Some refinements of asymptotic stability theory.

JOHN R. HADDOCK (U.S.A.) (*)

Summary. - *The concept of a Liapunov function with strongly negative definite derivative on an annulus is introduced. Sufficient conditions for stability and asymptotic stability are given without using any "definiteness" properties of Liapunov functions.*

1. - Introduction.

Let G be an open subset of R^n containing the origin and consider the system of ordinary differential equations

$$(1) \quad x' = f(t, x) \quad (d/dt)$$

where $f: [0, \infty) \times G \rightarrow R^n$, $f(t, 0) = 0$, and (1) is of CARATHÉODORY type [2, p. 43]. Thus, (1) possesses the zero solution, $x(t) \equiv 0$, and for each (t_0, x_0) in $[0, \infty) \times G$, there exists at least one solution $x(t, t_0, x_0)$ defined on some t interval containing t_0 .

For a set B , we shall denote by B^c the complement of B . Also, for $\varepsilon > 0$, we define $S(\varepsilon) = \{x: |x| < \varepsilon\}$ and $H(\varepsilon) = \{x: \varepsilon/2 \leq |x| \leq \varepsilon\}$ where $|\cdot|$ is any suitable norm in R^n .

The purpose of this paper is to refine fundamental asymptotic stability theorems by improving standard conditions on LIAPUNOV functions which insure asymptotic stability (which is not necessarily uniform) of the zero solution of (1). For stability definitions we refer the reader to [8, pp. 27-28]. For other definitions of terms and symbols we refer to [8, Chapter I] and [4, Chapter I]. To avoid confusion we mention that "infinitesimal upper bound" and "decreascent" are used interchangeably in the literature.

We shall say that V is a *Liapunov function* for (1) on G if

- (i) $V(t, x) \geq 0$ for all (t, x) in $[0, \infty) \times G$,
- (ii) V is C' (has continuous first partials), and
- (iii)
$$V'(t, x) = \partial V(t, x)/\partial t + \sum_{i=1}^n [\partial V(t, x)/\partial x_i] f_i(t, x) \leq 0$$
for all (t, x) in $[0, \infty) \times G$.

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Although one need only assume that V is absolutely continuous in t uniformly at any point (t_0, x_0) and that V is locally LIPSCHITZIAN in x (cf. [8, pp. 10-11]), we ask that V be C' . Also, we could replace $V(t, x) \geq 0$ by V bounded from below, but by choosing an appropriate constant c , we can define a new LIAPUNOV function $W = V + c$ where $W(t, x) \geq 0$. Clearly, $W' = V'$.

Henceforth, when we refer to a LIAPUNOV function, we mean a LIAPUNOV function for (1) on G . Also, when we ask that certain conditions hold on $[0, \infty) \times S(A)$, we mean for some $A > 0$ where $S(A)$ is contained in G . By a *wedge* we mean a continuous, strictly increasing scalar function $g: [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$.

The following is the classical asymptotic stability theorem in LIAPUNOV theory.

THEOREM 1.1. - ([4, Theorem II.4.2]). Suppose there exist wedges g_1, g_2, g_3 and a LIAPUNOV function V which satisfy the following conditions on $[0, \infty) \times S(A)$;

- (i) $g_1(|x|) \geq V(t, x) \geq g_2(|x|)$ and
- (ii) $V'(t, x) \leq -g_3(|x|)$.

Then the zero solution of (1) is asymptotically stable.

Observing that the conditions of Theorem 1.1 are sufficient to imply uniform asymptotic stability of the zero solution of (1), MARACHKOFF [5] (see also ANTOSIEWICZ [1] and MASSERA [6]) obtained the following result which does not necessarily imply uniform asymptotic stability.

THEOREM 1.2. - ([6, Theorem 4]). Suppose there exist $A > 0$ and $M > 0$ such that $|f(t, x)| \leq M$ for all (t, x) in $[0, \infty) \times S(A)$. Further, suppose there exist wedges g_2, g_3 and a LIAPUNOV function V which satisfy the following conditions on $[0, \infty) \times S(A)$;

- (i) $V(t, 0) \equiv 0$,
- (ii) $V(t, x) \geq g_2(|x|)$, and
- (iii) $V'(t, x) \leq -g_3(|x|)$.

Then the zero solution of (1) is asymptotically stable.

In this paper we seek, among other things, to improve the conditions of Theorem 1.1 and Theorem 1.2. We shall see that if V' satisfies certain conditions in an annulus, then the entire weight of asymptotic stability can be shifted to V' .

