Absolutely continuous solutions of Cauchy's problem for $u_{xy} = f(x, y, u, u_x, u_y)$.

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Summary: - *See Introduction*

Let $I = [0, b] \times [0, b]$ be an interval in the (x, y) — plane and $y = \alpha(x)$ a strictly monotonic decreasing curve with endpoints $(0, b)$ and $(a, 0)$. We look for a function $u(x, y)$, absolutely continuous in I, satisfying the differential equation

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{1}$$

almost everywhere in I and taking on prescribed values on the curve, together with its first derivatives

$$\begin{aligned} u(x, \alpha(x)) &= \sigma_1(x), \\ u_x(x, \alpha(x)) &= \sigma_2(x). \\ u_y(x, \alpha(x))\alpha'(x) &= \sigma_1'(x) - \sigma_2(x). \end{aligned} \tag{2}$$

At first, we show that problem (1), (2) is solvable if the given functions satisfy conditions similar to those of CARATHÉODORY (cp. introduction of [1]) for the initial value problem

$$u' = g(x, u), \; u(0) = u_0 . \tag{3}$$

Then we admit linear growth of f with respect to u, u_x and u_y, reducing this case to an existence theorem of [1] by means of a priori estimates. Next, we prove a theorem on dependence on parameters for (3). Essentially, it says that the solution of (3) depends absolutely continuously on the parameter if the right hand side g has the same property and satisfies a LIPSCHITZ condition with respect to u. Applyng this theorem, we state some conditions sufficient

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for the solution u of problem (1), (2) to have integrable derivatives u_{xx} and u_{yy}. In case of classical solutions of DARBOUX's problem, *i.e.* (1) together with $u(x, 0) = \sigma(x)$ and $u(0, y) = \tau(y)$, the latter question was considered in [5] and [6].

Let us note that we have chosen the somewhat complicated uniqueness conditions in Theorem 1, which are the weakest known ones, since it is still an open question whether (1), (2) has an absolutely continuous solution if, for example, f is only continuous and bounded. It is well known that these conditions are not sufficient for the existence of a classical solution, as was shown in [3].

1. - Let R^p be the p-dimensional, real Euclidean space. A set F of functions $f(x)$ is called almost uniformly bounded on $A \subset R^p$, if to every $\varepsilon > 0$ there is a set $A_\varepsilon \subset A$ with $\mu(A - A_\varepsilon) < \varepsilon$ and a constant $M_\varepsilon > 0$ such that $|f(x)| \leq M_\varepsilon$ for all $f \in F$ and $x \in A_\varepsilon$(μ denotes the p-dimensional LEBESGUE measure) If $A \subset R^p$ is compact then $C(A)$ denotes the space of all real valued functions $\varphi(x)$ continuous on A with norm $|\varphi|_0 = \max_A |\varphi(x)|$. If A is measurable then $L(A)$ stands for the space of all functions LEBESGUE integrable over A. In the sequel, we always have $I_x = [0, a]$, $I_y = [0, b]$ and $I = I_x \times \times I_y \subset R^2$.

$C_x(I)$ is the space of all functions $u(x, y)$, defined in I, continuous in x and measurable in y with

$$|u|_x = \int_{I_y} \max_{I_x} |u(x, \eta)| \, d\eta < \infty.$$

$C_y(I)$ is defined correspondingly.

The inverse function of $\alpha(x)$ is denoted by $\beta(y)$. 'Almost everywhere' and 'for almost all' is shortened by a.e. and f.a.a. respectively.

2. - The formal integration of (1), (2) yields

$$u_1(x, y) = \sigma_1(\beta(y)) + \int_{\beta(y)}^{x} \sigma_2(\xi) d\xi + \int_{\beta(y)}^{x} \int_{\alpha(\xi)}^{y} f(\xi, \eta, u_1, u_2, u_3) d\xi d\eta$$

$$u_2(x, y) = \sigma_2(x) + \int_{\alpha(x)}^{y} f(x, \eta, u_1, u_2, u_3) d\eta \tag{4}$$

$$u_3(x, y) = \sigma_1'(\beta(y))\beta'(y) - \sigma_2(\beta(y))\beta'(y) + \int_{\beta(y)}^{x} f(\xi, y, u_1, u_2, u_3) d\xi,$$

where the arguments of u_i in the integrals coincide with the first ones of f, and (u_1, u_2, u_3) stands for (u, u_x, u_y).

