C^n: \|x\| < M\}$, there exists $\delta > 0$ such that $\|h(t_1, x_1) - h(t_2, x_2)\| < \varepsilon$ if $|t_1 - t_2| + \|x_1 - x_2\| \leqslant \delta$. Choose n_0 such that $\|u_m - u\| < \delta$ if $m \geqslant n_0$. Hence, if $m > n_0$, $\|Fu_m - Fu\| < \varepsilon$. Therefore $Fu_m \to Fu$ in $C[0, 2\pi]$ and so we have proved that F is compact. A similar, but simpler, argument shows that F is continuous.
(*b*) now follows immediately from theorem 2.2.

We shall now discuss two alternative formulations of the rather cumbersome condition (2) of theorem 4.2.

(1) Firstly we study a system of second order equations similar to that studied by Leach in [9]. Consider

$$u_j''(t) + g_j(t, u_j(t)) = h_j(t, u_1(t), \dots, u_n(t), u_1'(t), \dots, u_n'(t)) \tag{4.8}$$

$$u_j(0) = u_j(2\pi)\,; \quad u_j'(0) = u_j'(2\pi) \tag{4.9}$$

for $j = 1, \dots, n$ where u_j: $[0, 2\pi] \to R$, g_j: $[0, 2\pi]\times R \to R$ and h_j: $[0, 2\pi]\times R^{2n} \to R$ and the following conditions are satisfied:

(i) the function $(t, x) \to (\partial g_j/\partial x)(t, x)$ is continuous; there exists $\delta > 0$ and an integer m_j such that $(m_j + \delta)^2 < (\partial g_j/\partial x)(t, x) < (m_j + 1 - \delta)^2$, for all $t \in [0, 2\pi]$ and all $x \in R$.

(ii) h_j is continuous and has bounded range.

If we let $u_k = v_k$ and $u_k' = w_k$, we see that (4.8), (4.9) is equivalent to the system

$$\begin{aligned} u'(t) &= A(t, u(t)) + h(t, u(t)) \\ u(0) &= u(2\pi) \end{aligned} \tag{4.10}$$

where u: $[0, 2\pi] \to R^{2n}$ such that $u(t) = (v_1(t), w_1(t), \dots, v_n(t), w_n(t))$, A: $[0, 2\pi]\times$

$\times R^{2n} \to R^{2n}$ such that

$$A(t, x_1, y_1, \ldots, x_n, y_n) = \big(y_1, -g(t, x_1), \ldots, y_n, -g(t, x_n)\big)$$

and $h\colon [0, 2\pi]\times R^{2n} \to R^{2n}$ such that $h = (0, h_1, 0, h_2, \ldots, 0, h_n)$.

It is easily verified that $x \to A(t, x)$ is Fréchet differentiable with $A'(t, x) = \operatorname{diag}(B_1, B_2, \ldots, B_n)$ where

$$B_j = \begin{pmatrix} 0 & 1 \\ -\dfrac{\partial g_j}{\partial x}(t, v_j(t)) & 0 \end{pmatrix}.$$

Hence, if $u \in C[0, 2\pi]$, $\int_0^{2\pi} A'(s, u(s))\, ds = \operatorname{diag}(C_1, \ldots, C_n)$ where

$$C_j = \begin{pmatrix} 0 & 2\pi \\ k_j & 0 \end{pmatrix} \quad \text{and} \quad k_j = -\int_0^{2\pi} \frac{\partial g_j}{\partial x}(s, v_j(s))\, ds\,.$$

The eigenvalues of $\int_0^{2\pi} A'(s, u(s))\, ds$ are $\pm\sqrt{-2\pi k_j}\, i$. By (i) $2\pi(m_j+\delta)^2 < -k_j < 2\pi(m_j + 1 - \delta)^2$. Hence dist $[\{2\pi n i\colon n \text{ is an integer}\}, \{\lambda\colon \lambda \text{ is an eigenvalue of } \int_0^{2\pi} A'(s, u(s))\, ds\}] > 2\pi\delta$.

It is now clear that (4.10) satisfies all the hypotheses of theorem 4.2 and so (4.8), (4.9) has at least one solution.

(2) We can replace condition (2) in theorem 4.2 by a condition on the numerical range of the Fréchet derivatives of A.

If $T \in M_n$, the numerical range of T, denoted by $\theta(T)$, is defined as $\{(Tu, u)\colon u \in C^n\ \|u\| = 1\}$. It can be shown that $\theta(T)$ is convex (KATO [6]).