2. - Sufficient conditions for asymptotic stability.

We now introduce the concept of V' strongly negative definite in an annulus and show that "under certain conditions" asymptotic stability can be concluded without knowledge of any "definiteness" properties of V .

DEFINITION 2.1. - $V'(t, x)$ is *strongly negative definite on the annulus* $a \leq |x| \leq b$ ($0 < a \leq b$) if there exists $\delta > 0$ such that $a \leq |x| \leq b$ implies $V'(t, x) \leq -\delta |f(t, x)|$ for all $t \geq 0$. V' is *strongly negative definite on $S(A)$* if, for every, $a, b > 0$ ($a \leq b < A$), there exists $\delta = \delta(a, b) > 0$ such that $a \leq |x| \leq b$ implies $V'(t, x) \leq -\delta |f(t, x)|$ for all $t \geq 0$.

THEOREM 2.1. - Suppose there exists $A > 0$ such that, for each $\varepsilon > 0$ ($\varepsilon < A$), there exists a LIAPUNOV function V_ε which satisfies the following conditions;

- (i) $V_\varepsilon(t, 0) = 0$ and
- (ii) V'_ε is strongly negative definite on $H(\varepsilon)$.

Then the zero solution of (1) is stable.

PROOF. - Suppose there exists $t_0 \geq 0$ such that $x(t) = 0$ is not stable past t_0 . Then there exists $\varepsilon > 0$ such that, for every $\eta > 0$ there exists $|x_0| < \eta < \varepsilon$ and $t^* > t_0$ such that $|x(t^*, t_0, x_0)| = \varepsilon$. Let $V = V_\varepsilon$ be a LIAPUNOV function such that V' is strongly negative definite on $H(\varepsilon)$; that is, $V'(t, x) \leq -\delta |f(t, x)|$ for $\varepsilon/2 \leq |x| \leq \varepsilon$, $t \geq 0$ and some $\delta > 0$.

Now V is continuous and $V(t_0, 0) = 0$ so there exists $\delta_1 > 0$ such that $V(t_0, x) < \delta\varepsilon/2$ whenever $|x| < \delta_1 < \varepsilon/2$. Also, there exists x_0, t_1 and t_2 such that $|x_0| < \delta_1$, $|x(t_1, t_0, x_0)| = \varepsilon/2$, $|x(t_2, t_0, x_0)| = \varepsilon$, and $\varepsilon/2 \leq |x(t, t_0, x_0)| \leq \varepsilon$ whenever $t_0 < t_1 \leq t \leq t_2$. Integrating V' along $x(t, t_0, x_0) = x(t)$ we obtain

$$\begin{aligned} V(t_2, x(t_2)) &= V(t_1, x(t_1)) + \int_{t_1}^{t_2} V'(s, x(s)) ds \\ &\leq V(t_0, x_0) - \int_{t_1}^{t_2} \delta |f(s, x(s))| ds \\ &\leq V(t_0, x_0) - \delta \left| \int_{t_1}^{t_2} f(s, x(s)) ds \right| \end{aligned}$$

$$\leq V(t_0, x_0) - \delta |x(t_2) - x(t_1)| < \delta\varepsilon/2 - \delta\varepsilon/2 = 0.$$

The following example illustrates that the conclusion of Theorem 2.1 is This contradicts $V \geq 0$ and hence, the zero solution is stable.

not asymptotic stability. Consider the scalar linear equation

$$(2) \quad x' = -p(t)x$$

where $p(t)$ is continuous on $[0, \infty)$, nonnegative and nonincreasing, and $\int_0^\infty p(t)dt < \infty$. The zero solution of this equation is not asymptotically stable. However, for $V(t, x) = x^2/2$, we have $V'(t, x) = -p(t)x^2$ and the conditions of Theorem 2.1 are satisfied, since for any $\varepsilon > 0$, we have $V'(t, x) \leq -(\varepsilon/4)|p(t)x|$ whenever $\varepsilon/2 \leq |x| \leq \varepsilon$.

THEOREM 2.2. - In addition to the hypothesis of Theorem 2.1, suppose for each $t_0 \geq 0$ there exists $\delta = \delta(t_0) > 0$ such that $|x_0| < \delta$ implies $x(t, t_0, x_0)$ is not bounded strictly away from zero. Then the zero solution of (1) is asymptotically stable.