Let $E = C(I) \times C_y(I) \times C_x(I)$ be normed by $|u| = |u_1|_0 + |u_2|_y + |u_3|_x$. An element $u \in E$ is called a solution of system (4) if it satisfies the first equation on I, the second one on I_y (f.a.a. x) and the third one on I_x (f.a.a. y). On account of HILFSSATZ 5 in [2], u_1 is then a solution of CAUCHY's problem (1), (2).

THEOREM 1. - *Let $\alpha(x)$ be strictly decreasing and absolutely continuous in I_x with $\alpha(0) = b$ and $\alpha(a) = 0$; $\sigma_1(x)$ absolutely continuous in I_x and $\sigma_2(x) \in L(I_x)$; $f(x, y, z)$ defined in $I \times R^3$, measurable in (x, y) and continuous in z with $|f(x, y, z)| \leq M(x, y) \in L(I)$. The equation*

$$v(y) = \sigma_2(x) + \lim_{n\to\infty} \int_{\alpha(x)}^{y} f(x, \eta, \varphi(\eta), v(\eta), \varphi_n(\eta))d\eta$$

have (f.a.a.$x \in I_x$) at most one solution $v(y)$ continuous in I_y, if an arbitrary $\varphi \in C(I_y)$ and an arbitrary sequence $(\varphi_n(y)) \subset L(I_y)$, almost uniformly bounded on I_y, are fixed. The equation

$$w(x) = \sigma_1'(\beta(y))\beta'(y) - \sigma_2(\beta(y))\beta'(y) + \int_{\beta(y)}^{x} f(\xi, y, \varphi(\xi), \psi(\xi), w(\xi))d\xi$$

have (f.a.a. $y \in I_y$) at most one solution $w(x)$, continuous on I_x, if an arbitrary $\varphi \in C(I\ )$ and an arbitrary $\psi \in L(I_x)$ are fixed.

Then, system (4) has a solution $u \in E$.

This theorem is a particular case of Theorem 1 in [1, §2]. The condition

$$|f(x, y, z)| \leq M(x, y) \in L(I) \tag{5}$$

excludes, for example, linear equations (1). In the following two sections, (5) is weakened so that the linear case is included.

3. - We start with two propositions that yield a priori estimates for the solutions of (4).

PROPOSITION 1. - (a) *If $\varphi(x)$, $g(x) \in C(I_x)$ and $M(x) \in L(I_x)$ are non-negative and if*

$$\varphi(x) \leq g(x) + \int_{x}^{a} M(\xi)\varphi(\xi)d\xi, \tag{6}$$

then there is a constant $c_1 > 0$ *such that*

$$\varphi(x) \leq c_1 \cdot \max \{g(\xi) : x \leq \xi \leq a\}.$$

(b) *Let* $\alpha(x)$ *be as in Theorem* 1; $\varphi(x, y), g(x, y) \in C(I)$ *and* $M(x, y) \in L(I)$ *non-negative;*

$$I_0 = \{(\xi, \eta) \in I : \eta \leq \alpha(\xi)\}; \; I_0(x, y) = I_0 \cap \{(\xi, \eta) \in I : \xi \geq x \text{ and } \eta \geq y\}$$

and

$$\varphi(x, y) \leq g(x, y) + \int_{\beta(y)}^{x} \int_{\alpha(\xi)}^{y} M(\xi, \eta)\varphi(\xi, \eta) d\xi d\eta \text{ for } (x, y) \in I_0.$$

Then there is a constant $c_2 > 0$ *such that* $\varphi(x, y) \leq c_2 \cdot \max \{g(\xi, \eta) : (\xi, \eta) \in I_0(x,y)\}$.

