Invariant Sets for Perturbed Semigroups of Linear Operators (*) (**).

ROBERT H. MARTIN, JR. (Raleigh, North Carolina, U.S.A.)

Summary. – *Let E be a Banach space and consider the initial value problem* (*) $u'(t) = Au(t) + B(t, u(t))$, $t \geq 0$, $u(0) = z$; *where A is the generator of a linear contraction semigroup and B: $[0, \infty) \times E \to E$ is continuous. The main results of this paper deal with criteria insuring that a closed subset Ω of E is invariant for* (*)—*that is, $z \in \Omega$ implies that a solution u to* (*) *satisfies $u(t) \in \Omega$ for all $t \geq 0$.*

1. – Introduction.

Let E be a real or complex Banach space with norm $|\cdot|$, and let $\{T(t): t \geq 0\}$ be a strongly continuous semigroup of linear contractions on E. Now suppose that Ω is a closed subset of E with the property that if $x \in \Omega$ then $T(t)x \in \Omega$ for all $t \geq 0$. In this paper we consider the existence of a solution u with values in Ω to the integral equation

$$\text{(IE)} \qquad u(t) = T(t)z + \int_0^t T(t-\tau) B(\tau, u(\tau))\, d\tau\,, \qquad z \in \Omega,$$

where B is a continuous function from $[0, \infty) \times \Omega$ into E. If A is the infinitesimal generator of T, then solutions to (IE) may be regarded as generalized or mild solutions to the abstract Cauchy problem

$$\text{(ACP)} \qquad u'(t) = Au(t) + B(t, u(t))\,, \qquad u(0) = z \in D(A) \cap \Omega\,.$$

In particular, if u is a solution to (ACP) then u is a solution to (IE), and if a solution to (IE) is differentiable, then $u(t) \in D(A) \cap \Omega$ for all $t \geq 0$ and u is a solution to (ACP).

In § 2 we use the techniques of WEBB [11] to set up approximate solutions to (IE), and criteria for the existence of solutions is given in § 3. Some examples illustrating these techniques are indicated in § 4.

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2. – Approximate solutions.

Let $T=\{T(t): t\geqslant 0\}$ be a strongly continuous semigroup of linear contractions on the Banach space E (i.e., $T(0)=I$, where I is the identity mapping on E; $T(t+s)=T(t)T(s)$ for all $t,s\geqslant 0$; $|T(t)x|\leqslant|x|$ for all $(t,x)\in[0,\infty)\times E$; and $t\to T(t)x$ is continuous on $[0,\infty)$ for each $x\in E$). Also let A be the infinitesimal generator of T $\left(\text{i.e., } Ax=\lim_{t\to 0+} t^{-1}(T(t)x-x), \text{ with } D(A) \text{ the set of all } x \text{ for which this limit exists}\right)$. Recall that $D(A)$ is dense in E. Throughout this paper we assume the following conditions hold:

(C1) Ω is a closed subset of E.

(C2) If $x\in\Omega$ then $T(t)x\in\Omega$ for all $t\geqslant 0$.

(C3) $D(A)\cap\Omega$ is dense in Ω.

Note that is many cases condition (C2) implies condition (C3). In particular, if Ω is the closure of an open set in E or if Ω is convex, then (C2) implies (C3).

For each $y\in E$ let $d(y;\Omega)=\inf\{|y-x|: x\in\Omega\}$. In addition to (C1)-(C3) we frequently assume the following conditions hold:

(C4) $\Lambda\supset\Omega$ and B is a continuous function from $[0,\infty)\times\Lambda$ into E.

(C5) $\liminf_{h\to 0+} d(x+hB(t,x);\Omega)/h=0$ for all $(t,x)\in[0,\infty)\times\Omega$.

To employ our techniques we need the following result concerning the uniformess of the limit in (C5).

Lemma 1. – If conditions (C4) and (C5) are fulfilled then

$$\lim_{h\to 0+} d(x+hB(t,x\Omega));\,/h=0 \qquad \text{for all } (t,x)\in[0,\infty)\times\Omega\,,$$

and this limit is uniform on compact subsets of $[0,\infty)\times\Omega$.

Proof. – Let K be a compact subset of Ω and let $\beta,\varepsilon>0$. By [7, Theorem 2], we have for each $t\in[0,\beta]$ a $\delta(t,\varepsilon)>0$ such that

$$d(x+hB(t,x);\Omega)\leqslant h\varepsilon/2 \qquad \text{for all } (h,x)\in[0,\delta(t,\varepsilon)]\times K\,.$$

Let $\eta>0$ be such that $|B(t,x)-B(s,x)|\leqslant\varepsilon/2$ if $x\in K$ and $t,s\in[0,\beta]$ with $|t-s|\leqslant\eta$, and let $\{t_i\}_0^n$ be a partition of $[0,\beta]$ such that $t_i-t_{i-1}\leqslant\eta$ for $i=1,\dots,n$. Set $\delta=\min\{\delta(t_i,\varepsilon): i=0,\dots,n\}$. If $h\in[0,\delta]$ and $(t,x)\in[t_i,t_{i+1}]\times K$, then

$$\begin{aligned} d(x+hB(t,x);\Omega) &\leqslant d(x+hB(t_i,x);\Omega)+h|B(t,x)-B(t_i,x)| \\ &\leqslant h\varepsilon/2+h\varepsilon/2=h\varepsilon\,, \end{aligned}$$

and the assertion of Lemma 1 follows.

Under the conditions (C1)-(C5) we consider the existence of « approximate solutions » to the integral equation (IE). So assume that (C1)-(C5) hold and $z \in \Omega$. Now choose positive numbers R, M, η and β such that the following is satisfied:

(D1) If $(t, x) \in [0, \beta] \times A$ with $|x - z| \leqslant R$ then $|B(t, x)| \leqslant M$.

(D2) If $|x - z| \leqslant \eta$ and $|y| \leqslant \beta(M + 2)$ then $|T(t)x + y - z| \leqslant R$ for all $t \in [0, \beta]$.