THEOREM 4.3. – *If $\theta(A'(t, x))$ is contained in a closed convex set K for all $t \in [0, 2\pi]$ and all $x \in C^n$, then $\theta\Big[\int_0^{2\pi} A'(s, u(s))\, ds\Big\}$ is contained in the convex set $K_1 = \{2\pi x\colon x \in K\}$ for all $u \in C[0, 2\pi]$.*

PROOF. – Let $u \in C[0, 2\pi]$ and $x \in C^n$ with $\|x\| = 1$.

$$\int_0^{2\pi} \big(A'(s, u(s))\, x, x\big)\, ds$$

is the limit of Riemann sums of the form

$$\sum_{i=0}^{n-1} (s_{i+1} - s_i)\big(A'(s_i, u(s_i))\, x, x\big) \qquad \text{where } 0 = s_0 < s_1 < \ldots < s_n = 2\pi\,.$$

Since $\sum_{i=0}^{n-1}(2\pi)^{-1}(s_{i+1}-s_i)=1$ and $\big(A'(s_i,u(s_i))x,x\big)\in K$ for $i=0,\dots,n-1$,

$$\sum_{i=0}^{n-1}(2\pi)^{-1}(s_{i+1}-s_i)\big(A'(s_i,u(s_i))x,x\big)\in K \text{ i.e.}$$

$$\sum_{i=0}^{n-1}(s_{i+1}-s_i)\big(A'(s_i))x,x\big)\in K_1 .$$

Since K_1 is closed, $\int_0^{2\pi}\big(A'(s,u(s))x,x\big)\,ds\in K_1$. By considering Riemann sums it can easily be shown that

$$\int_0^{2\pi}\big(A'(s,u(s))x,x\big)\,ds=\Big(\int_0^{2\pi}A'(s,u(s))\,ds\,x,x\Big)\in K_1$$

and this completes the proof.

Since, for $T\in M_n$, $\theta(T)$ clearly contains all the eigenvalues of T, by theorem 4.3 we can replace condition (2) by

(2′) there exists a closed convex set K and $\delta>0$ such that $\theta\big(A'(t,x)\big)\subset K$ for all $t\in[0,2\pi]$ and all $x\in C^n$ and

$$\operatorname{dist}\,[K_1,\{ni\colon n \text{ is an integer}\}]>\delta .$$

5. – In this section we shall apply the results of section 2 to obtain existence and uniqueness theorems for nonlinear boundary value problems associated with elliptic and ordinary differential expressions.

Let Ω be an open subset of R^n with smooth boundary denoted by $bd(\Omega)$. We shall consider only linear differential operators generated by the Laplacian, Δ, but it is easy to see how our results can be extended to higher order expressions or to elliptic expressions with variable coefficients. We shall apply theorem 2.2 where $X=Y=L_2(\Omega)$—the set of all real-valued Lebesgue square integrable functions on Ω with norm $\|u\|_0=\Big\{\int_\Omega|u(x)|^2\,dx\Big\}^{\frac12}$. We shall also require the space $C_2^0(\bar\Omega)=$ $=\{u\colon\bar\Omega\to R\colon u$ has continuous second order derivatives on $\bar\Omega$ and $u(x)=0$ for all $x\in bd(\Omega)\}$ and the space $N_2(\Omega)=\{u\colon\Omega\to R\colon$ all generalized partial derivatives of u of order $\leqslant 2$ are in $L_2(\Omega)\}$ with norm $\|u\|_2=\Big\{\sum_{\alpha\leqslant 2}\|D^\alpha u\|_0^2\Big\}^{\frac12}$.

Let $L\colon D(L)\to L_2(\Omega)$ be such that $Lu=-\Delta u(x)+q(x)u(x)$ where $q\colon\Omega\to R$ is measurable and $u\in D(L)$ if and only if $u\in C_0^2(\bar\Omega)$ and $-\Delta u+qu\in L_2(\Omega)$. In the examples which we shall discuss L is essentially self adjoint with closure $\bar L$. Moreover we shall have that $D(\bar L)=H_2^0(\Omega)$—the closure of $C_2^0(\bar\Omega)$ in $\|\;\|_2$—and that the graph norm on $D(\bar L)$ is equivalent to $\|\;\|_2$. In this case we have as an immediate consequence of theorem 2.2:

Theorem 5.1. – If

(1) $N: L_2(\Omega) \to L_2(\Omega)$ is continuously Fréchet differentiable and there exists $K > 0$ such that $\|N'(u)\| \leqslant K$ for all $u \in L_2(\Omega)$,

(2) $\bar{L} + N'(u)$ has a bounded everywhere defined inverse on $L_2(\Omega)$ and there exists $\delta > 0$ such that $\|[L + N'(u)]^{-1}\| \leqslant \delta$ for all $u \in L_2(\Omega)$,

(3) $F: H_2^0(\Omega) \to L_2(\Omega)$ is compact, continuous and has bounded range, then there exists at least one solution of $\bar{L}u + Nu = Fu$.