Proof. - (a) The function $\psi(x) = \max \left\{\varphi(\xi) \exp\left(-2\int_{\xi}^{a} M(\tau)d\tau\right) : x \leq \xi \leq a\right\}$ is decreasing. By (6), we have

$$\varphi(x) \leq g(x) + \psi(x)\int_{x}^{a} M(\xi) \exp\left(2\int_{\xi}^{a} M(\tau)d\tau\right)d\xi \leq g(x) + \frac{1}{2}\psi(x) \exp\left(2\int_{x}^{a} M(\xi)d\xi\right).$$

Therefore, $\psi(x) \leq 2 \cdot \max \{g(\xi) : x \leq \xi \leq a\}$, hence

$$\varphi(x) \leq 2 \exp\left(2\int_{0}^{a} M(\xi)d\xi\right) \cdot \max \{g(\xi) : x \leq \xi \leq a\}.$$

(b) Can be verified in the same way, using

$$\psi(x, y) = \max \left\{\varphi(\xi, \eta) \exp\left(-2\int_{\beta(\eta)}^{\xi} \int_{\alpha(s)}^{\eta} M_1(s, t) ds dt\right) : (\xi, \eta) \in I_0(x, y)\right\}$$

instead of $\psi(x)$.

PROPOSITION 2. - *Let the conditions of Theorem* 1 *on* $\alpha(x)$, $\sigma_1(x)$, $\sigma_2(x)$ *and* $f(x, y, z)$ *be satisfied; but, instead of* (5), *let us assume that*

$$|f(x, y, z)| \leq M_0(x, y) + M_1(x, y)z_1 + M_2(y)z_2 + M_3(x)z_3 \tag{7}$$

holds with $M_i(x, y) \in L(I)$ for $i = 0, 1$, $M_2(y) \in L(I_y)$, $M_3(x) \in L(I_x)$ and $M_i \geq 0$ for $i = 0, \ldots, 3$.

Then there exist a constant $K_1 > 0$ *and functions* $K_2(x) \in L(I_x)$, $K_3(y) \in L(I_y)$ *such that*

$$\begin{gathered} |u_1(x, y)| \leq K_1\varphi(x, y)\psi(x, y), \ |u_2(x, y)| \leq K_2(x)\varphi(x, y)\psi(x, y) \ (\text{f.a.a. } x \in I_x) \\ |u_3(x, y)| \leq K_3(y)\varphi(x, y)\psi(x, y) \ (\text{f.a.a. } y \in I_y) \end{gathered} \tag{8}$$

holds for every solution u of system (4); *here,*

$$\varphi(x, y) = \exp\left(\left|\int_{\beta(y)}^{x} K_2(\xi)d\xi\right|\right) \text{ and } \psi(x, y) = \exp\left(\left|\int_{\alpha(x)}^{y} K_3(\eta)d\eta\right|\right).$$

Proof. - Let u be a solution of (4). By (7), we have

$$\begin{aligned} |u_1(x, y)| \leq |\sigma_1|_0 + \int_0^a |\sigma_2(\xi)|\, d\xi + \int_I M_0(\xi, \eta)d\xi d\eta + \int_{\beta(y)}^{x} \int_{\alpha(\xi)}^{y} M_1(\xi, \eta)|u_1(\xi, \eta)|\, d\xi d\eta \,+ \\ + \int_{\beta(y)}^{x} \int_{\alpha(\xi)}^{y} \{M_2(\eta)\,|u_2(\xi, \eta)| + M_3(\xi)\,|u_3(\xi, \eta)|\}d\xi d\eta. \end{aligned} \tag{9}$$

Applying the Lemma of GRONWALL for two variables (see [7, 19 III]) in case $x \geq \beta(y)$ and Proposition 1 (b) in case $x < \beta(y)$, we obtain, since in the latter case the last integral in (9) is decreasing in (x, y)

$$|u_1(x, y)| \leq c_1 + d_1 \int_{\beta(y)}^{x} \int_{\alpha(\xi)}^{y} \{M_2(\eta)\,|u_2(\xi, \eta)| + M_3(\xi)\,|u_3(\xi, \eta)|\}d\xi d\eta \tag{10}$$

with positive constants c_1 and d_1.