(D3) $\{z_n\}_1^\infty$ is a sequence in $D(A) \cap \Omega$ such that $|z_n - z| \leqslant \eta$ and $\lim_{n\to\infty} z_n = z$.

Note that one is assured from (C1)-(C5) that such numbers R, M, η and β can be found so that (D1)-(D3) hold. Our fundamental result on approximate solutions is the following:

PROPOSITION 1. – Suppose that (C1)-(C5) and (D1)-(D3) are fulfilled and that $\{\varepsilon_n\}_1^\infty$ is a sequence in $(0, 1]$ with $\lim_{n\to\infty} \varepsilon_n = 0$. Then for each positive integer n there is an ε_n-approximate solution $u_n\colon [0, \beta] \to D(A)$ to (IE) in the following sense: there is a positive integer $N = N(n)$ and a partition $\{t_i^n\}_{i=0}^N$ of $[0, \beta]$ with $t_{i+1}^n - t_n^i \leqslant \varepsilon_n$ such that

(i) $u_n(0) = z_n$, $u_n(t_i^n) \in \Omega$ with $|u_n(t_i^n) - z| \leqslant R$, and if $t \in [t_i^n, t_{i+1}^n)$ then

$$u_n'(t) = Au_n(t) + B(t_i^n, u_n(t_i^n))$$

and

$$u_n(t) = T(t - t_i^n)\, u_n(t_i^n) + \int_{t_i^n}^t T(t - \tau)\, B(t_i^n, u_n(t_i^n))\, d\tau \;;$$

(ii) u_n is continuous on $[t_n^i, t_{i+1}^n)$, $u_n(t_{i+1}^n -)$ exists,

$$|u_n(t_{i+1}^n -) - u_n(t_{i+1}^n)| \leqslant \varepsilon_n (t_{i+1}^n - t_i^n)\,,$$

and if $t \in [t_i^n, t_{i+1}^n)$ then $d(u_n(t); \Omega) \leqslant (t - t_i^n)\, \varepsilon_n$;

(iii) if $y_n\colon [0, \beta] \to \{t_i^n\}$ is defined by $\gamma_n(\beta) = \beta$ and $\gamma_n(t) = t_i^n$ whenever $t \in [t_i^n, t_{i+1}^n)$, then

$$|u_n(t) - T(t)\, z_n - \int_0^t T(t - \tau)\, B\big(\gamma_n(\tau), u_n(\gamma_n(\tau))\big)\, d\tau| \leqslant t_i^n\, \varepsilon_n$$

whenever $t \in [t_i^n, t_{i+1}^n)$; and

(iv) if $(t, y) \in [t_i^n, t_{i+1}^n] \times A$ with

$$|y - u_n(t_i^n)| \leqslant (t_{i+1}^n - t_i^n)(M + 1) + \max\Big\{\big|(T(h) - I)\, u_n(t_i^n)\big|\Big\}\, 0 \leqslant b \leqslant t_{i+1}^n - t_i^n\}\,,$$

then $|B(t, y) - B(t_i^n, u_n(t_i^n))| \leqslant \varepsilon_n$ and $|(T(t - t_i^n) - I)\, u_n(t_i^n)| \leqslant \varepsilon_n$.

The construction of the approximate solution u_n is patterned very closely to that of WEBB [11, Proposition (3.1)], and we only indicate it here. In particular we use Webb's construction in the interval $[t_i^n, t_{i+1}^n)$ and, as opposed to defining $u_n(t_{i+1}^n)$ so that u_n is continuous at t_{i+1}^n, we define $u_n(t_{i+1}^n)$ so that it is in $D(A) \cap \Omega$. This is reflected by the jump discontinuity of u_n in part (ii). Before indicating the construction of u_n we first establish the following result:

LEMMA 2. – Let the suppositions of Proposition 1 be fulfilled, let $K \subset \Omega$ be compact, and let $\varepsilon > 0$. Then there is a $\delta = \delta(\varepsilon, K) > 0$ such that

$$d\Big(T(h)x + \int_t^{t+h} T(t+h-\tau)B(t,x)\,d\tau;\,\Omega\Big) < h\varepsilon$$

for all $x \in K$, t, $t+h \in [0, \beta]$, and $h \in [0, \delta]M$

PROOF. – By continuity there is a $\delta_1 > 0$ such that

$$\big|B(t, T(h_1)x) - T(h_2)B(t,x)\big| < \varepsilon/2 \qquad \text{for } t \in [0,\beta],\ h_2,\, h_2 \in [0, \delta_1], \text{ and } x \in K\,.$$

(Recall $T(t)x \in \Omega$ by (C2)). Therefore,

$$\Big|hB(t, T(h)x) - \int_t^{t+h} T(t+h-\tau)B(t,x)\,d\tau\Big| < h\varepsilon/2$$

for all $h \in [0, \delta_1]$, $t \in [0, \beta]$ and $x \in K$. Also, by Lemma 1, there is a $\delta \in (0, \delta_1]$ such that

$$d\big(T(h)x + hB(t, T(h)x);\,\Omega\big) < h\varepsilon/2 \qquad \text{for } t \in [0,\beta],\ h \in [0,\delta], \text{ and } x \in K\,.$$

Consequently, if $t \in [0, \beta]$, $h \in [0, \delta]$ and $x \in K$ then

$$d\Big(T(h)x + \int_t^{t+h} T(t+h-\tau)B(t,x)\,d\tau;\,\Omega\Big) < d\big(T(h)x + hB(t, T(h)x);\,\Omega\big) + \Big|\int_t^{t+h} T(t+h-\tau)B(t,x)\,d\tau - hB(t, T(h)x)\Big| < h\varepsilon\,,$$

and the proof of Lemma 2 is complete.

