Linear Integro-Differential-Boundary-Parameter Problems (*).

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Summary. – *Necessary and sufficient conditions for a linear vector differential system, involving integral, boundary, and vector parameter terms, to be symmetric (self-adjoint) are developed and applied to obtain canonical forms for symmetric problems. In addition, the concept of the equivalence of two such linear problems under nonsingular transformations is examined, and a relationship between equivalence of a problem with its adjoint and symmetry is obtained.*

0. – Introduction.

The recent results of VEJVODA and TVRDÝ [8] and the author [9] will be extended to obtain the most general form of symmetric (self-adjoint) problems of vector form

$$A_1(x)y' + [A_0(x) - \lambda B(x)]y + H(x)[M_2 y(a) + N_2 y(b)] + K(x)\int_a^b F(\xi) y\, d\xi + L(x)p = 0\,, \qquad p' = 0\,, \tag{0.1a}$$

$$My(a) + Ny(b) + \int_a^b F(\xi) y\, d\xi = 0\,. \tag{0.1b}$$

In considering the concept of self-adjointness for problems with general integro-boundary conditions (0.1b), the introduction of a term involving a vector parameter p appears as a natural consequence of the form of the adjoint problem as previously obtained by COLE [1], JONES [2] and VEJVODA and TVRDÝ [8]. Moreover, a reformulation of the integro-boundary conditions yields a simplification of one of the conditions for self-adjointness deduced by KRALL [4] for problems (0.1a, b) with $K(x) \equiv 0$.

The notation and hypotheses under which the problems are considered will be noted in Section 1. In Section 2 the problem adjoint to (0.1a, b) will be developed, and necessary and sufficient conditions for symmetry (self-adjointness) will be obtained in Section 3, together with canonical forms of symmetric problems. In Section 4 the equivalence of two integro-differential-boundary-parameter problems under

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(*) Entrata in Redazione il 5 dicembre 1973.

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nonsingular transformations will be discussed, and, in particular, the equivalence of a problem (0.1a, b) with its adjoint. In this connection, the conclusion of Reid [5, Problem III.11, no. 5] for two-point boundary problems (*i.e.*, problems (0.1a, b) with $H(x)$, $K(x)$, $L(x)$ and $F(x)$ as zero matrices) equivalent to their adjoint will be extended to remove the restriction that the matrix coefficient of the derivative term be the identity matrix. A generalization of this result then yields a relationship between the concept of equivalence of a problem (0.1a, b) with its adjoint under sets containing a nonsingular skew-Hermitian transformation and that of symmetry.

1. – Notations and hypotheses.

Vector and matrix notation will be employed throughout the paper. Matrices will be denoted by Roman and Greek capital letters, vectors by lower-case Roman letters, while lower-case Greek letters will usually be employed as scalars. Vector operators will be indicated by capitals or lower-case letters in script type. The $\varrho \times \varrho$ identity matrix will be represented by I_ϱ, and, as is customary, 0 will be used indiscriminately to denote either the number zero, a zero vector or a zero matrix. Further, the operations of conjugate-transpose and differentiation, applied to both vectors and matrices, will be indicated by * and ′, respectively. Finally, when row and column dimensions agree, $[M; N]$ will denote the matrix $[M^* \; N^*]^*$, $[M, N; P, Q]$ will represent the matrix

$$\begin{bmatrix} M & N \\ P & Q \end{bmatrix},$$

and, more generally, $[M, N; P, Q; R, S]$ will indicate the matrix wherein block partitioning produces successive row block-matrices $[M \; N]$, $[P \; Q]$ and $[R \; S]$ in that order.

For problem (0.1a, b) it will be assumed that the elements of the $n \times n$ matrix $A_1(x)$ are complex-valued functions continuously differentiable on the finite interval $a \leqslant x \leqslant b$, the elements of the $n \times n$ matrices $A_0(x)$ and $B(x)$, the $n \times \nu$ matrix $H(x)$, the $n \times m$ matrix $K(x)$, the $n \times \varkappa$ matrix $L(x)$, and the $m \times n$ matrix $F(x)$ are all complex-valued functions continuous on $[a, b]$, and, further, that the $\varkappa$ columns of $L(x)$ are linearly independent on $[a, b]$. Moreover, M_2 and N_2 are each $\nu \times n$ constant matrices, and M and N are each $m \times n$ constant matrices with $0 \leqslant m \leqslant 2n$ and such that the m integro-boundary forms (0.1b) are linearly independent forms. The vectors y and p are, respectively, n- and $\varkappa$-dimensional vectors, and λ is a scalar constant. In addition, it is to be noted that a necessary and sufficient condition for the linear independence of the m boundary forms (0.1b) is the linear independence on $[a, b]$ of the rows of the $m \times 3n$ matrix $[M \; N \; F(x)]$ (see JONES [2, Theorem 2.1]).

Now, let D be an $m \times m$ nonsingular constant matrix such that the m integro-boundary conditions (0.1b), on multiplication on the left by D, reduce to the equivalent form

$$(1.1b)\qquad \begin{aligned} \vartheta_0[y] &\equiv M_0 y(a) + N_0 y(b) &&= 0\,,\\ \vartheta_1[y] &\equiv M_1 y(a) + N_1 y(b) + \int_a^b F_1(\xi) y\, d\xi &&= 0\,,\\ \vartheta_2[y] &\equiv \int_a^b F_2(\xi) y\, d\xi &&= 0\,, \end{aligned}$$

of ϱ, σ and τ conditions, respectively, $\varrho + \sigma + \tau = m$, with the $(\varrho + \sigma) \times 2n$ matrix $[M_0, N_0;\ M_1, N_1]$ of rank $\varrho + \sigma$, the $\sigma + \tau$ rows of $[F_1(\xi);\ F_2(\xi)]$ linearly independent on $[a, b]$, and, additionally, such that $\int_a^b F_1(\xi) F_2^*(\xi)\, d\xi = 0$. (This latter orthogonality condition can be assured by effecting the replacement of $F_1(x)$ by $F_1(x) - \left(\int_a^b F_1 F_2^*\, d\xi\right)\left(\int_a^b F_2 F_2^*\, d\xi\right)^{-1} F_2(x)$ where the rows of $[F_1(x);\ F_2(x)]$ are, a priori, linearly independent on $[a, b]$). Moreover, without loss of generality, we may consider that the $\nu \times 2n$ matrix $[M_2\ N_2]$ has rank $\nu = 2n - \varrho - \sigma$ with the $2n \times 2n$ matrix $[M_0, N_0;\ M_1, N_1;\ M_2, N_2]$ nonsingular (see, for example, [8, Remark 6.2]), and that $(0.1a)$ is reduced to the form

$$(1.1a)\qquad \begin{aligned} \mathfrak{L}[y;\, p\colon \lambda] \equiv A_1(x) y' + [A_0(x) - \lambda B(x)] y + H(x)[M_2 y(a) + N_2 y(b)]\\ + K_1(x) \int_a^b F_1(\xi) y\, d\xi + L(x) p = 0\,, \qquad p' = 0\,, \end{aligned}$$

where $K_1(x)$ is the $n \times \sigma$ matrix consisting of the $\varrho + 1, \ldots, \varrho + \sigma$ columns of $K(x) D^{-1}$.