From the second equation of (4), we have

$$\begin{aligned} |u_2(x, y)| \leq |\sigma_2(x)| + \int_0^b M_0(x, \eta)d\eta + \left|\int_{\alpha(x)}^{y} M_2(\eta)\,|u_2(x, \eta)|\, d\eta\right| \\ + \left|\int_{\alpha(x)}^{y} \{M_1(x, \eta)\,|u_1(x, \eta)| + M_3(x)\,|u_3(x, \eta)|\} d\eta.\right| \end{aligned}$$

Applying GRONWALL's Lemma for one variable (see [7, 1 III]) and Proposition 1 (*b*) respectively, we obtain

$$(11) \qquad |u_2(x, y)| \leq c_2(x) + d_2 \left| \int_{\alpha(x)}^{y} \left\{ M_1(x, \eta) \, |u_1(x, \eta)| + M_3(x) \, |u_2(x, \eta)| \right\} d\eta \right|,$$

with $0 \leq c_2(x) \in L(I_x)$ and $d_2 > 0$. By a similar conclusion

$$(12) \qquad |u_3(x, y)| \leq c_3(y) + d_3 \left| \int_{\beta(y)}^{x} \left\{ M_1(\xi, y) \, |u_1(\xi, y)| + M_2(y) \, |u_2(\xi, y)| \right\} d\xi \right|.$$

Substituting $|u_2(\xi, \eta)|$ and $|u_3(\xi, \eta)|$ in (10) by the bounds (8), an easy calculation yields

$$(13) \qquad |u_1(x, y)| \leq \left\{ c_1 + d_1 \int_0^a M_3(\xi) d\xi + d_1 \int_0^b M_2(\eta) d\eta \right\} \varphi(x, y) \psi(x, y).$$

Thus, from (11) and (12), we obtain the following conditions (14) on $K_2(x)$ and $K_3(y)$

$$(14) \quad \begin{aligned} c_2(x) + \left\{ c_1 d_2 + d_1 d_2 \left(\int_0^b M_2(\eta) d\eta + \int_0^a M_3(\xi) d\xi \right) \right\} \int_0^b M_1(x, \eta) d\eta + d_2 M_3(x) \leq K_2(x) \\ c_3(y) + \left\{ c_1 d_3 + d_1 d_3 \left(\int_0^b M_2(\eta) d\eta + \int_0^a M_3(\xi) d\xi \right) \right\} \int_0^a M_1(\xi, y) d\xi + d_3 M_2(y) \leq K_3(y). \end{aligned}$$

Hence, if we define $K_2(x)$ and $K_3(y)$ by (14) with the sign of equality and if we take the wavy brackett in (13) for K_1, Proposition 2 is proved.

4. - As a consequence of the a priori estimates just established, we have the following extension of Theorem 1.

THEOREM 2. - *Let all conditions of Theorem* 1 *be satisfied, with* (5) *replaced by* (7). *Then there exists an absolutely continuous solution of* CAUCHY's *problem* (1), (2).

Proof. - Let $K_i(x, y) (i = 1, 2, 3)$ be the bounds in (8) which are determined by (13) and (14). Let

$$\varphi(s; c, d) = \begin{cases} c & \text{for } s < c \\ s & \text{" } c \leq s \leq d \\ d & \text{" } s > d \end{cases}$$

and define $z_i(x, y) = \varphi(z_i; -K_i(x, y), K_i(x, y))$ for $i = 1, 2, 3$.

Then, $f^*(x, y, z) = f(x, y, z(x, y))$ satisfies (5). Hence, system (4), with f^* instead of f, has a solution u (Theorem 1) since both uniqueness conditions remain valid (cp. Theorem 1 in [1, §3]). On account of the a priori estimates, u is also a solution of the original system (4).