We now indicate the construction of u_n, which is by induction. Define $t_0^n = 0$ and $u_n(t_0^n) = z_n$, and assume that u_n is constructed on $[0, t_i^n]$. If $t_i^n < \beta$ choose the number δ_i^n as follows:

(1) $\delta_i^n \in [0, \varepsilon_n]$ and $t_i^n + \delta_i^n < \beta$;

(2) if $(t, y) \in [t_i^n, t_i^n + \delta_i^n] \times \Lambda$ and

$$|y - u_n(t_i^n)| \leqslant \delta_i^n(M + 1) + \max\left\{\left|(T(h) - I)\, u_n(t_i^n)\right|\right\} 0 \leqslant h \leqslant \delta_i^n\}$$

then $|B(t, y) - B(t_i^n, u_n(t_i^n))| \leqslant \varepsilon_n$ and $|(T(t - t_i^n) - I)\, u_n(t_i^n)| \leqslant \varepsilon_n$;

(3) $d\left(T(h)\, u_n(t_i^n) + \int\limits_{t_i^n}^{t_{i+h}^n} T(t_i^n + h - \tau)\, B(t_i^n, u_n(t_i^n))\, d\tau;\ \Omega\right) \leqslant h\varepsilon_n/2$

for all $h \in [0, \delta_i^n]$; and

(4) δ_i^n is the largest number such that (1)-(3) hold.

Using the continuity of T and B and Lemma 2, we see that $\delta_i^n > 0$. Let $t_{i+1}^n = t_i^n + \delta_i^n$ and for each $t \in [t_i^n, t_{i+1}^n)$ define

$$u_n(t) = T(t - t_i^n)\, u_n(t_i^n) + \int\limits_{t_i^n}^{t} T(t - \tau)\, B(t_i^n, u_n(t_i^n))\, d\tau\ .$$

It follows from the construction of u_n and the induction hypothesis that the properties listed in Proposition 1 are fulfilled on $[0, t_{i+1}^n)$. By (3) we have that

$$d\left(u_n(t_{i+1}^n -);\ \Omega\right) \leqslant \varepsilon_n(t_{i+1}^n - t_i^n)/2\ ,$$

and since $D(A) \cap \Omega$ is dense in Ω there is a $w \in D(A) \cap \Omega$ with $|u_n(t_{i+1}^n -) - w| \leqslant \varepsilon_n(t_{i+1}^n - t_i^n)$. It then follows that $|w - z| \leqslant R$ and if we define $u_n(t_{i+1}^n) = w$, the properties of u_n are valid on $[0, t_{i+1}^n]$. We now show that $t_N^n = \beta$ for some positive integer N. Assume, for contradiction, that $t_i^n < \beta$ for all i and let $s_0 = \lim\limits_{i\to\infty} t_i^n$. Again us ng the techniques of WEBB [11, Proposition (3.1)], it follows that $w = \lim\limits_{i\to\infty} u_n(t_i^n)$ also exists, and that $w \in \Omega$ since Ω is closed. Thus $K = \{w\} \cup \{u_n(t_0^n), u_n(t_1^n), \ldots\}$ is compact. The continuity of T and B and Lemma 2 shows that (2) and (3) hold with δ_i^n replaced by $s_0 - t_i^n$ for all large i. Since $\delta_i^n < s_0 - t_i^n$ we have a contradiction to (4). Thus $t_N^n = \beta$ for some positive integer N and the indication of the proof of Proposition 1 is complete.

PROPOSITION 2. – Let the suppositions of Proposition 1 be fulfilled and let $\{u_n\}_1^\infty$ be as constructed in Proposition 1. If $u(t) = \lim\limits_{n\to\infty} u_n(t)$ exists uniformly for $t \in [0, \beta]$, then u is a continuous function from $[0, \beta]$ into Ω and u is a solution to (IE) on $[0, \beta]$.

PROOF. – If $t \in [0, \beta]$ we have from (ii) of Proposition 1 that

$$d(u(t);\ \Omega) = \lim_{n\to\infty} d(u_n(t);\ \Omega) \leqslant \lim_{n\to\infty} \varepsilon_n^2 = 0\ ,$$

so u maps $[0,\beta]$ into Ω. If γ_n is as in (iii) of Proposition 1 and

$$w_n(t) = T(t)z_n + \int_0^t T(t-\tau)\,B(\gamma_n(\tau))\,d\tau \qquad \text{for } t\in[0,\beta],$$

then $|w_n(t)-u_n(t)|\leqslant\beta\varepsilon_n$, and so $w_n\to u$ as $n\to\infty$ uniformly. Since each w_n is continuous, it follows that u is continuous. Also, if $t\in[t_i^n, t_{i+1}^n)$ then

$$(2.1)\qquad \big|u_n(t)-u_n(\gamma_n(t))\big| = \Big|\big(T(t-t_i^n)-I\big)\,u_n(t_i^n) + \int_{t_i^n}^t T(t-\tau)\,B\big(\gamma_n(\tau), u_n(\gamma_n(\tau))\big)\,d\tau\Big|$$
$$\leqslant \varepsilon_n + (t-t_i^n)\,M \leqslant \varepsilon_n(1+M)\,,$$

and we have that $\big(\gamma_n(t), u_n(\gamma_n(t))\big)\to\big(t, u(t)\big)$ as $n\to\infty$ uniformly on $[0,\beta]$. Since $\big\{\big(t, u(t)\big): t\in[0,\beta]\big\}$ is compact, we see that $B\big(\gamma_n(t), u_n(\gamma_n(t))\big)\to B\big(t, u(t)\big)$ as $n\to\infty$ uniformly on $[0,\beta]$. Thus, by (iii) of Proposition 1,

$$\Big|u(t) - T(t)z - \int_0^t T(t-\tau)B\big(\tau, u(\tau)\big)\,d\tau\Big|$$
$$= \lim_{n\to\infty}\Big|u_n(t) - T(t)z_n - \int_0^t T(t-\tau)B\big(\big(\gamma_n(\tau), u_n(\gamma_n(\tau))\big)\,d\tau\Big| \leqslant \lim_{n\to\infty}\beta\varepsilon_n = 0\,,$$

for all $t\in[0,\beta]$, and the proof of Proposition 2 is complete.