PROOF. - By Theorem 2.1, the zero solution is stable. Suppose there exists $t_0 \geq 0$ such that $x(t) = 0$ is not asymptotically stable past t_0 . Let $\delta_1 > 0$ and x_0 be chosen such that $|x_0| < \delta_1$, $|x_0| < \delta$ implies $|x(t, t_0, x_0)| < A$ for $t \geq t_0$, and $x(t, t_0, x_0) = x(t) \rightarrow 0$ as $t \rightarrow \infty$. Since $x(t)$ is not bounded strictly away from zero, there exists $\varepsilon > 0$ ($\varepsilon < A$) and sequences $\{t'_n\}, \{t''_n\} \rightarrow \infty$ such that $|x(t'_n)| = \varepsilon/2$, $|x(t''_n)| = \varepsilon$ for each n and $\varepsilon/2 \leq |x(t)| \leq \varepsilon$ whenever $t'_n \leq t \leq t''_n$. Let $V = V_\varepsilon$ be a LIAPUNOV function such that V' is strongly negative definite on $H(\varepsilon)$. Since $V'(t, x) \leq 0$ on $[0, \infty) \times S(A)$, we have

$$(3) \quad V'(t, x) \leq V(t_0, x_0) + \sum_{n=1}^k \int_{t'_n}^{t''_n} V'(s, x(s)) ds$$

where $t''_k \leq t \leq t'_{k+1}$. Hence,

$$\begin{aligned} V(t, x(t)) &\leq V(t_0, x_0) + \sum_{n=1}^k \int_{t'_n}^{t''_n} V'(s, x(s)) ds \\ &\leq V(t_0, x_0) - \delta \sum_{n=1}^k \int_{t'_n}^{t''_n} |f(s, x(s))| ds \\ &\leq V(t_0, x_0) - \delta \sum_{n=1}^k \left| \int_{t'_n}^{t''_n} f(s, x(s)) ds \right| \end{aligned}$$

$$\begin{aligned}
&= V(t_0, x_0) - \delta \sum_{n=1}^k |x(t_n'') - x(x(t_n'))| \\
&\leq V(t_0, x_0) - \delta k\varepsilon/2 \rightarrow -\infty \text{ as } t \rightarrow \infty.
\end{aligned}$$

This contradicts V bounded from below and the proof is complete.

The following simple example assures that uniform asymptotic stability cannot be concluded in Theorem 2.2. In fact, we shall later see that the conclusion is not equiasymptotic stability and hence, by Theorem 3 of [7], not uniform stability.

The zero solution of the scalar linear equation

$$(4) \quad x' = -x/(t+1)$$

is not uniformly asymptotically stable. However, for $V(t, x) = x^2/2$, we have $V'(t, x) = -x^2/(t+1)$ and the conditions of Theorem 2.2 are satisfied.

We now examine two cases in which the conditions of Theorem 2.2 are satisfied.

COROLLARY 2.1. - Suppose there exist continuous, nonnegative functions h_3, h_4 , wedges g_3, g_4 and a LIAPUNOV function V such that the following conditions are satisfied on $[0, \infty) \times S(A)$;

- (i) $V(t, 0) = 0$,
- (ii) $V'(t, x) \leq -h_3(t)g_3(|x|)$, and
- (iii) $|f(t, x)| \leq h_4(t)g_4(|x|)$.

If there exists $a > 0$ such that $h_3(t) \geq ah_4(t)$ and if $\int_0^\infty h_3(t)dt = \infty$, then the zero solution of (1) is asymptotically stable.

PROOF. - We show that the conditions of Theorem 2.2 are satisfied. Let $\varepsilon > 0$ be given such that $\varepsilon < A$ and let $c > 0$ be chosen such that $ca > g_4(\varepsilon)/g_3(\varepsilon/2)$. We now show that V' is strongly negative definite on $H(\varepsilon)$. Let $\delta = 1/c$ and suppose $\varepsilon/2 \leq |x| \leq \varepsilon$. Then

$$\begin{aligned}
V(t, x) &\leq -h_3(t)g_3(|x|) \leq -h_3(t)g_3(\varepsilon/2) \\
&\leq -ah_4(t)g_4(\varepsilon)/ca = -(1/c)h_4(t)g_4(\varepsilon) \\
&\leq -(1/c)h_4(t)g_4(|x|) \leq -(1/c)|f(t, x)| = -\delta|f(t, x)|.
\end{aligned}$$

Hence, since ε was arbitrary, the conditions of Theorem 2.1 are satisfied and the zero solution is stable.