2. – The adjoint problem.

With the introduction of new vectors

$$\begin{aligned} u_1 &\equiv M_1 y(a) + \int_a^x F_1(\xi) y\, d\xi\,,\\ u_2 &\equiv \int_a^x F_2(\xi) y\, d\xi\,,\\ s_1 &\equiv M_1 y(a) + N_1 y(b)\,,\\ s_2 &\equiv M_2 y(a) + N_2 y(b)\,, \end{aligned}$$

problem (1.1a, b) is equivalent to the differential system consisting of $(3n+\sigma+\tau+\varkappa-\varrho)$ linear differential equations and $(2n+2\sigma+2\tau)$ end-point conditions:

$$
\begin{aligned}
&A_1(x)y'+A_0(x)y-K_1(x)s_1+H(x)s_2+L(x)p=\lambda B(x)y\,,\\
&u_1'-F_1(x)y=0\,,\\
&u_2'-F_2(x)y=0\,,\\
&s_1'=0\,,\\
&s_2'=0\,,\\
&p'=0\,,\\
&M_0y(a)+N_0y(b)=0\,,\\
&M_1y(a)-u_1(a)=0\,,\\
&N_1y(b)+u_1(b)=0\,,\\
&u_2(a)=0\,,\\
&u_2(b)=0\,,\\
&M_1y(a)-s_1(a)+N_1y(b)=0\,,\\
&M_2y(a)-s_2(a)+N_2y(b)=0\,.
\end{aligned}
\tag{2.1}
$$

Now, if constant matrices P_α, Q_α $(\alpha=0,1,2)$, of dimensions $n\times\varrho, n\times\sigma$ and $n\times\nu$, respectively, are defined by

$$
[M_0, N_0;\ M_1, N_1;\ M_2, N_2]\cdot[-P_0, -P_1, -P_2;\ Q_0, Q_1, Q_2]=I_{2n}\,,
\tag{2.2}
$$

the differential system adjoint to (2.1) is comprised of $(3n+\sigma+\tau+\varkappa-\varrho)$ linear differential equations together with $(4n+2\varkappa-2\varrho)$ end-point conditions (see, for example, [5, § 3.6]):

$$
\begin{aligned}
&-[A_1^*(x)z]'+A_0^*(x)z-F_1^*(x)v_1-F_2^*(x)q=\lambda B^*(x)z\,,\\
&v_1'=0\,,\\
&q'=0\,,\\
&-t_1'-K_1^*(x)z=0\,,\\
&-t_2'+H^*(x)z=0\,,\\
&-v_2'+L^*(x)z=0\,,\\
&P_2^*A_1^*(a)z(a)+Q_2^*A_1^*(b)z(b)+P_2^*M_1^*v_1(a)-Q_2^*N_1^*v_1(b)-t_2(a)=0\,,\\
&P_1^*A_1^*(a)z(a)+Q_1^*A_1^*(b)z(b)+P_1^*M_1^*v_1(a)-Q_1^*N_1^*v_1(b)-t_1(a)=0\,,\\
&t_1(b)=0\,,\\
&t_2(b)=0\,,\\
&v_2(a)=0\,,\\
&v_2(b)=0\,.
\end{aligned}
\tag{2.3}
$$

On eliminating v_1, v_2, t_1 and t_2 it follows from relation (2.2) that the system (2.3) is equivalent to the integro-differential-boundary-parameter problem:

$$(2.4)\quad \begin{cases} (a) & \begin{aligned} \mathcal{M}[z;\, q\colon \lambda] \equiv -[A_1^*(x)z]' + [A_0^*(x) - \lambda B^*(x)]z - \\ - F_1^*(x)[P_1^* A_1^*(a)z(a) + Q_1^* A_1^*(b)z(b)] \\ + F_1^*(x)\int_a^b K_1^*(\xi)z\, d\xi - F_2^*(x)q = 0\,, \qquad q' = 0\,, \end{aligned} \\ (b) & \begin{aligned} P_2^* A_1^*(a)z(a) + Q_2^* A_1^*(b)z(b) + \int_a^b H^*(\xi)z\, d\xi = 0\,, \\ \int_a^b L^*(\xi)z\, d\xi = 0\,, \end{aligned} \end{cases}$$

which will be termed the *adjoint* to problem (1.1a, b). Further, the adjoint problem (2.4a, b) remains invariant under equivalent reformulations of problem (1.1a, b) which adds to (1.1a) a term $\sum_{\alpha=0}^{2} J_\alpha(x)\vartheta_\alpha[y]$, $J_\alpha(x)$ $(\alpha = 0, 1, 2)$ appropriately dimensioned matrices with elements continuous functions on $[a, b]$. This invariance property is employed in the next section in obtaining canonical forms.

3. – Symmetric problems.

An integro-differential-boundary-parameter problem (1.1a, b) will be termed *symmetric* if the integro-boundary forms (1.1b) and (2.4b) are equivalent forms, and if, for each value of λ, the operators $\mathcal{L}$ and $\mathcal{M}$ coincide in the sense that $\varkappa = \tau$ and there exists a $\tau \times \tau$ nonsingular constant matrix Υ such that $\mathcal{L}[y;\, p\colon \lambda] \equiv \mathcal{M}[y;\, \Upsilon p\colon \lambda]$ for arbitrary vector pairs $\{y, p\}$, p constant and y of class C' on $[a, b]$ and satisfying (1.1b). Inasmuch as the replacement of $F_2(x)$ in (1.1b) by $\Upsilon^* F_2(x)$, Υ a $\tau \times \tau$ nonsingular constant matrix, yields an equivalent set of integral-boundary forms, symmetry of a problem (1.1a, b) is synonymous with equivalence of (1.1b) and (2.4b) and the existence of a $\tau \times \tau$ nonsingular constant matrix Υ such that, after replacement of $F_2(x)$ be $\Upsilon^* F_2(x)$, the operators $\mathcal{L}$ and $\mathcal{M}$ coincide for sets $\{y, p, \lambda\}$ with y of class C' on $[a, b]$ and satisfying (1.1b), p a constant vector and λ a scalar. This terminology, employed for differential expressions in [5, p. 122], is an extension of that introduced for the integro-differential-boundary problems in [9].

The initial result of this section provides necessary and sufficient conditions for the equivalence of the integro-boundary forms of a problem and its adjoint.