For linear equations (1), we have the following

COROLLARY. - *Let the conditions on* $\alpha(x)$, $\sigma_1(x)$ *and* $\sigma_2(x)$ *in Theorem* 1 *be satisfied and let*

$$f(x, y, z) = a_0(x, y) + \sum_{i=1}^{3} a_i(x, y) z_i \text{ wilh } a_i(x, y) \in L(I),$$

$$\sup_{I_x} |a_2(x, y)| \in L(I_y) \text{ and } \sup_{I_y} |a_3(x, y)| \in L(I_x).$$

Then problem (1), (2) *has an absolutely continuous solution.*

Remarks. - (a) If the coefficient $a_2(x, y)$ or $a_3(x, y)$ is only integrable then problem (1), (2) need not have an absolutely continuous solution, as the following example shows: $I = [0, 1] \times [0, 1]$, $\alpha(x) = (1-x)^{1/2}$, $\sigma_1(x) = 0$, $\sigma_2(x) = 1$, $f(x, y, z) = (1-x)^{-1/2} z_2$; if a solution exists, then $u_x(x, y)$ is a solution of $\varphi' = (1-x)^{-1/2}\varphi$ and $\varphi((1-x)^{1/2}) = 1$ (f.a.a. $x \in I_x$), hence $u_x(x, y) = \exp(y(1-x)^{-1/2} - 1)$ and therefore $u_{xy} \notin L(I)$.

(b) The method applied in the proof of Proposition 2 can also be used to improve a known existence theorem for DARBOUX's problem (see [2,$\alpha(y) = 0$]), assuming (7) instead of (5).

(c) By means of the estimates (10)-(12), one can verify immediately that (4) has exactly one solution, if $f(x, y, z)$ satisfies a LIPSCHITZ condition

$$|f(x, y, z) - f(x, y, \bar{z}| \leq M_1(x, y)|z_1 - \bar{z}_1| + M_2(y)|z_2 - \bar{z}_2| + M_3(x)|z_3 - \bar{z}_3|$$

with the functions M_i from (7): If w is the difference of two solutions of (4) then we have (10)-(12) with $c_i = 0$; for $\varepsilon > 0$, we choose bounds of the form

$$|w_2(x, y)| \leq \varepsilon K_2(x)\varphi(x, y)\psi(x, y), \quad |w_3(x, y)| \leq \varepsilon K_3(y)\varphi(x, y)\psi(x, y)$$

(cp. (8)) and we obtain (14) with $c_i = 0$ for $i = 1, 2, 3$; hence, there is a constant $c > 0$ with

$$\max(|w_1|_0, |w_2|_y, |w_3|_x) \leq c\cdot\varepsilon, \text{ i.e. } w = 0 \in E.$$

5. - Under the conditions of Theorem 2 there exists a solution $u(x, y)$ with $u_{xy} \in L(I)$. Now, we give some conditions sufficient for u to have integrable derivatives u_{xx} and u_{yy}. To this end we prove at first a theorem on dependence on parameters for problem (3).

THEOREM 3. - *Let $\gamma(y)$ be absolutely continuous and monotone on I_y with values in I_x; $\sigma(y)$ absolutely continuous on I_y; $g(x, y, z)$ defined on $I \times R^1$, measurable in x and absolutely continuous in y with sup $\{|g_y(x, y, z) : |z| \leq c\} \in L(I)$ for every $c > 0$, and $g(x, 0, 0) \in L(I_x)$. Moreover, g satisfy the* LIPSCHITZ *condition*

$$|g(x, y, z) - g(x, y, \bar{z})| \leq k(x)|z - \bar{z}| \text{ with } k(x) \in L(I_x). \tag{15}$$

Then, for every $y \in I_y$, the initial value problem

$$u' = g(x, y, u) \quad u(\gamma(y)) = \sigma(y) \tag{16}$$

has exactly one solution $u(x, y)$, absolutely continuous in x. Moreover, $u(x, y) \in C(I)$ and $u(x, y)$ is absolutely continuous in y for every $x \in I_x$. If, in addition, $g_y(x, y, z)$ is continuous in z then $u(x, y)$ is absolutely continuous on I.