Suppose there exists $t_0 \geq 0$ such that, for each $\eta > 0$ there exists $|x_0| < \eta$ such that $x(t, t_0, x_0)$ is bounded strictly away from zero. Let $\delta_1 > 0$ and x_0 be chosen such that $|x_0| < \delta_1$ and $b \leq |x(t, t_0, x_0)| < A$ for all $t \geq t_0$. Integrating V' along $x(t, t_0, x_0) = x(t)$ we obtain

$$\begin{aligned} V(t, x(t)) &\leq V(t_0, x_0) - \int_{t_0}^t h_3(s) g_3(|x(s)|) ds \\ &\leq V(t_0, x_0) - g_3(b) \int_{t_0}^t h_3(s) ds \rightarrow -\infty \text{ as } t \rightarrow \infty. \end{aligned}$$

This contradicts V bounded from below and hence, $x(t)$ cannot be bounded strictly away from zero. Therefore, the conditions of Theorem 2.2 are satisfied and the conclusion follows.

It is easily seen that the wedge g_4 in the above corollary can be replaced by a continuous function $g: R^n \rightarrow [0, \infty)$ where $g(x) > 0$ if $x \neq 0$. In this case we ask that $c > 0$ be chosen such that $ca > m/g_3(\varepsilon/2)$ where $m = \max g(x)$ on the annulus $\varepsilon/2 \leq |x| \leq \varepsilon$.

The following corollary shows that Theorem 1.2 (MARACHKOFF and MASSERA's result) is a special case of Theorem 2.2. In fact, we see that Theorem 1.2 can be proven without the use of condition (ii).

COROLLARY 2.2. - Suppose there exists $A > 0$ and $M > 0$ such that $|f(t, x)| \leq M$ for all (t, x) in $[0, \infty) \times S(A)$. Further, suppose there exist a LIAPUNOV function V and a wedge g_3 such that the following conditions are satisfied on $[0, \infty) \times S(A)$;

- (i) $V(t, 0) = 0$ and
- (ii) $V'(t, x) \leq -g_3(|x|)$.

Then the zero solution of (1) is asymptotically stable.

PROOF. - Again, we show that the conditions of Theorem 2.2 are satisfied. Let $\varepsilon > 0$ be given such that $\varepsilon < A$ and let $\delta = g_3(\varepsilon/2)/M$. Then, for $\varepsilon/2 \leq |x| \leq \varepsilon$ we have

$$\begin{aligned} V'(t, x) &\leq -g_3(|x|) \leq g_3(\varepsilon/2) = -[g_3(\varepsilon/2)/M]M \\ &\leq [g_3(\varepsilon/2)/M]|f(t, x)| = -\delta|f(t, x)|. \end{aligned}$$

Hence, V' is strongly negative definite on $H(\varepsilon)$ for each $\varepsilon < A$. As in Corollary 2.1, it is easily shown that for each $t_0 \geq 0$ there exists $\delta > 0$ such that

$x(t, t_0, x_0)$ is not bounded away from zero whenever $|x_0| < \delta$. Therefore, the conditions of Theorem 2.2 are satisfied and the proof is complete.

We are now ready to mention an example which illustrates that the conclusion of Theorem 2.2 is not equiasymptotic stability. Clearly, it suffices to show that equiasymptotic stability cannot be concluded in Corollary 2.2. The following example is discussed at some length in [6, Example 4] and, in view of this, we shall omit many of the details.

Consider the second order system given in polar coordinates:

$$(5) \quad \begin{aligned} r' &= r \frac{\partial g(t, \theta)/\partial t}{g(t, \theta)} \\ \theta' &= 0 \end{aligned}$$

$$\text{where } g(t, \theta) = \frac{\sin^4 \theta}{\sin^4 \theta + (1 - t \sin^2 \theta)^2} + \frac{1}{1 + \sin^4 \theta} \cdot \frac{1}{1 + t^2}.$$

The solution $r = 0$ is asymptotically stable but not equiasymptotically stable. Also, the assumptions of Corollary 2.2 are satisfied for the LIAPUNOV function

$$V(t, r, \theta) = \frac{r^2}{g^2(t, \theta)} \cdot [4 + U(t, \theta)]$$

where

$$U(t, \theta_0) = \begin{cases} \int_t^\infty g^2(t, \theta_0) dt & \text{if } \theta_0 \neq k\pi \\ \int_t^\infty g^2(t, \theta_0) dt + \int_{-\infty}^\infty \frac{dt}{(1 + t^2)^2} & \text{if } \theta_0 = k\pi. \end{cases}$$

In this case, $V' = -r^2$ is negative definite. We note that V is positive definite but, according to Corollary 2.2, this condition need not be checked. For a discussion of another example similar to the above see [4, pp. 259-260].