LEMMA 3.1. – *For a problem* (1.1a, b) *the integro-boundary forms* (1.1b) *and* (2.4b) *are equivalent if and only if there exist a* $\tau\times\tau$ *nonsingular constant matrix* Υ *and a* $\tau\times n$ *constant matrix* G *such that*

$$(3.1)\qquad \begin{aligned} &(a)\ \varrho+\sigma=\nu=n \ \textit{and}\ \tau=\varkappa\,,\\ &(b)\ [M_0, N_0;\ M_1, N_1]\cdot\operatorname{diag}\{-A_1^{*-1}(a), A_1^{*-1}(b)\}\cdot[M_0, N_0;\ M_1, N_1]^*=0\,,\\ &\left.\begin{aligned}&(c)\ L(x)\equiv -F_2^*(x)\Upsilon\\ &(d)\ H(x)\equiv F_1^*(x)E_1+F_2^*(x)G\end{aligned}\right\}\ \textit{on}\ [a,b]\,,\end{aligned}$$

where $E_1\equiv[0\ I_\sigma]C^{-1}$ *and* $C\equiv[M_2\ N_2]\cdot\operatorname{diag}\{-A_1^{-1}(a), A_1^{-1}(b)\}\cdot[M_0, N_0;\ M_1, N_1]^*$.

As the $\nu\times 2n$ matrix $[P_2^*A_1^*(a)\ Q_2^*A_1^*(b)]$ has rank ν, equivalence of (1.1b) and (2.4b) necessitates that $\varrho+\sigma=\nu$ and that $\tau=\varkappa$. Then, as $\varrho+\sigma+\nu=2n$, condition (3.1a) follows; and, thus, a necessary and sufficient condition for equivalence of (1.1b) and (2.4b) is the existence of an $(n+\tau)\times(n+\tau)$ nonsingular constant matrix $[E, 0;\ G, -\Upsilon]$, with nonsingular constant matrices E and Υ of dimensions $n\times n$ and $\tau\times\tau$ respectively, and G of dimension $\tau\times n$, such that

$$(3.2)\qquad [P_2^*A_1^*(a)\ Q_2^*A_1^*(b)]=E^*[M_0, N_0;\ M_1, N_1]\,,$$

$$\left.\begin{aligned}&(3.3)\qquad L^*(x)\equiv-\Upsilon^*F_2(x)\,,\\ &(3.4)\qquad H^*(x)\equiv E^*[0;\ F_1(x)]+G^*F_2(x)\end{aligned}\right\}\ \text{on}\ [a,b]\,.$$

Then, (3.1b) follows from (3.2) in view of (2.2); and, hence, C, defined above in the lemma, is nonsingular as the n rows of $[M_2\ N_2]$ are linearly independent of the n rows of $[M_0, N_0;\ M_1, N_1]$. (An alternate proof can be provided by an argument similar to that employed in [3, p. 445]). On the other hand, with $E\equiv C^{-1}$, (3.2) follows from (3.1b) and relation (2.2). Moreover, relation (3.1d) is equivalent to (3.4) under the definition $E_1\equiv[0\ I_\sigma]E$, $E=C^{-1}$.

THEOREM 3.1. – *A problem* (1.1a, b) *is symmetric if and only if there exist a* $\tau\times\tau$ *nonsingular constant matrix* Υ *and a* $\sigma\times\sigma$ *constant Hermitian matrix* Γ *such that*

$$(3.5)\qquad \begin{aligned} &(a)\ \varrho+\sigma=\nu=n\ \textit{and}\ \tau=\varkappa\,,\\ &(b)\ [M_0, N_0;\ M_1, N_1]\cdot\operatorname{diag}\{A_1^{-1}(a), -A_1^{-1}(b)\}\cdot[M_0, N_0;\ M_1, N_1]^*=0\,,\\ &\left.\begin{aligned}&(c)\ A_1^*(x)\equiv-A_1(x),\ A_0^*(x)\equiv A_0(x)-A_1'(x),\ B^*(x)\equiv B(x),\\ &(d)\ L(x)\ \equiv-F_2^*(x)\Upsilon\,,\\ &(e)\ H(x)\ \equiv F_1^*(x)E_1\,,\\ &(f)\ K_1(x)\equiv F_1^*(x)[\Gamma+(\tfrac{1}{2})\Theta]\end{aligned}\right\}\ \textit{on}\ [a,b]\,,\end{aligned}$$

where E_1 *is defined in Lemma* 3.1 *and* Θ *is a skew-Hermitian matrix given by*

$$\Theta \equiv E_1[M_2\ N_2]\cdot \operatorname{diag}\{A_1^{-1}(a), -A_1^{-1}(b)\}\cdot[M_2^*;\ N_2^*]E_1^* . \tag{3.6}$$

For a problem (1.1a, b) with integro-boundary forms (1.1b) and (2.4b) equivalent, relations (3.5a, b, d) and (3.1d) are immediate from Lemma 3.1, while relations (3.5c) follow from the equivalence of the operators $\mathfrak{L}$ and $\mathcal{M}$ for sets $\{y, p, \lambda\}$ with $p = 0$, λ arbitrary, and y of class C' on $[a, b]$ satisfying $y(a) = y(b) = \int_a^b K_1^*(\xi) y\, d\xi = \int_a^b F_\alpha(\xi) y\, d\xi = 0$, $\alpha = 1, 2$. Then, under (3.5a, b, c, d) and (3.1d), a necessary and sufficient condition for (1.1a, b) to be symmetric is that

$$\begin{aligned} F_1^*(x)\Big[\big(E_1M_2 + P_1^*A_1^*(a)\big)y(a) + \big(E_1N_2 + Q_1^*A_1^*(b)\big)y(b) - \int_a^b K_1^*(\xi)y\, d\xi\Big] \\ + K_1(x)\int_a^b F_1(\xi)y\, d\xi + F_2^*(x)G[M_2y(a) + N_2y(b)] \equiv 0 \end{aligned} \tag{3.7}$$

on $[a, b]$ for arbitrary vectors y of class C' satisfying (1.1b). In particular, for vectors y of class C' with $y(a) = y(b) = 0$ condition (3.7) on $[a, b]$ implies that $F_1^*(x)\int_a^b K_1^*(\xi)y\, d\xi \equiv 0$ on $[a, b]$, and, consequently, $\int_a^b K_1^*(\xi)y\, d\xi = 0$ whenever also $\int_a^b F_\alpha(\xi)y\, d\xi = 0$, $\alpha = 1, 2$. Now, as $\int_a^b F_1(\xi)F_2^*(\xi)\, d\xi = 0$, an argument similar to that preceding Theorem 2.1 of [9] assures that $K_1(x) \equiv F_1^*(x)\Phi_1 + F_2^*(x)\Phi_2$ on $[a, b]$, where the $\sigma\times\sigma$ and $\tau\times\sigma$ constant matrices Φ_α, $\alpha = 1, 2$, respectively, are given by $\Phi_\alpha \equiv \Big[\int_a^b F_\alpha(\xi)F_\alpha^*(\xi)\, d\xi\Big]^{-1}\int_a^b F_\alpha(\xi)K_1(\xi)\, d\xi$, $\alpha = 1, 2$. Then, in view of the linear independence of the columns of $F_1^*(x)$ and $F_2^*(x)$ on $[a, b]$, condition (3.7) reduces to the requirements that

$$\big(E_1M_2 + P_1^*A_1^*(a) - \Theta M_1\big)y(a) + \big(E_1N_2 + Q_1^*A_1^*(b) - \Theta N_1\big)y(a) = 0\ , \tag{3.8}$$