REMARK 1. – Proposition 1 and 2 may be regarded as extension of Webb's techniques [11] for the case that $\Omega = E$ and as extensions of the techniques in MARTIN [6] for the case that $T(t) = I$ for all $t\geqslant 0$. The condition (C5) on B goes back to NAGUMO [8]. In the case that $T(t) = I$ for all $t\geqslant 0$ and $\Omega = E$, setting up this type of approximate solution is done in CARTAN [3, Theorem 1.3.1]. For higher order equations, see PAVEL [9].

3. – Existence criteria.

In this section we place conditions on B which insures the existence of solutions to (IE). The first result is of a classical nature and employs Lipschitz and compactness criteria on B.

THEOREM 1. – In addition to the suppositions and notations of Proposition 1, suppose that $B(t,x) = B_1(t,x) + B_2(t,x)$ for all $(t,x)\in[0,\infty)\times\Omega$, where B_1 and B_2 satisfy (*a*) there is an $L>0$ such that $|B_1(t,x)-B_1(t,y)|\leqslant L|x-y|$ for all (t,x), $(t,y)\in[0,\beta]\times\Omega$ with $|x-z|$, $|y-z|\leqslant R$; and

(*b*) there is a compact subset K of E such that $B_2(t,x)\in K$ for all $(t,x)\in[0,\beta]\times\Omega$ with $|x-z|\leqslant R$.

Then (IE) has a solution u on $[0,\beta]$ with values in Ω.

PROOF. – Let γ_n be as in (iii) of Proposition 1 and define $v_n(t)=\big(\gamma_n(t),u_n(\gamma_n(t))\big)$ for all $t\in[0,\beta]$ and $n\geqslant1$. Also let

$$\psi_n(t)=\int_0^t T(t-\tau)B_2v_n(\tau)\,d\tau \qquad \text{for all } t\in[0,\beta] \text{ and } n\geqslant1.$$

By assumption (*b*) and the continuity of T we have that

$$K_1=\{T(s)B_2(t,x):s,t\in[0,\beta],\ x\in\Omega \text{ and } |x-z|\leqslant R\}$$

is relatively compact, so if K_2 is the closed convex hull of K_1 and $K_3\equiv=\{tx:(t,x)\in[0,\beta]\times K_2\}$, then K_3 is compact. It then follows routinely that $\psi_n(t)\in K_3$ for all $t\in[0,\beta]$ and $n>1$. Moreover, if $0\leqslant s\leqslant t\leqslant\beta$ then

$$|\psi_n(t)-\psi_n(s)|\leqslant\left|\int_s^t T(t-\tau)B_2v_n(\tau)\,d\tau\right|+\left|\int_0^s[T(t-\tau)-T(s-\tau)]\,B_2v_n(\tau)\,d\tau\right|$$
$$\leqslant|t-s|M_2+\int_0^s|[T(t-s)-I]\,T(s-\tau)B_2v_n(\tau)|\,d\tau\,,$$

where $M_2=\max\{|y|:y\in K\}$. Since $\{T(\tau)x:(\tau,x)\in[0,\beta]\times K\}$ is compact, we see that $\{\psi_n\}_1^\infty$ is equicontinuous. Hence $\{\psi_n\}_1^\infty$ has a uniformly convergent subsequence by Ascoli's theorem. Relabeling if necessary we may assume that $\{\psi_n\}_1^\infty$ is uniformly convergent. If $g:[0,\beta]\to E$ we write $\|g\|=\sup\{|g(t)|:t\in[0,\beta]\}$. Now for each $n\geqslant1$ and $t\in[0,\beta]$ define

$$w_n(t)=T(t)z_n+\int_0^t T(t-\tau)B_1v_n(\tau)\,d\tau+\psi_n(t)\,.$$

It follows from (iii) of Proposition 1 that $\|w_n-u_n\|\leqslant\beta\varepsilon_n$. Moreover, if n and m are positive integers and $p_{n,m}(t)=|w_n(t)-w_m(t)|$ for all $t\in[0,\beta]$, we have from assumption (*a*) that

$$p_{n,m}(t)\leqslant|T(t)z_n-T(t)z_m|+\left|\int_0^t T(t-\tau)[B_1v_m(\tau)-B_1v_m(\tau)]\,d\tau\right|+|\psi_n(t)-\psi_m(t)|$$
$$\leqslant|z_n-z_m|+\|\psi_n-\psi_m\|+\int_0^t L|u_n(\gamma_n(\tau))-u_m(\gamma_m(\tau))|\,d\tau+\beta(\varepsilon_n+\varepsilon_m)\,.$$

Also, by (2.1) in the proof of Proposition 2 it follows that

$$\begin{aligned}|u_n(\gamma_n(\tau)) - u_m(\gamma_m(\tau))| &\leq p_{n,m}(\tau) + |w_n(\tau) - u_n(\gamma_n(\tau))| + |w_m(\tau) - u_m(\gamma_m(\tau))| \\ &\leq p_{n,m}(\tau) + \|w_n - u_n\| + \varepsilon_n(M+1) + \|w_m - u_m\| + \varepsilon_n(M+1) \\ &\leq p_{n,m}(\tau) + (\varepsilon_n + \varepsilon_m)(\beta + M + 1)\,.\end{aligned}$$

Thus

$$p_{n,m}(t) \leq \delta_{n,m} + \int_0^t L p_{n,m}(\tau)\, d\tau \qquad \text{for all } t \in [0, \beta]\,,$$

where $\delta_{n,m} = |z_n - z_m| + \|\psi_n - \psi_m\| + 2L\beta(\varepsilon_n + \varepsilon_m)(\beta + M + 1)$. It now follows from Gronwall's inequality that

$$\|w_n - w_m\| = \|p_{n,m}\| \leq \delta_{n,m} \exp[L\beta]\,,$$

and since $\delta_{n,m} \to 0$ as $n, m \to \infty$ we have that $\{w_n\}_1^\infty$ is uniformly Cauchy. The inequality $\|w_n - u_n\| \leq \beta\varepsilon_n$ shows that $\{u_n\}_1^\infty$ is also uniformly Cauchy, and Theorem 1 follows from Proposition 2.