We notice that the LIAPUNOV function in Corollary 2.1 (also Corollary 2.2) has strongly negative definite derivative on each annulus $H(\varepsilon)$, $\varepsilon < A$. However, in Theorem 2.1 (Theorem 2.2) we do not ask for a LIAPUNOV function with strongly negative definite derivative on each annulus $H(\varepsilon)$. Indeed, we only ask that for each $\varepsilon > 0$, $0 < \varepsilon < A$, there exists a LIAPUNOV function V_ε such that V'_ε is strongly negative definite on $H(\varepsilon)$. It is currently unknown if the conditions of Theorem 2.1 actually imply the existence of a LIAPUNOV function V with V' strongly negative definite on each $H(\varepsilon)$. It is easily shown, however, that if the conditions of Theorem 2.1 hold, then for each annulus there is a LIAPUNOV function V such that V' is strongly negative definite on this annulus.

THEOREM 2.3. - Suppose for each ε , $0 < \varepsilon < A$, there exists a LIAPUNOV function V_ε such that V'_ε is strongly negative definite on $H(\varepsilon)$. Then, for any $a, b > 0$ ($a \leq b < A$) there exists a LIAPUNOV function V (depending on a and b) such that V' is strongly negative definite on $a \leq |x| \leq b$.

PROOF. - Let $a, b > 0$ be given such that $a \leq b < A$ and choose n , a positive integer, sufficiently large such that $b/2n \leq a$. For each $k = 1, 2, \dots, n$ let V_k be a LIAPUNOV function such that $V'_k(t, x) \leq -\delta_k |f(t, x)|$ for $b/2k \leq |x| \leq b/k$ and some $\delta_k > 0$. Define $V = \sum_{k=1}^n V_k$ and let $\delta = \min(\delta_1, \dots, \delta_n)$. Let x be given such that $a \leq |x| \leq b$. Then $b/2j \leq |x| \leq b/j$ for some j , $1 \leq j \leq n$, and

$$V'(t, x) = \sum_{k=1}^n V'_k(t, x) \leq -\delta_j |f(t, x)| \leq -\delta |f(t, x)|.$$

Hence, $V'(t, x)$ is strongly negative definite on $a \leq |x| \leq b$.

COROLLARY 2.3. - Suppose for each ε , $0 < \varepsilon < A$, there exists a LIAPUNOV function V_ε such that $V_\varepsilon(t, 0) = 0$ and V'_ε is strongly negative definite on $H(\varepsilon)$. Further, suppose for some $b < A$ and each a , $0 < a \leq b$, there exists a continuous function $h_a(t)$ such that $\int_0^\infty h_a(t) dt = \infty$ and $|f(t, x)| \geq h_a(t)$ for $a \leq |x| \leq b$ and $t \geq 0$. Then the zero solution of (1) is asymptotically stable.

PROOF. - By Theorem 2.1, $x(t) = 0$ is stable. Suppose it is not asymptotically stable past some $t_0 \geq 0$. Let $\delta > 0$ be chosen such that $|x_0| < \delta$ implies $|x(t, t_0, x_0)| < b$ for all $t \geq t_0$. Suppose there exists $|x_0| < \delta$ such that $x(t, t_0, x_0) = x(t)$ is bounded away from zero. Then there exists $a > 0$ and $h_a(t)$ such that $a \leq |x(t)| \leq b$ and $|f(t, x(t))| \geq h_a(t)$ for all $t \geq t_0$. By Theorem 2.3, there exists a LIAPUNOV function V and a $\delta > 0$ such that $V'(t, x) \leq -\delta |f(t, x)|$ for all $t \geq t_0$ and $a \leq |x| \leq b$. Integrating V' along $x(t)$ we obtain

$$\begin{aligned} V(t, x(t)) &= V(t_0, x_0) + \int_{t_0}^t V'(s, x(s)) ds \\ &\leq V(t_0, x_0) - \delta \int_{t_0}^t |f(s, x(s))| ds \\ &\leq V(t_0, x_0) - \delta \int_{t_0}^t h_a(s) ds \rightarrow -\infty \text{ as } t \rightarrow \infty. \end{aligned}$$

This contradicts V bounded from below and again, the conditions of Theorem 2.2 have been established and the corollary is proven.

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