$$(GM_2 - \Phi_2M_1)y(a) + (GN_2 - \Phi_2N_1)y(b) = 0\ , \tag{3.9}$$

where $\Theta \equiv \Phi_1 - \Phi_1^*$, for arbitrary vectors $y(a), y(b)$ satisfying $\vartheta_0[y] \equiv M_0y(a) + N_0y(b) = 0$. However, as the $n+\sigma$ columns of $[-P_1, -P_2; Q_1, Q_2]$ form a maximal set of linearly independent vectors orthogonal to the ϱ rows of $[M_0\ N_0]$, relations (3.8), (3.9) hold for end-values $y(a), y(b)$ satisfying $\vartheta_0[y] = 0$ if and only if

$$[GM_2 - \Phi_2M_1\ \ GN_2 - \Phi_2N_1]\cdot[-P_1, -P_2; Q_1, Q_2] = 0;$$

and, hence, $G = 0$ and $\Phi_2 = 0$, and, as in the derivation of Theorem 2.1 of [9],

$$\Theta = -P_1^* A_1^*(a) P_1 + Q_1^* A_1^*(b) Q_1 ,$$
$$E_1 = P_1^* A_1^*(a) P_2 - Q_1^* A_1^*(b) Q_2 .$$

The latter relation, however, is an identity in view of the skew-Hermitian character of $A_1(x)$ and relation (3.2). Setting $\Gamma \equiv (\frac{1}{2})(\Phi_1 + \Phi_1^*)$, condition (3.5*f*) holds, and the theorem then follows by a argument similar to that used in establishing Theorem 2.1 of [9].

COROLLARY 1. – *A problem* (1.1*a*, *b*) *with* $K_1(x) \equiv 0$ *on* $[a, b]$ *is symmetric if and only if there exists a* $\tau \times \tau$ *nonsingular constant matrix* Υ *such that relations* (3.5*a*, *b*, *c*, *d*, *e*) *prevail and* $\theta = 0$, θ *given by* (3.6) *wherein* E_1 *is defined in Lemma* 3.1.

The above result, which follows from the linear independence of the columns of $F_1^*(x)$ on $[a, b]$, provides a simplification of condition (5) of Theorem 3.1 of Krall [4].

COROLLARY 2. – *The differential-parameter problem*

$$\begin{aligned} A_1(x)y' + [A_0(x) - \lambda B(x)]y + L(x)p &= 0 , \qquad p' = 0 , \\ M_0 y(a) + N_0 y(b) &= 0 , \\ \int_a^b F_2(\xi) y \, d\xi &= 0 , \end{aligned} \tag{3.10}$$

is symmetric if and only if there exists a $\tau \times \tau$ *nonsingular constant matrix* Υ *such that*

(*a*) $\varrho = n$ *and* $\tau = \varkappa$,

(*b*) $M_0 A_1^{*-1}(a) M_0^* = N_0 A_1^{*-1}(b) N_0^*$,

$$\left.\begin{aligned} &(c) \quad A_1^*(x) \equiv -A_1(x),\ A_0^*(x) \equiv A_0(x) - A_1'(x),\ B^*(x) \equiv B(x) , \\ &(d) \quad L(x) \equiv -F_2^*(x)\Upsilon \end{aligned}\right\} \textit{on } [a, b] .$$

COROLLARY 3. – *An* $(n+\tau)$*-dimensional symmetric differential-parameter problem* (3.10) *is equivalent to the* $(n+2\tau)$*-dimensional self-adjoint (symmetric) two-point boundary problem*

$$\begin{aligned} A_1(x)y' + A_0(x)y + L(x)p &= \lambda B(x) y , \\ \Upsilon^* u_2' - \Upsilon^* F_2(x) y &= 0 , \\ -\Upsilon p' &= 0 , \\ M_0 y(a) + N_0 y(b) &= 0 , \\ u_2(a) &= 0 , \\ u_2(b) &= 0 , \end{aligned}$$

under the introduction of $u_2(x) \equiv \int_a^x F_2(\xi) y \, d\xi$ *on* $[a, b]$, *with* Υ *as in Corollary* 2 *above.*

Further, the development of canonical forms for the symmetric problems considered in [9] extends to symmetric problems (1.1a, b), and, in particular, the following extension of the Corollary to Theorem 3.1 of [9] obtains.

THEOREM 3.2. – *Every $(n+\tau)$-symmetric integro-differential-boundary-parameter problem* (1.1a, b) *is reducible to the form*

$$iR_1(x)[R_1(x)y]' + [R_0(x) - \lambda B(x)]y + iF_1^*(x)[-M_1R_1^2(a)y(a) + N_1R_1^2(b)y(b)] + F_1^*(x)[\Psi + i\Lambda]\int_a^b F_1(\xi)y\,d\xi - F_2^*(x)p = 0\,, \qquad p' = 0\,,$$

$$M_0y(a) + N_0y(b) = 0\,,$$

$$M_1y(a) + N_1y(b) + \int_a^b F_1(\xi)y\,d\xi = 0\,,$$

$$\int_a^b F_2(\xi)y\,d\xi = 0\,,$$

where on $[a, b]$ $R_0(x)$, $R_1^2(x)$ and $B(x)$ are each $n\times n$ Hermitian matrices, $R_1(x)$ is nonsingular with $R_1^2(x) \equiv -iA_1(x)$, $R_0(x) \equiv A_0(x) - (\frac{1}{2})A_1'(x)$, the $\sigma+\tau$ rows of $[F_1(x); F_2(x)]$ are linearly independent, Ψ is a $\sigma\times\sigma$ constant Hermitian matrix, the $\sigma\times\sigma$ Hermitian matrix $\Lambda \equiv (\frac{1}{2})(-M_1R_1^2(a)M_1^* + N_1R_1^2(b)N_1^*)$,

$$[M_0, N_0;\ M_1, N_1]\cdot\operatorname{diag}\{R_1^{-2}(a), -R_1^{-2}(b)\}\cdot[M_0, N_0;\ M_1, N_1]^* = 0\,,$$

and the rows of the $n\times 2n$ matrix $[M_0, N_0;\ M_1, N_1]$ are orthonormed, in the sense that $M_0M_0^* + N_0N_0^* = I_\varrho$, $M_1M_1^* + N_1N_1^* = I_\sigma$ and $M_0M_1^* + N_0N_1^* = 0$.

For a symmetric problem (1.1a, b) the matrix $-iA_1(x)$ is Hermitian on $[a, b]$ and, hence, there exist $n\times n$ positive matrices $C(x)$ and $D(x)$, with continuously differentiable elements, such that $-iA_1(x) \equiv C(x) - D(x)$ and $C(x)D(x) \equiv D(x)C(x) \equiv 0$ on $[a, b]$ (see, for example, [7, Section 108]). Then, a solution for $R_1(x)$ is afforded by $R_1(x) \equiv C^{(\frac{1}{2})}(x) + iD^{(\frac{1}{2})}(x)$, where $C^{(\frac{1}{2})}(x)$ and $D^{(\frac{1}{2})}(x)$ denote the unique positive square roots of $C(x)$ and $D(x)$, respectively. Moreover the continuous differentiability of the elements of $C^{(\frac{1}{2})}(x)$ and $D^{(\frac{1}{2})}(x)$ and, consequently, of $R_1(x)$ on $[a, b]$ follows, in particular, either from a theorem of Rellich [6, pp. 57-58] or from Reid [5, Problem **F**. 1.6, pp. 524-525].