REMARK 2. – If the function B_2 in Theorem 1 is the zero function, then the solution u to (IE) on $[0, \beta]$ is easily seen to be unique.

Now we place a dissipative type condition on B to insure the existence and uniqueness of solutions to (IE). Again our techniques follow WEBB [11]. If $x, y \in E$ define

$$m_-[x, y] = \lim_{h \to 0-} (|x + hy| - |x|)/h \qquad \text{and} \qquad m_+[x, y] = \lim_{h \to 0+} (|x + hy| - |x|)/h\,.$$

It follows that if $x, y, z \in E$ then

$$m_-[x, y + z] \leq m_+[x, y] + m_-[x, z] \leq |y| + m_-[x, z]\,.$$

Moreover, if $u\colon [0, b] \to E$ has a derivative at $s \in (0, b)$ and $p(t) = |u(t)|$ for all $t \in [0, b]$, then p has a left derivative at s and $p'_-(s) = m_-[u(s), u'(s)]$. Also, if A is as in § 2, then

$$m_+[x, Ax] \leq 0 \qquad \text{for all } x \in D(A)\,.$$

We use the above properties of m_- and m_+ frequently and without comment in our proofs.

THEOREM 2. – In addition to the suppositions and notations of Proposition 1, suppose that $\Lambda \supset \{x \in E: |x-z| \leqslant R\}$ and there is an $L > 0$ such that

$$(3.1)\qquad m_-[x-y, B(t,x)-B(t,y)] \leqslant L|x-y| \quad \text{for all } (t,x),(t,y) \in [0,\beta]\times E \text{ with } |x-z|, |y-z| \leqslant R.$$

Then (IE) has a unique solution u on $[0,\beta]$ with values in Ω.

PROOF. – Let n and m be positive integers and define $p(t) = |u_n(t)-u_m(t)|$ for all $t \in [0,\beta]$. If i and j are positive integers and $t \in (t_i^n, t_{i+1}^n) \cap (t_j^m, t_{j+1}^m)$, then it follows from (3.1) and part (ii) of Proposition 1 that

$$\begin{aligned} p'_-(t) &= m_-\Big[u_n(t)-u_m(t), Au_n t) - Au_m(t) + B(t_i^n, u_n(t_i^n)) - B(t_j^m, u_m(t_j^m))\Big] \\ &\leqslant m_+\Big[u_n(t)-u_m(t), A\big(u_n(t)-u_m(t)\big)\Big] + m_-\Big[u_n(t)-u_m(t), B(t_i^n, u_n(t_i^n)) - B(t_j^m, u_m(t_j^m)\Big] \\ &\leqslant Lp(t) + |B(t_i^n, u_n(t_i^n)) - B(t, u_n(t))| + |B(t, u_m(t)) - B(t_j^m, u_m(t_j^m))| \\ &\leqslant Lp(t) + \varepsilon_n + \varepsilon_m, \end{aligned}$$

where the last inequality follows from (iv) of Proposition 1 and the fact that $u_n(t)$ $u_m(t) \in \Lambda$. Solving this differential inequality we have that

$$(3.2)\qquad p(t) \leqslant \Big[(\varepsilon_n+\varepsilon_m)L^{-1} + \sum_{s\in[0,t]} |p(s)-p(s-)|\Big]\exp[Lt] \qquad \text{for all } t \in [0,\beta].$$

(See Lemma 2 of [6]). Using (ii) of Proposition 1 it follows that

$$\begin{aligned} \sum_{s\in[0,\beta]} |p(s)-p(s-)| &\leqslant \sum_{k=1}^{N(n)} |u_n(t_k^n) - u_n(t_k^n-)| + \sum_{k=1}^{N(m)} |u_m(t_k^m) - u_m(t_k^m-)| \\ &\leqslant \sum_{k=1}^{N(n)} \varepsilon_n(t_k^n - t_{k-1}^n) + \sum_{k=1}^{N(m)} \varepsilon_m(t_k^m - t_{k-1}^m) \\ &= \beta(\varepsilon_n+\varepsilon_m). \end{aligned}$$

Substituting this estimate into (3.2) shows that

$$|u_n(t)-u_m(t)| = p(t) \leqslant (\varepsilon_n+\varepsilon_m)(L^{-1}+\beta)\exp[Lt] \qquad \text{for all } t \in [0,\beta].$$

Thus $\{u_n\}_1^\infty$ is uniformly Cauchy on $[0,\beta]$ and the existence of a solution to (IE) follows from Proposition 2. The uniqueness assertion follows easily from the techniques used in the proof of Theorem 3 below, and is omitted.

THEOREM 3. – Suppose that conditions (C1)-(C5) are fulfilled, the set Λ in (C4) is open, and there is a continuous real valued function ϱ on $[0, \infty)$ such that

$$m_-[x-y, B(t,x)-B(t,y)] \leq \varrho(t)|x-y| \quad \text{for all } (t,x), (t,y) \in [0,\infty)\times\Lambda .$$

Then for each z in Ω there is a unique noncontinuable solution u_z to (IE) on $[0, b_z)$. Also, if $a, w \in \Omega$ then

$$(3.3) \qquad |u_z(t)-u_w(t)| \leq |z-w| \exp\left(\int_0^t \varrho(\tau)\,d\tau\right) \quad \text{for all } t\in[0,b_z)\cap[0,b_w) .$$