Proof. - Let us assume that $\gamma(y)$ is decreasing. At first, we have

$$|g(x, y, z)| \leq |g(x, 0, 0)| + \int_0^b |g_y(x, \eta, 0)| d\eta + k(x)|z|,$$

i.e. $|g(x, y, z)| \leq M(x)(1 + |z|)$ with $M(x) \in L(I_x)$. Hence, the first two assertions of the theorem are valid (cp. the proof of Theorem 4, next sec.). Since

$$u(x, y) = \sigma(y) + \int_{\gamma(y)}^x g(\xi, y, u(\xi, y))d\xi \text{ for } (x, y) \in I,$$

it is obvious that u is absolutely continuous in x. Now, let $x \in I_x$ be fixed, $y_0 < y_1 < ... < y_n$ points of I_y, $\gamma_i = \gamma(y_i)$ and $\sigma_i = \sigma(y_i)$. Then, we have

$$|u(x, y_i) - u(x, y_{i-1})| \leq |\sigma_i - \sigma_{i-1}| + \int_{\gamma_i}^{\gamma_{i-1}} M(\xi)(1 + |u|_0)d\xi +$$

$$+ \int_{y_{i-1}}^{y_i} \int_0^a \sup_{|z| \leq |u|_0} |g_y(\xi, \eta, z)| d\xi d\eta + \left| \int_{\gamma_{i-1}}^x k(\xi)|u(\xi, y_i) - u(\xi, y_{i-1})| d\xi \right|.$$

Therefore, GRONWALL's Lemma and Proposition 1 (a) yield

$$|u(x, y_i) - u(x, y_{i-1})| \leq c|\sigma_i - \sigma_{i-1}| + c\left(1 + |u|_0\right) \int_{\gamma_i}^{\gamma_{i-1}} M(\xi)d(\xi) +$$

$$+ c\int_{y_{i-1}}^{y_i} \int_0^a \sup_{|z| \leq |u|_0} |g_y(\xi, \eta, z)| d\xi d\eta,$$

for some constant $c > 0$. Since the functions on the right hand side are absolutely continuous in y, so is $u(x, y)$. Especially, we have

$$(17)\quad |u_y(x, y)| \leq c\,|\sigma'(y)| + c(1 + |u|_0)M(\gamma(y))\gamma'(y) + c\int_0^a \sup_{|z|\leq|u|_0} |g_y(\xi, y, z)|\, d\xi \in L(I_y).$$

To prove the last assertion, let us note that $g(x, y, \varphi(t))$ is absolutely continuous in t for any absolutely continuous function $\varphi(t)$, by (15) (see [4, sec. 38.1]). Hence, if g_y is continuous in z, we obtain

$$u_{xy} = g_y(x, y, u(x, y)) + g_z(x, y, u(x,y))u_y(x, y) \text{ a.e. on } I$$

from (16). Thus, $u_{xy} \in L(I)$ follows from (15), (17) and the assumption on g_y.

6. - We use Theorem 3 to prove the following

THEOREM 4. - *Let $\sigma_1'(x)$ and $\sigma_2(x)$ be absolutely continuous on I_x; $\alpha(x)$ strictly decreasing, $\alpha'(x)$ absolutely continuous, $\alpha(0) = b$ and $\alpha(a) = 0$; $f(x, y, z)$ defined on $I \times R^3$, absolutely continuous in x and y (separately) with*

$$\sup\{|f_x(x, y, z)| : |z_i| \leq c,\ i = 1, 2, 3\} \in L(I) \text{ and}$$

$$\sup\{|f_y(x, y, z)| : |z_i| \leq c\} \in L(I) \text{ for every } c > 0;$$

$$|f(x, y, z) - f(x, y, \bar{z})| \leq k(x, y) \cdot \sum_{i=1}^{3} |z_i - \bar{z}_i| \text{ with } \sup_{I_x} k(x, y) \in L(I_y)$$

and $\sup\limits_{I_y} k(x, y) \in L(I_x)$.