4. – Equivalent linear integro-differential-boundary-parameter problems.

The concept of equivalence of two differential systems, introduced by REID [5, Ch. III, § 11], can be extended to equivalence of a problem (1.1*a*, *b*) with

$$(4.1a)\qquad \mathfrak{L}^0[w;\, r\colon \lambda] \equiv A_1^0(x)w' + [A_0^0(x) - \lambda B^0(x)]w + H^0(x)[M_2^0 w(a) + N_2^0 w(b)] + K_1^0(x)\int_a^b F_1^0(\xi)w\,d\xi + L^0(x)r = 0\,, \qquad r' = 0\,,$$

$$(4.1b)\qquad \begin{aligned} \vartheta_0^0[w] &\equiv M_0^0 w(a) + N_0^0 w(b) = 0\,,\\ \vartheta_1^0[w] &\equiv M_1^0 w(a) + N_1^0 w(b) + \int_a^b F_1^0(\xi)w\,d\xi = 0\,,\\ \vartheta_2^0[w] &\equiv \int_a^b F_2^0(\xi)w\,d\xi = 0\,,\end{aligned}$$

with coefficient matrices of the same dimensions, with maximal rank character and elements of similar continuity and row-independence character, on $[a, b]$ as pertain to the corresponding matrices without the superscript ^{0}, enumerated in Sect. 1. Further, let P_α^0, Q_α^0, $\alpha = 0, 1, 2$, satisfy the inverse relation (2.2) with respect to M_α^0, N_α^0, $\alpha = 0, 1, 2$, wherein the superscript is added to each submatrix in (2.2). In addition, let the dimensions of the matrix coefficients in (4.1*a*, *b*) be designated by the surperscript ^{0} after the same letters designating the dimensions of the corresponding matrices in (1.1*a*, *b*); with, however $n^0 = n$ and $\varkappa^0 = \varkappa$. Thus, $H^0(x)$ is an $n \times \nu^0$ matrix, $F_1^0(x)$ a $\sigma^0 \times n$ matrix, M_0^0 a $\varrho^0 \times n$ matrix, etc..

Problem (1.1*a*, *b*) will be termed *equivalent* to (4.1*a*, *b*) *under the transformations*

$$w = T(x)y \ \text{ for } x \in [a, b]\,, \qquad r = \Pi p\,,$$

where $T(x)$ is an $n \times n$ nonsingular matrix function with elements continuously differentiable on $[a, b]$ and Π is a $\varkappa \times \varkappa$ nonsingular constant matrix, if the integral-boundary forms (1.1*b*) and (4.1*b*) are equivalent and, for arbitrary λ, $\mathfrak{L}[y;\, p\colon \lambda] = 0$ if and only if $\mathfrak{L}^0[w;\, r\colon \lambda] = 0$ for vector sets $\{y, p\}$, p constant and y of class C' satisfying (1.1*b*).

LEMMA 4.1. – *For $T(x)$ an $n \times n$ matrix nonsingular on $[a, b]$, the integro-boundary forms* (1.1*b*) *and* (4.1*b*) *are equivalent under $w = T(x)y$ if and only if there exist a $\tau \times \tau$ nonsingular constant matrix Σ_2 and a $\sigma \times \tau$ constant matrix Δ_2 such that*

$$(4.2)\qquad \begin{aligned} &(a)\quad \varrho^0 = \varrho,\ \sigma^0 = \sigma \ \text{and}\ \tau^0 = \tau\,,\\ &(b)\quad [M_0, N_0;\, M_1, N_1]\cdot \operatorname{diag}\{-T^{-1}(a), T^{-1}(b)\}\cdot[P_2^0; Q_2^0] = 0\,,\\ &(c)\quad [M_0\ N_0]\cdot \operatorname{diag}\{-T^{-1}(a), T^{-1}(b)\}\cdot[P_1^0; Q_1^0] = 0\,,\\ &\left.\begin{aligned}&(d)\quad F_2^0(x)T(x) \equiv \Sigma_2 F_2(x)\,,\\ &(e)\quad F_1^0(x)T(x) \equiv \Sigma_1 F_1(x) + \Delta_2 F_2(x)\end{aligned}\right\}\ \text{on } [a, b]\,,\end{aligned}$$

where $\Sigma_1^{-1} \equiv [M_1\ N_1]\cdot \operatorname{diag}\{-T^{-1}(a), T^{-1}(b)\}\cdot[P_1^0; Q_1^0]$.

Conditions (4.2a) hold from the maximal row ranks of several combinations of the boundary matrix coefficients, and relations (4.2b-e) are then synonymous with the existence of nonsingular constant matrices Σ_α, $\alpha = 0, 1, 2$, of dimensions $\varrho \times \varrho$, $\sigma \times \sigma$ and $\tau \times \tau$, respectively, and constant matrices Δ_0 and Δ_2 of dimensions $\sigma \times \varrho$ and $\sigma \times \tau$, respectively, such that $\mathscr{s}_0^0[w] = \Sigma_0 \mathscr{s}_0[y]$, $s_1^0[w] = \Sigma_1 \mathscr{s}_1[y] + \Delta_0 \mathscr{s}_0[y] + \Delta_2 \mathscr{s}_2[y]$ and $\mathscr{s}_2^0[w] = \Sigma_2 \mathscr{s}_2[y]$ under $w = T(x)y$. Herein, it is to be noted that as the columns of $[P_0^2; Q_0^2]$ form a maximal set of $\nu^0 = 2n - (\varrho + \sigma)$ linearly independent vectors orthogonal to the $\varrho + \sigma$ rows of $[M_0, N_0; M_1, N_1] \cdot \operatorname{diag}\{-T^{-1}(a), T^{-1}(b)\}$, and as the $\varrho + \sigma$ columns of $[P_0^0, P_1^0; Q_0^0, Q_1^0]$ are linearly independent of the columns of $[P_0^2; Q_0^2]$, then, in view of (4.2c), Σ_1 is well-defined; and, moreover, there exist a $\varrho \times \varrho$ nonsingular matrix Σ_0 and a $\sigma \times \varrho$ matrix Δ_0 such that

$$[M_0, N_0; M_1, N_1] \cdot \operatorname{diag}\{-T^{-1}(a), T^{-1}(b)\} \cdot [P_0^0, P_1^0; Q_0^0, Q_1^0] = [\Sigma_0, 0; \Delta_0, \Sigma_1]^{-1}.$$

Furthermore, as equivalence of integro-boundary forms is a symmetric relation in that equivalence of (1.1b) to (4.1b) under $w = T(x)y$, with $T(x)$ nonsingular on $[a, b]$, holds if and only if (4.1b) is equivalent to (1.1b) under $y = T^{-1}(x)w$, conditions (4.2b, c) may be replaced by the set

$$(4.2b') \qquad [M_0^0, N_0^0; M_1^0, N_1^0] \cdot \operatorname{diag}\{-T(a), T(b)\} \cdot [P_2; Q_2] = 0\,,$$

$$(4.2c') \qquad [M_0^0\ N_0^0] \cdot \operatorname{diag}\{-T(a), T(b)\} \cdot [P_1; Q_1] = 0\,,$$

wherein P_α, Q_α, $\alpha = 1, 2$ are determined by relation (2.2).