We consider the case where N is a Nemytskii operator i.e. there exists $g: R \to R$ such that $(Nu)(x) = g(u(x))$. We shall give conditions on Ω and g to ensure that the conditions of theorem 5.1 are satisfied.

Suppose that g has a continuous derivative and that there exists $M > 0$ such that $|g'(x)| \leqslant M$ for all $x \in R$. Let $u \in L_2(\Omega)$. Then

$$|(Nu)(x)| = |g(u(x))| \leqslant |g(u(x)) - g(0)| + |g(0)| = |u(x)||g'(\xi)| + |g(0)| \leqslant M|u(x)| + |g(0)|.$$

Hence, if $g(0) = 0$ or if Ω is bounded, $N: L_2(\Omega) \to L_2(\Omega)$. It follows from the Lebesgue dominated convergence theorem that N is continuously differentiable with $(N'(u)h)(x) = g'(u(x))h(x)$. Hence $\|N'(u)\| \leqslant M$.

The following lemma is useful for verifying that condition (2) is satisfied.

Lemma 5.2. Let A be a densely defined self-adjoint operator on $L_2(\Omega)$ which is bounded below. Suppose that there exists $a \in R$ such that the essential spectrum of A is contained in $[a, \infty)$ and A has a finite or infinite number of eigenvalues $\lambda_1 < \lambda_2 < \ldots < a$. If there exists an integer k, $\delta > 0$ and $p: \Omega \to R$ such that $\lambda_k + \delta < p(x) < \lambda_{k+1} - \delta$ for all $x \in \Omega$, then, if $A_1 u = Au - pu$, A_1 has a bounded everywhere defined inverse and $\|A_1^{-1}\| \leqslant \delta^{-1}$.

Proof. – It is obvious that A_1 is self-adjoint. Let F_k be the family of all $(k-1)$ dimensional subspaces of $L_2(\Omega)$ and, if $F \in F_k$, let $F^\perp$ denote the orthogonal complement of F. It is well known that

$$\lambda_k = \sup_{F \in F_k} \inf \{(Au, u): \|u\| = 1 \text{ and } u \in F^\perp\}$$

Define

$$\mu_k = \sup_{F \in F_k} \inf \{(A_1 u, u): \|u\| = 1 \text{ and } u \in F^\perp\}.$$

If $F \in F_k$,

$$\inf \{(A_1 u, u): \|u\| = 1 \text{ and } u \in F^\perp\} = \inf \{(Au, u) - (pu, u): \|u\| = 1 \text{ and } u \in F^\perp\} \leqslant$$
$$\leqslant \inf \{(Au, u): \|u\| = 1 \text{ and } u \in F^\perp\} - (\lambda_k + \delta) \leqslant \lambda_k - (\lambda_k + \delta) = -\delta.$$

Hence

$$\mu_k \leqslant -\delta\,. \tag{5.1}$$

Let $\{w_n\}$ be an orthonormal sequence of eigenvectors of A such that $Aw_i = \lambda_i w_i$. Let $P = [w_1, \ldots, w_k] \in F_{k+1}$. Then

$$\begin{aligned}\mu_{k+1} &\geqslant \inf\{(A_1 u, u)\colon \|u\| = 1 \text{ and } u \in P^\perp\}\\ &\geqslant \inf\{(Au, u)\colon \|u\| = 1 \text{ and } u \in P^\perp\} - (\lambda_{k+1} - \delta)\end{aligned}$$

Since $\inf\{(Au, u)\colon \|u\| = 1 \text{ and } u \in P^\perp\} = \lambda_{k+1}$,

$$\mu_{k+1} \geqslant \delta\,. \tag{5.2}$$

If A_1 has essential spectrum in $(-\infty, \delta - \varepsilon)$, where $\varepsilon > 0$ it is easy to see that $\mu_j < \delta - \varepsilon$ for all integers j. Since ε is arbitrary, (5.2) shows that A_1 has only eigenvalues in $(-\infty, \delta)$ and that the dimension of the subspace spanned by eigenvectors corresponding to these eigenvalues is less than or equal to k. By (5.1) there is a subspace of dimension k consisting of eigenvectors of A_1 corresponding to eigenvalues in $(-\infty, -\delta]$. Hence $(-\delta, \delta)$ contains no points in the spectrum of A_1 and so 0 is in the resolvent set of A_1. Therefore A_1 is invertible and, by KATO [6], V 3.16,

$$\|A_1^{-1}\| = \text{dist}[0, \text{ spectrum of } A_1] \leqslant \delta^{-1}\,.$$