PROOF. – Since Λ is open, local existence of solutions follows from Theorem 2, so let u be a solution to (IE) and let v be a solution to (IE) with z replaced by w, and let $\beta > 0$ be such that u and v are defined on $[0, \beta]$. We use the techniques of WEBB [11, Proposition (3.6)] to establish (3.3) on $[0, \beta]$. For each positive integer n be $\{t_i^n\}_0^n$ be the partition of $[0, \beta]$ such that $t_i^n - t_{i-1}^n = \beta/n$ for $i = 1, \dots, n$, and define $\gamma_n: [0,\beta] \to \{t_i^n\}_0^n$ by $\gamma_n(\beta) = \beta$ and $\gamma_n(t) = t_i^n$ if $t \in [t_i^n, t_{i+1}^n)$. Now let $\{z_n\}_1^\infty$ and $\{w_n\}_1^\infty$ be sequences in $D(A)\cap\Omega$ such that $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$. For each $t\in[0,\beta]$ define

$$u_n(t) = T(t)\,z_n + \int_0^t T(t-\tau)\,B(\gamma_n)\tau),\, u(\gamma_n(\tau))\,d\tau \quad \text{and}$$

$$v_n(t) = T(t)\,w_n + \int_0^t T(t-\tau)\,B\big(\gamma_n(\tau), v(\gamma_n(\tau))\big)\,d\tau .$$

It follows easily that $u_n \to u$ and $v_n \to v$ uniformly as $n\to\infty$. Thus $u_n(t), v_n(t) \in \Lambda$ when n is sufficiently large. Moreover, if $t \notin \{t_i^n\}_1^n$ then

$$u_n'(t) = Au_n(t) + B\big(\gamma_n(t), u(\gamma_n(t))\big) \quad \text{and} \quad v_n'(t) = Av_n(t) + B\big(\gamma_n(t), v(\gamma_n(t))\big) ;$$

so if $p_n(t) = |u_n(t) - v_n(t)|$ it follows that

$$\begin{aligned}(p_n)_-'(t) &= m_-\Big[u_n(t)-v_n(t), Au_n(t)-Av_n(t)+B\big(\gamma_n(t), u(\gamma_n(t))\big)-B\big(\gamma_n(t))\big)\Big]\\ &\leq m_-\Big[u(t)-v_n(t), B(t,u_n(t))-B\big(t,v_n(t)\big)\Big]\\ &\quad+\Big|B\big(\gamma_n(t), u(\gamma_n(t))\big)-B(t,u_n(t))\Big|+\Big|B(t,v_n(t))-B\big(\gamma_n(t), v(\gamma_n(t))\big)\Big|\\ &\leq \varrho(t)\,p_n(t)+\varepsilon_n ,\end{aligned}$$

where

$$\varepsilon_n = \sup\left\{\Big|B\big(\gamma_n(t), u(\gamma_n(t))\big)-B(t,u_n(t))\Big| + \Big|B(t,v_n(t))-B\big(\gamma_n(t), v(\gamma_n(t))\big)\Big| : t\in[0,\beta]\right\} .$$

Since $\varepsilon_n \to 0$ as $n \to \infty$ we conclude from the above differential inequality that

$$|u(t) - v(t)| = \lim_{n\to\infty} p_n(t) \leqslant |z - w| \exp\left(\int_0^t \varrho(\tau)\, d\tau\right)$$

for all $t \in [0, \beta]$. The inequality (3.3) now follows immediately and the uniqueness assertion is evident by setting $z = w$ in (3.3). This completes the proof of Theorem 3.

REMARK 3. – If the suppositions of Theorem 3 are fulfilled with $\Lambda = E$, then the results of LOVELADY [5] show that each of the noncontinuable solutions u_z to (IE) is defined on $[0, \infty)$. In this case Theorem 3 may be regarded as a criteria for the invariance of Ω.

In [11] Webb gives an example of a solution to (IE) that is in $D(A)$ initially, but not in $D(A)$ for any time $t > 0$. A convenient criteria which assures that a solution u to (IE) is also a solution to the Cauchy problem (ACP) is given by SEGAL [10, Theorem 3, Lemma 3.1], and we record it here for future reference:

PROPOSITION 3 (SEGAL [10]). – In addition to the suppositions of Theorem 3 suppose that B is continuously differentiable on $(0, \infty) \times \Lambda$. Then for each $z \in D(A) \cap \Omega$, the solution u_z to (IE) is differentiable on $[0, b_z)$, u_z maps $[0, b_z)$ into $D(A) \cap \Omega$, and $u_z'(t) = Au_z(t) + B(t, u_z(t))$ for all $t \in [0, b_z)$.

If, in Theorem 3, the function B is independent of $t \in [0, \infty)$, then one may use Webb's techiques in [11] to show that the noncontinuable solutions u_z to (IE) exists on $[0, \infty)$. Since this leads to some interesting results on semigroups of nonlinear operators and existence results for nonlinear operator equations, we indicate these ideas here. A family $U = \{U(t) : t \geqslant 0\}$ of functions each mapping Ω into Ω is said to be a *semigroup of type α on Ω* if α is a real number and each of the following is fulfilled: (a) $U(0)x = x$ for all $x \in \Omega$; (b) $U(t + s)x = U(t)\,U(s)x$ for all $x \in \Omega$, t, $s \geqslant 0$; (c) $t \to U(t)x$ is continuous on $[0, \infty)$ for each $x \in \Omega$; and (d) $|U(t)x - U(t)y| \leqslant |x - y| \exp[\alpha t]$ for all $x, y \in \Omega$, $t \geqslant 0$. The *generator* of U is the function G defined by $Gx = \lim_{t\to 0+} t^{-1}(U(t)x - x)$, with the domain $D(G)$ being the set of all $x \in \Omega$ for which this limit exists.

Now we assume that (C1)-(C3) are valid and each of the following conditions hold:

(C4)′ Λ is an open subset of E, $\Omega \subset \Lambda$, and C is a continuous function from Λ into E.

(C5)′ $\liminf_{h\to 0+} d(x + hCx; \Omega)/h = 0$ for each $x \in \Omega$.

(C6)′ there is a real number α such that $m_-[x - y, Cx - Cy] \leqslant \alpha|x - y|$ for all x, $y \in \Lambda$.