THEOREM 4.1. – *Problem* (1.1a, b) *is equivalent to problem* (4.1a, b) *under the transformations* $w = T(x)y$, $r = \Pi p$ *if and only if* $T(x)$ *is an* $n \times n$ *nonsigular continuously differentiable matrix function on* $[a, b]$, Π *is a* $\varkappa \times \varkappa$ *nonsingular constant matrix, and there exist a* $\tau \times \tau$ *nonsingular constant matrix* Σ_2 *and a* $\sigma \times \tau$ *constant matrix* Δ_2 *such that*

$$(4.3)\quad \left.\begin{array}{ll}
(a) & \varrho^0 = \varrho,\ \sigma^0 = \sigma,\ \tau^0 = \tau \ \text{and}\ \nu^0 = \nu\,, \\
(b) & [M_0, N_0; M_1, N_1] \cdot \operatorname{diag}\{-T^{-1}(a), T^{-1}(b)\} \cdot [P_2^0; Q_2^0] = 0\,, \\
(c) & [M_0\ N_0] \cdot \operatorname{diag}\{-T^{-1}(a), T^{-1}(b)\} \cdot [P_1^0; Q_1^0] = 0\,, \\
(d) & T'(x) - T(x) A_1^{-1}(x) A_0(x) + A_1^{0^{-1}}(x) A_0^0(x) T(x) \equiv 0\,, \\
(e) & A_1^{0^{-1}}(x) B(x) T(x) \equiv T(x) A_1^{-1}(x) B(x)\,, \\
(f) & A_1^{0^{-1}}(x) L^0(x) \Pi = T(x) A_1^{-1}(x) L(x)\,, \\
(g) & F_2^0(x) T(x) \equiv \Sigma_2 F_2(x)\,, \\
(h) & F_1^0(x) T(x) \equiv \Sigma_1 F_1(x) + \Delta_2 F_2(x)\,, \\
(i) & A_1(x) T^{-1}(x) A_1^{0^{-1}}(x) K_1^0(x) \\
& \quad \equiv K_1(x) \Sigma_1^{-1} + H(x)[M_2 T^{-1}(a) P_1^0 - N_2 T^{-1}(b) Q_1^0]\,, \\
(j) & A_1(x) T^{-1}(x) A_1^{0^{-1}}(x) H^0(x) \equiv H(x)[-M_2 T^{-1}(a) P_2^0 + N_2 T^{-1}(b) Q_2^0]
\end{array}\right\} \text{on } [a, b]\,,$$

where $\Sigma_1 \equiv (-M_1 T^{-1}(a) P_1^0 + N_1 T^{-1}(b) Q_1^0)^{-1}$.

For a problem (1.1*a*, *b*) equivalent to (4.1*a*, *b*) under $w = T(x)y$, $r = \Pi p$, relations (4.3*a*, *b*, *c*, *g*, *h*) hold from Lemma 4.1, and for vectors y with continuously differentiable components and satisfying $y(a) = y(b) = \int_a^b F_1(\xi) y\, d\xi = \int_a^b F_2(\xi) y\, d\xi = 0$ we have that

$$[A_1(x), 0; 0, I][y; p]' + [A_0(x) - \lambda B(x), L(x); 0, 0][y; p] = 0$$

for a value λ if and only if

$$[A_0^1(x), 0; 0, I][w; r]' + [A_0^0 - \lambda B^0(x), L^0(x); 0, 0][w; r] = 0$$

for the same value λ under $[w; r] = [T(x), 0; 0, \Pi][y; p]$. Consequently, relations (4.3*d*, e, *f*) follow from Theorem 11.1 of REID [5, Chapter III]. Then, for arbitrary λ and arbitrary vectors y with continuously differentiable components and satisfying (1.1*b*),

$$\begin{aligned} (4.4)\qquad & A_1^0(x)\, T(x) A_1^{-1}(x) \mathfrak{L}[y; p: \lambda] - \mathfrak{L}^0[w; r: \lambda] \\ &\equiv \{A_1^0(x)\, T(x) A_1^{-1}(x)[H(x)\, M_2 - K_1(x)\, M_1] T^{-1}(a) + \\ &\qquad + [K_1^0(x)\, M_1^0 - H^0(x)\, M_2^0]\} w(a) \\ &\qquad + \{A_1^0(x)\, T(x) A_1^{-1}(x)[H(x) N_2 - K_1(x) N_1] T^{-1}(b) + \\ &\qquad + [K_1^0(x) N_1^0 - H^0(x) N_2^0]\} w(b) \end{aligned}$$

under $w = T(x)y$, $r = \Pi p$. Now, as the $2n - \varrho$ columns of $[-P_1^0, -P_2^0; Q_1^0, Q_2^0]$ form a maximal set of vectors orthogonal to the ϱ rows of $[M_0^0\; N_0^0]$ it follows from (4.3*b*, *c*) that, for each x on $[a, b]$, the right-hand side expression in (4.4) vanishes for arbitrary vectors $w(a)$, $w(b)$ satisfying $\mathscr{B}_0^0[w] = 0$ if and only if relations [4.3*i*, *j*] hold; and, the necessity of relations (4.3) are established.

Conversely, if all the relations (4.3) hold for $T(x)$ an $n \times n$ nonsingular, continuously differentiable matrix function on $[a, b]$, Π and Σ_2 nonsingular constant matrices of dimensions $\varkappa \times \varkappa$ and $\tau \times \tau$, respectively, and Δ_2 a $\sigma \times \tau$ constant matrix, then the boundary forms (1.1*b*) and (4.1*b*) are equivalent under $w = T(x)y$ and $A_1^0(x)\, T(x) \cdot A_1^{-1}(x) \mathfrak{L}[y; p: \lambda] \equiv \mathfrak{L}^0[Ty, \Pi p: \lambda]$ on $[a, b]$ for arbitrary vectors y of class C' satisfying (1.1*b*).