Then CAUCHY'S *Problem* (1), (2) *has exactly one solution which is absolutely continuous in I and such that u_{xx} and u_{yy} exist a.e. on I and are in $L(I)$.*

Proof. - We have

$$(18)\quad |f(x, y, z)| \leq |f(0, 0, 0)| + \int_0^a |f_x(\xi, 0, 0)|\, d\xi + \int_0^b |f_y(x, \eta, 0)|\, d\eta + k(x, y) \sum_1^3 |z_i|,$$

i.e., f satisfies an inequality of the form (7). Since the LIPSCHITZ condition of the theorem is a particular case of the uniqueness conditions in Theorem 1, every condition of Theorem 2 is satisfied. Hence, problem (1), (2) has exactly one absolutely continuous solution $u(x, y)$ (cp. remark (c)).

Moreover, we have f.a.a. $y \in I_y$

$$u_y(x, y) = \sigma_3(y) + \int_{\beta(y)}^{x} f(\xi, y),\ u(\xi, y),\ u_x(\xi, y)\ u_y(\xi, y))\, d\xi,$$

where $\sigma_3(y) = \sigma_1'(\beta(y))\beta'(y) - \sigma_2(\beta(y))\beta'(y)$. Since $f^*(\xi, y, z_3) = f(\xi, y, u(\xi, y), u_x(\xi, y), z_3)$ is continuous in y and measurable in ξ and satisfies a LIPSCHITZ condition in z_3 (with $\sup_{I_y} k(\xi, y)$), the equation

$$w(x, y) = \sigma_3(y) + \int_{\beta(y)}^{x} f^*(\xi, y, w(\xi, y))d\xi$$

has exactly one solution $w \in C(I)$ which, in addition, is absolutely continuous in x(for every y) and coincides with $u_y(x, y)$ a. e. in I. Let us add that the continuity of w follows easily from BANACH's fixed point theorem: Let y be fixed, consider $v \in C(I_x)$ with norm $|v|_2 = \max_{I_x} \left\{ |v(x)| \exp\left(-2\left|\int_{\beta(y)}^{x} k_1(\tau)d\tau\right|\right)\right\}$, where $k_1(\tau) = \sup_{I_y} k(\tau, y)$, and the operator $T: C(I_x) \to C(I_x)$, defined by $(Tv)(x) = \sigma_3(y) + \int_{\beta(y)}^{x} f^*(\xi, y, v(\xi))d\xi$; an easy calculation yields $|Tv - T\bar{v}|_2 \leq \frac{1}{2}|v - \bar{v}|_2$; hence, starting successive approximation with $v^0(x, y) = \sigma_3(y)$, all approximations $v^n(x, y)$ are in $C(I)$; this implies continuity of the limit function $w(x, y)$, since

$$|v^{n+1}(x, y) - w(x, y)| \leq 2^{1-n}|v^1 - v^0|_0 \cdot \exp\left(2\int_0^a k_1(\tau)d\tau\right).$$

Thus, the function $u_y(x, y)$ can be redefined on a set of measure zero such that it becomes continuous on I. The same conclusion holds for u_x. Now, $g(x, y, z_3) = f(x, y, u(x, y), u_x(x, y), z_3)$ satisfies the conditions of Theorem 3, which implies the existence of u_{yy} a.e. in I and $u_{yy} \in L(I)$. Existence and integrability of u_{xx} are shown correspondingly.

For the linear equation

$$u_{xy} = a_0(x, y) + a_1(x, y)u + a_2(x, y)u_x + a_3(x, y)u_y$$

our condition on f in Theorem 4 means that the coefficients $a_i(x, y)$ are absolutely continuous in x and y (separately), with $\frac{\partial}{\partial x} a_i(x,y) \in L(I)$ and $\frac{\partial}{\partial y} a_i(x, y) \in L(I)$. This implies neither continuity nor boundedness of $a_i(x, y)$, as the example

$$a(x, y) = \begin{cases} (x^5 y^{-3})^{1/2} \exp(-xy^{-1/2}) & \text{for } 0 < y \leq 1 \\ 0 & \text{for } y = 0 \end{cases} \quad \text{and } 0 \leq x \leq 1$$

shows.

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