Suppose that Ω is bounded and q is bounded and measurable. The following results can be found in AGMON [1]. L is essentially self-adjoint, $\bar{L}$ is bounded below and the spectrum of $\bar{L}$ consists only of the eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots$. The coerciveness inequality $\|u\|_2 < C(\|Lu\|_0 + \|u\|_0)$ implies that $D(\bar{L}) = H_2^0(\Omega)$ and that the graph norm of $\bar{L}$ is equivalent to $\|\ \|_2$.

If $g\colon R \to R$ has a continuous derivative and there exists an integer k and $\delta > 0$ such that $\lambda_k + \delta < -g'(t) < \lambda_{k+1} - \delta$ for all $t \in R$ it follows from lemma 5.2 that $\|[L + N'(u)]^{-1}\| < \delta^{-1}$. Hence conditions (1) and (2) are satisfied.

If $(Fu)(x) = f(x, u(x), u'(x))$ where $f\colon \Omega \times R \times R \to R$ is continuous and has bounded range, $F\colon H_1(\Omega) \to L_2(\Omega)$ is continuous by LANDESMAN and LAZER [7] proposition 3.1. Since the embedding from $H_2^0(\Omega)$ to $H_1(\Omega)$ is compact and continuous, $F\colon H_2^0(\Omega) \to L_2(\Omega)$ is compact and continuous. Clearly F has bounded range and so condition (3) is satisfied.

The above results where Ω is bounded and N and F are Nemytskii operators are a special case of the results of LANDESMAN and LAZER [7]. Because of the general nature of our theorems 2.2 and 5.1, however, we can easily extend our results to the case where Ω is unbounded.

Consider, for example, the case where $\Omega = (0, \infty)$ and

$$(Lu)(x) = -u''(x) - \frac{4}{x}\,u(x)\,. \tag{5.3}$$

We recall that $u \in D(L)$ only if u'' is continuous and $u(0) = 0$. By KATO [6], V 4.4 and STUART [12] L is essentially self-adjoint, $D(\bar{L}) = H_0^2[0, \infty]$ and there exists $K > 0$ such that the graph norm of $\bar{L} + KI$ is equivalent to $\| \|_2$. Moreover, the spectrum of $\bar{L} = \{-4/n^2 : n = 1, 2, \ldots\} \cup [0, \infty)$ and, for each positive integer n, $-4/n^2$ is a simple (multiplicity one) eigenvalue.

Suppose that $g: R \to R$ has continuous derivative and there exists an integer k and $\delta > 0$ such that

$$-\frac{4}{k^2} + \delta < g'(t) < -\frac{4}{(k+1)^2} - \delta \quad \text{for all } t \in R;\ g(0) = 0\,. \tag{5.4}$$

If $(Nu)(x) = -g(u(x)) - Ku(x)$, then $\bar{L} + KI$ and N satisfy conditions (1) and (2) of theorem 5.1.

Define $F: H_2^0[0, \infty] \to L_2[0, \infty]$ by $(Fu)(x) = a(x) f(x, u(x), u'(x))$ where f is continuous and has bounded range and a is continuous and

$$a \in L_2[0, \infty]\,; \quad \lim_{x \to \infty} a(x) = 0 \tag{5.5}$$

If $|f(x, y, z)| \leqslant M$ for all $x, y, z \in R$ and $u \in L_2[0, \infty]$, then $\|Fu\|_0 < M\|a\|_0$ and so F has bounded range. If $T > 0$, $F: H_2^0[0, T] \to L_2[0, T]$ is compact and continuous by proposition 3.1 of LANDESMAN and LAZER [7] as we proved above. Since $\lim_{x \to \infty} a(x) = 0$, a simple subsequence argument, like that in BROWN [2] section 10, shows that

$$F: H_2^0[0, \infty] \to L_2[0, \infty] \quad \text{is compact and continuous}.$$

Hence theorem 5.1 proves that there exists at least one solution of the equation

$$(\bar{L}u)(x) - g(u(x)) = a(x) f(x, u(x), u'(x))$$

where L is given by (5.3), g has a continuous derivative and satisfies (5.4), a and f are continuous, f has bounded range and a satisfies (5.5).

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