Of special interest is the case in which system (4.1*a*, *b*) is the system (2.4*a*, *b*), the adjoint to problem (1.1*a*, *b*). In this instance, as the rank of $[P_2^* A_1^*(a)\; Q_2^* A_1^*(b)]$ is maximal, it follows from relation (4.2*a*) that $\varrho + \sigma = \nu = n$ and $\tau = \varkappa$, and, hence, that there exist an $n \times n$ nonsingular constant matrix V and an $n \times \tau$ constant matrix W such that the identifications

$$\begin{aligned} (4.5)\qquad [0; F_1^0(x)] &\leftrightarrow V H^*(x) + W L^*(x)\,, \\ [M_0^0, N_0^0; M_1^0, N_1^0] &\leftrightarrow V[P_2^* A_1^*(a)\;\; Q_2^* A_1^*(b)] \end{aligned}$$

may be set. Then, the further identifications

$$F_2^0(x) \quad \leftrightarrow L^*(x)\,,$$
$$[M_2^0\ N_2^0] \leftrightarrow [P_0^*, Q_0^*;\ P_1^*, Q_1^*]\cdot\operatorname{diag}\{A_1^*(a), A_1^*(b)\}$$

yield the identities

$$[P_0^0, P_1^0;\ Q_0^0, Q_1^0] \equiv \operatorname{diag}\{A_1^{*-1}(a), A_1^{*-1}(b)\}\cdot[M_2^*;\ N_2^*]V^{-1}\,,$$
$$[P_2^0;\ Q_2^0] \equiv \operatorname{diag}\{A_1^{*-1}(a), A_1^{*-1}(b)\}\cdot[M_0, N_0;\ M_1, N_1]^*$$

and the identification

$$H^0(x) \leftrightarrow [0 \quad -F_1^*(x)];$$

and, hence, the following preliminary result.

LEMMA 4.2. – *For a problem* (1.1*a*, *b*) *the integro-boundary forms* (1.1*b*) *and* (2.4*b*) *are equivalent under* $z = T(x)y$ *if and only if there exist an* $n\times n$ *matrix* $T(x)$ *nonsingular on* $[a, b]$, *a* $\tau\times\tau$ *nonsingular constant matrix* Σ_2 *and a* $\sigma\times\tau$ *constant matrix* Δ_2 *such that, with* $T_1(x) \equiv A_1^*(x)T(x)$ *on* $[a, b]$,

(*a*) $\varrho + \sigma = \nu = n$ *and* $\tau = \varkappa$,

(*b*) $[M_0, N_0;\ M_1, N_1]\cdot\operatorname{diag}\{-T_1^{-1}(a), T_1^{-1}(b)\}\cdot[M_0, N_0;\ M_1, N_1]^* = 0\,,$

(*c*) $[0\, I_\sigma]V^{*-1}\tilde{C}[I_\varrho;\ 0] = 0\,,$

$$\left.\begin{aligned}&(d)\quad T_1^*(x)A_1^{-1}(x)L(x) \equiv F_2^*(x)\,\Sigma_2^*\,,\\ &(e)\quad T_1^*(x)A_1^{-1}(x)H(x) \equiv F_1^*(x)[0\ \Sigma_1^*]V^{*-1} + F_2^*(x)G\end{aligned}\right\}\ \text{on } [a, b]\,,$$

where $\tilde{C} \equiv [M_2\ N_2]\cdot\operatorname{diag}\{-T_1^{*-1}(a), T_1^{*-1}(b)\}\cdot[M_0, N_0;\ M_1, N_1]^*$, $\Sigma_1^{*-1} \equiv [0\ I_\sigma]V^{*-1}\cdot$ $\cdot\tilde{C}[0;\ I_\sigma]$, $G^* = V^{-1}([0;\ \Delta_2] - W\Sigma_2)$, *and* V *and* W *defined through the identitification* (4.5).

It is to be noted that as the nonsingularity of $\tilde{C}$ may be established by an argument similar to that following Lemma 3.1 in establishing the nonsingularity of C, relation (4.6*c*) and the nonsingularity of V then insure that Σ_1 is well-defined. Consequently, with

$$\tilde{E}_1 \equiv [0\ I_\sigma]\tilde{C}^{-1}\,, \tag{4.7}$$

it follows from (4.6c) and the definition of Σ_1 that

$$(4.8)\qquad [0\ \Sigma_1^*]V^{*-1} = \Sigma_1^*[0\ I_\sigma]V^{*-1}\tilde{C}[I_\varrho,\, 0;\, 0,\, I_\sigma]\tilde{C}^{-1} = \Sigma_1^*[0\ \Sigma_1^{*-1}]\tilde{C}^{-1} = \tilde{E}_1;$$

and, hence, from Lemma 3.1 with the choices Υ as $-\Sigma_2^*$, C as $\tilde{C}$, E_1 as $\tilde{E}_1$ and G as in Lemma 4.2 above, we have the next result.

Lemma 4.3 – *For an $n\times n$ nonsingular matrix $T(x)$ with continuously differentiable components on $[a, b]$, the integro-boundary forms* (1.1b) *and* (2.4b) *of a problem* (1.1a, b) *and its adjoint* (2.4a, b) *are equivalent under $z = T(x)y$ if and only if the integro-boundary forms $\vartheta_\alpha[y]$, $\alpha = 0, 1, 2$, of the problem $T^*(x)\mathfrak{L}[y; p\colon \lambda] = 0$, $\vartheta_\alpha[y] = 0$, $\alpha = 0, 1, 2$,* and *those of its related adjoint are equivalent forms.*

Now, for the case where the system (4.1a, b) is the adjoint system (2.4a, b) the following additional identifications prevail:

$$(4.9)\qquad \begin{aligned}
&A_1^0(x) \leftrightarrow -A_1^*(x)\,,\\
&A_0^0(x) \leftrightarrow A_0^*(x) - A_1^{*'}(x)\,,\\
&B^0(x) \leftrightarrow B^*(x)\,,\\
&K_1^0(x) \leftrightarrow F_1^*(x)\Xi\,,\\
&F_1^0(x) \leftrightarrow \Xi^{-1}K_1^*(x)\,,\\
&L^0(x) \leftrightarrow -F_2^*(x)\,,
\end{aligned}$$

with Ξ a $\sigma\times\sigma$ nonsingular constant matrix. Consequently, the next result follows from Theorem 4.1 and Lemmas 4.2 and 4.3.

Theorem 4.2. – *A problem* (1.1a, b) *is equivalent to its adjoint* (2.4a, b) *under $z = T(x)y$, $q = \Pi p$ if and only if $T(x)$ is an $n\times n$ nonsingular continuously differentiable matrix function on $[a, b]$, Π is a $\varkappa\times\varkappa$ nonsingular constant matrix, and there exists a $\sigma\times\sigma$ nonsingular constant matrix Ξ such that the integro-boundary forms $\vartheta_\alpha[y]$, $\alpha = 0, 1, 2$, of the problem $T^*(x)\mathfrak{L}[y; p\colon \lambda] = 0$, $\vartheta_\alpha[y] = 0$, $\alpha = \overline{0}, 1, 2$, and those of its related adjoint are equivalent, and on $[a, b]$, for $T_1(x) \equiv A_1^*(x)T(x)$,*

$$(4.10)\qquad \begin{aligned}
&(a)\quad T_1'(x) - A_0^*(x)A_1^{*-1}(x)T_1(x) - T_1(x)A_1^{-1}(x)A_0(x) \equiv 0\,,\\
&(b)\quad T_1(x)A_1^{-1}(x)B(x) + B^*(x)A_1^{*-1}(x)T_1(x) = 0\,,\\
&(c)\quad T_1(x)A_1^{-1}(x)L(x) - F_2^*(x)\Pi \equiv 0\,,\\
&(d)\quad T_1(x)A_1^{-1}(x)\Big[-K_1(x) + H(x)[M_2\ N_2]\cdot\operatorname{diag}\{-T_1^{-1}(a),\, T_1^{-1}(b)\}\cdot\\
&\qquad\qquad \cdot[M_2\ N_2]^*\tilde{E}_1^*\Big] \equiv F_1^*(x)\Xi\Sigma_1\,,\\
&(e)\quad T_1(x)A_1^{-1}(x)H(x)[M_2\ N_2]\cdot\operatorname{diag}\{-T^{-1}(a),\, T^{-1}(b)\}\cdot[M_0,\, N_0;\, M_1,\, N_1]^*\\
&\qquad\qquad \equiv [0\ \ -F_1^*(x)]\,,
\end{aligned}$$

where $\tilde{E}_1$ is given by (4.7), *and Σ_1 and $\tilde{C}$ defined in Lemma* 4.2.

Moreover, in case the transformation matrix $T(x)$ is such that $T_1(x)$ is skew-Hermitian on $[a, b]$ the conditions for equivalence of a problem (1. 1*a*, *b*) with its adjoint are reducible. In this connection we first note that the result of REID [5, Problem III. 11, no. 5] may be extended in that, in the notation of the Problem, the assumption that $A_1(t)A_2(t)$ be the identity matrix may be dropped with the skew-Hermitian property of the result then pertaining to the matrix $\widehat{T}_1(x) \equiv A_2^*(x)A_1^*(x)\widehat{T}(x)$. In particular, $\widehat{T}(x)$ may be selected as of the form (11.19) of Reid [5, Chapter III] with $c_1 = \cos\theta + i\sin\theta$, $c_2 = -\cos\theta + i\sin\theta$, and θ a real value such that $\widehat{T}_1^*(x)\cdot \cdot\widehat{T}_1^{-1}(x) - \exp[2i\theta]I$ is nonsingular for some value of x on $[a, b]$.

LEMMA 4.4. – *If the two-point differential problem*

$$\begin{aligned} \mathfrak{L}[y:\lambda] &\equiv A_1(x)y' + A_0(x)y - \lambda B(x)y = 0\,, \\ \vartheta_0[y] &\equiv M_0 y(a) + N_0 y(b) = 0 \end{aligned} \tag{4.11}$$

is equivalent to its adjoint under a nonsingular transformation matrix $T(x)$, *with elements continuously differentiable on* $[a, b]$, *then there exists a nonsingular transformation matrix* $\widehat{T}(x)$, *with elements continuously differentiable and such that* $A_1^*(x)\widehat{T}(x)$ *is skew-Hermitian on* $[a, b]$, *under which problem* (4.11) *is equivalent to its adjoint*; *moreover, the system* $\widehat{T}^*\mathfrak{L}[y:\lambda] = 0$, $\vartheta_0[y] = 0$ *is self-adjoint.*

The above result can be extended to differential-parameter problems (3.10). More generally, the following result relates the concepts of equivalence of a problem (1.1*a*, *b*) with its adjoint, under transformation sets $\{T(x), \Pi\}$ with $A_1^*(x)T(x)$ skew-Hermitian on $[a, b]$, with that of symmetry.

THEOREM 4.3. – *For an integro-differential-boundary-parameter problem* $\mathfrak{L}[y; p:\lambda] = 0$, $\vartheta_\alpha[y] = 0$, $\alpha = 0, 1, 2$, *equivalent to its adjoint* (2.4*a*, *b*) *under* $z = T(x)y$, $q = \Pi p$, *with* $T(x)$ *an* $n \times n$ *nonsingular matrix with elements continuously differentiable and* $A_1^*(x)T(x)$ *skew-Hermitian on* $[a, b]$, *and* Π *a* $\varkappa \times \varkappa$ *nonsingular constant matrix, the system*

$$T^*(x)\mathfrak{L}[y; p:\lambda] = 0\,, \quad \vartheta_\alpha[y] = 0\,, \qquad \alpha = 0, 1, 2\,, \tag{4.12}$$

is symmetric.

If (1.1*a*, *b*) is equivalent to its adjoint under $z = T(x)y$, $q = \Pi p$, conditions (3.5*a*, *b*) for problem (4.11) follow from Theorem 4.2 and Lemmas 3.1 and 4.2, while relations (3.5*c*) for problem (4.12) follow from the assumption $A_1^*(x)T(x)$ skew-Hermitian and relations (4.10*a*, *b*) on $[a, b]$. Further, from (4.10*c*), relation (3.5*d*) for problem (4.12) holds for the choice $\Upsilon = \Pi$; moreover, in this case the matrix Σ_2 in (4.6) is given by $\Sigma_2 = -\Pi^*$.

Now, as $T_1(x) \equiv A_1^*(x)T(x)$ is skew-Hermitian on $[a, b]$ relation (4.10*e*) reduces to $T^*(x)H(x)\tilde{C} \equiv [0\ F_1^*(x)]$ on $[a, b]$; and, hence, $T^*(x)H(x) \equiv F_1^*(x)\tilde{E}_1$ on $[a, b]$, which is relation (3.5*e*) for problem (4.12). Moreover, it follows from relation (4.8) and the linear independence of the columns of $F_2^*(x)$ that the matrix G in (4.6*e*) is the zero matrix. Furthermore, relation (4.10*d*) reduces to

$$T^*(x)K_1(x) \equiv F_1^*(x)(\Xi\Sigma_1 + \tilde{\Theta}) \text{ on } [a, b], \tag{4.13}$$

with $\tilde{\Theta}$ the analogue of the matrix Θ in (3.6) for problem (4.12). However, the identifications (4.5) and (4.9) imply that

$$T^*(x)K_1(x)\Xi^{*-1}[0\, I_\sigma] \equiv T^*(x)\big(H(x)V^* + L(x)W^*\big) \text{ on } [a, b],$$

and, consequently,

$$F_1^*(x)\{\tilde{E}_1 V^* - (\Xi\Sigma_1 + \tilde{\Theta})\Xi^{*-1}[0\, I_\sigma]\} - F_2^*(x)\Pi W^* \equiv 0$$

on $[a, b]$. Then, as the columns of $F_1^*(x)$ and $F_2^*(x)$ are assumed to be linearly independent on $[a, b]$ we have that $W = 0$ and, from (4.8), that $\Sigma_1^* = (\Xi\Sigma_1 + \tilde{\Theta})\Xi^{*-1}$. Now, as $\tilde{\Theta}$ is skew-Hermitian it follows that the matrix $\big(\Xi\Sigma_1 + (\frac{1}{2})\tilde{\Theta}\big)$ is Hermitian; and, hence, from (4.13) we have that relation (3.5*f*) holds for problem (4.12) with the choice $\Gamma \equiv \Xi\Sigma_1 + (\frac{1}{2})\tilde{\Theta}$, and the theorem is established.

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