Under these assumptions we consider the existence of solutions to the autonomous integral equation

$$\text{(AIE)} \qquad u(t) = T(t)z + \int_0^t T(t - \tau)\, Cu(\tau)\, d\tau\,, \qquad z \in \Omega,\ t \geqslant 0.$$

Our fundamental result is the following extension of Theorem I in Webb [11]:

Theorem 4. – Suppose that conditions (C1)-(C3) and (C4)′-(C6)′ are fulfilled. Then for each $z \in \Omega$ (AIE) has a unique solution u_z on $[0, \infty)$ with values in Ω. Also, if $U(t)z = u_z(t)$ for all $(t, z) \in [0, \infty) \times \Omega$, then U is a semigroup of type α on Ω and $A + C$ with $D(A + C) = D(A) \cap \Omega$ is the generator of U.

Using Theorem 3, the proof follows that of Webb [11, Proposition (3.6), (3.9), and (3.10)] and is omitted. Concerning the existence of a critical point to (AIE) and its asymptotic stability, we have the following result:

Theorem 5. – In addition to the suppositions of Theorem 4, suppose that $\alpha < 0$. Then there is a unique point $x^* \in D(A) \cap \Omega$ such that $Ax^* + Cx^* = \theta$. Moreover, since $U(t)x^* = x^*$ for all $t \geqslant 0$, we have that $|U(t)z - x^*| \leqslant |z - x^*| \exp[\alpha t]$ for all $(t, z) \in [0, \infty) \times \Omega$.

The proof of Theorem 5 follows in a standard manner from Theorem 4 (see, e.g., [11, Proposition (3.15)]). In the case that Ω is convex, we can establish considerable information on the resolvent of $A + C$.

Theorem 6. – Suppose that (C1)-(C3) and (C4)′-(C6)′ are valid and that Ω is convex. For each $h > 0$ such that $h\alpha < 1$ define

$$\mathcal{R}\big(I - h(A + C)\big) = \{x - h(Ax + Cx) : x \in D(A) \cap \Omega\} .$$

Then $I - h(A + C)$ is injective on $D(A) \cap \Omega$, $\mathcal{R}\big(I - h(A + C)\big) \supset \Omega$, and

$$|[I - h(A + C)]^{-1}x - [I - h(A + C)]^{-1}y| \leqslant (1 - ha)^{-1}|x - y|$$

for all $x, y \in \Omega$.

Proof. – It follows easily from $(C6)'$ that

$$|[I - h(A + C)]x - [I - h(A + C)y]| \geqslant (1 - h\alpha)|x - y| \qquad \text{for all } x, y \in \Omega.$$

Thus, to complete the proof, we show that $\mathcal{R}\big(I - h(A + C)\big) \supset \Omega$. Let $w \in \Omega$ and define $C^*x = hCx - x + w$ for all $x \in \Lambda$. It follows that C^* is continuous and

$$m_-[x - y, C^*x - C^*y] = m_-[x - y, hCx - hCy] - |x - y| \leqslant (h\alpha - 1)|x - y|$$

for all $x, y \in \Lambda$. Also (see the proof of Theorem 6 in [6]),

$$\liminf_{h \to 0+} d(x + hC^*x; \Omega)/h = 0 \qquad \text{for all } x \in \Omega .$$

If $T_h(t)x = T(ht)x$ for all $(t, x) \in [0, \infty) \times E$, then T_h is a semigroup of linear contractions, hA is the generator of T_h, and if $x \in \Omega$ then $T_h(t)x = T(ht)x \in \Omega$ for all

$t \geqslant 0$. Thus, with T replaced by T_h, C by C^*, and α by $(h\alpha - 1)$, the suppositions of Theorem 5 are fulfilled. Hence there is an $x^* \in D(hA) \cap \Omega = D(A) \cap \Omega$ such that $hAx^* + C^*x^* = \theta$. It follows that $x - h(Ax + Cx) = w$ and the proof of Theorem 6 is complete.

REMARK 4. – Except in Theorem 1, we used very heavily in our proofs that z is an interior point of Λ. It is the case that if we replace m_- by m_+, then Theorems 2-6 are valid when $\Lambda = \Omega$ (and this modification is not needed when Ω is convex—see [6, Theorems 5 and 6]). In the case tht E has a uniformly convex dual sace, the proofs are not difficult, since the mapping $(x, y) \to |x|m_+[x, y]$ is uniformly continuous on bounded subsets of $E \times E$. However, in the general case, the proofs are very tedious, and use the fact that the mapping $(x, y) \to |x|m_+[x, y]$ is upper semicontinuous on $E \times E$ (see the proof of Theorem 3 in [6] for the case that $T(t) = I$ for all $t \geqslant 0$).

REMARK 5. – In the case that Ω is convex, we may use Theorem 6, a result of CRANDALL and LIGGETT [4], and the techniques of WEBB [11, Proposition (3.18)] to show that U is as in Theorem 4, then $U(t)z = \lim_{n\to\infty} [I - tn^{-1}(A + B)]^{-n}z$ for all $z \in \Omega$, $t \geqslant 0$. Also, for $\Omega = E$ and $\alpha = 0$, BARBU [1] shows $\mathcal{R}(I - (A + C))$ is E when A is a nonlinear (multivalued) m-dissipative operator.

4. – Examples.

In this section we indicate some situations where these techniques may be applied Throughout this section we assume that J is a closed number interval and F is a Banach space with norm denoted $\|\cdot\|$. Also, it is assumed that V is a nonempty open subset of F and f is a continuous function from $J \times V$ into F, and K is a nonempty, closed subset of F such that $K \subset V$. Now let $\mathcal{F}n(J, F)$ be the vector space of all function from J into F. We suppose that E is a Banach subspace of $\mathcal{F}n(J, F)$ (with the norm on E denoted $|\cdot|$), that T is a semigroup of linear contractions on E with generator A, and that conditions (C1)-(3) are fulfilled with

$$\Omega = \{x \in E : x(s) \in K \text{ for all } s \in J\}. \tag{4.1}$$

Set $\Lambda = \{x \in E : x(s) \in V \text{ for all } s \in J\}$ and define the mapping C from Λ into $\mathcal{F}n(J, F)$ by

$$[Cx](s) = f(s, x(s)) \qquad \text{for all } (s, x) \in J \times \Lambda. \tag{4.2}$$

Throughout this section it is assumed that f has the following properties:

(P1) There is a number α such that if $h > 0$ and $(s, \xi), (s, \eta) \in J \times V$ then

$$\|\xi - \eta - h[f(s, \xi) - f(s, \eta)]\| \geqslant (1 - h\alpha)\|\xi - \eta\|.$$

(P2) If $d_1(\xi, K) = \inf\{\|\xi - \eta\| : \eta \in K\}$ for each $\xi \in F$, then

$$\liminf_{h\to 0+} d_1(\eta + hf(s, \eta); K)/h = 0 \qquad \text{for all } (s, \eta) \in J \times K .$$

We now consider various Banach spaces E in $\mathcal{F}n(J, F)$ as well as further conditions on f which insure that C satisfies contitions (C4)′-(C6)′ in §3. These results then lead to the existence of solutions of abstract partial differential equations of the form

$$(4.3) \qquad \begin{cases} \varphi_t(t, s) = [A\varphi(t, \cdot)](s) + f(s, \varphi(t, s)) & (t, s) \in [0, \infty) \times J \\ \varphi(0, s) = z(s), \quad \text{where } z \in D(A) \cap \Omega \end{cases}$$

If u is a solution to the autonomous integral equation (AIE) in § 3, then the function $\varphi(t, s) = [u(t)](s)$ for all $(t, s) \in [0, \infty) \times J$ is called a *mild solution* to (4.3). We remark here that many of our results apply equally well when f is time dependent—$f: [0, \infty) \times J \times V \to F$ is continuous and $[B(t, x)](s) = f(t, s, x(s))$ for all $(t, s, x) \in$ $\in [0, \infty) \times J \times \Lambda$. However, for simplicity, we restrict our attention to the time independent case. Before establishing our results, note that (P2) and Lemma 1 imply that

(P2)′ $\lim_{h\to 0+} d_1(\eta + hf(s, \eta); K)/h = 0$, uniformly for (s, η) in a compact subset of $J \times K$.

If J is a compact interval then $\mathcal{C}(J, F)$ is the Banach space of all continuous functions $x: J \to F$ with $|x| = \max\{\|x(s)\| : s \in J\}$.

PROPOSITION 4. – Suppose that J is compact, K is convex, and $E = \mathcal{C}(J, F)$. Then Λ is open, Ω is closed, convex, and nonempty, and the operator $\mathcal{C}$ defined of Λ by (4.2) satisfies conditions (C4)′-(C6)′.

PROOF. – It is immediate that Λ is open, Ω is closed, convex and nonempty, and that C is continuous. If $x, y \in \Lambda$ then

$$\|x(s) - y(s) - h\big[f(s, x(s)) - f(s, y(s))\big]\| \geq (1 - h\alpha)\|x(s) - y(s)\| \qquad \text{for all } h > 0,\ s \in J$$

by (P1). Taking the maximum over s of each side of this inequality shows that $|x - y - h[Cx - Cy]| \geq (1 - h\alpha)|x - y|$, and this is easily seen to imply that (C6)′ Now let $x \in \Omega$ and let $\varepsilon > 0$. By (P2)′ there is an $h \in (0, \varepsilon)$ and a family $\{\mu(s): s \in J\}$ of members of K such that

$$\|x(s) + hf(s, x(s)) - \mu(s)\| \leq h\varepsilon/7 \qquad \text{for all } s \in J .$$

For notational convenience set $z(s) = x(s) + hf(s, x(s))$ for all $s \in J$. By uniform continuity there is a $\delta > 0$ such that $\|z(s) - z(t)\| \leq h\varepsilon/7$ whenever $|s - t \leq \delta$. Let $\{s_i\}_0^n$ be a partition of J with $|s_i - s_{i-1}| \leq \delta$ for $i = 1, \ldots, n$ and define $y: J \to F$ by

$$y(s) = \mu(s_{i-1}) + (s - s_{i-1})[\mu(s_i) - \mu(s_{i-1})]/(s_i - s_{i-1})$$

whenever $s \in [s_{i-1}, s_i]$ and $i = 1, \ldots, n$. Then y is continuous and $y(s) \in K$ for all $s \in J$ since K is convex. Thus $y \in \Omega$ and

$$d(x + hCx; \Omega) \leqslant |x + hCx - y| \leqslant h\varepsilon/7 + \sup\{\|\mu(s) - y(s)\| : s \in J\}.$$

However, if $s \in [s_{i-1}, s_i]$ then

$$\begin{aligned}\|\mu(s) - y(s)\| &\leqslant \|\mu(s) - \mu(s_{i-1})\| + \|\mu(s_{i-1}) - y(s)\| \\ &\leqslant \|\mu(s) - \mu(s_{i-1})\| + \|\mu(s_i) - \mu(s_{i-1})\| \\ &\leqslant 2 \sup\{\|\mu(s) - \mu(s_{i-1})\| : s \in [s_{i-1}, s_i]\}.\end{aligned}$$

Since $|s_i - s_{i-1}| \leqslant \delta$, we have from the choice of and μ that if $s \in [s_{i-1}, s_i]$ then

$$\begin{aligned}\|\mu(s) - \mu(s_{i-1})\| &\leqslant \|\mu(s) - z(s)\| + \|z(s) - z(s_{i-1})\| + \|z(s_{i-1}) - \mu(s_{i-1})\| \\ &\leqslant h\varepsilon/7 + h\varepsilon/7 + h\varepsilon/7 = 3h\varepsilon/7.\end{aligned}$$

Thus $\|\mu(s) - y(s)\| \leqslant 6h\varepsilon/7$ for all $s \in J$ and we have that $d(x + hCx, \Omega) \leqslant h\varepsilon$. This shows that (C5)$'$ holds and the proof of Proposition 4 is complete.

REMARK 6. – The proof techniques of Proposition 4 may be applied to various Banach subspaces of $\mathcal{C}(J, F)$ with no essential changes. If $J = [a, b]$ the following assertions are easily seen to be true: (a) if $E = \{x \in \mathcal{C}(J, F) : x(a) = x(b)\}$ and $f(a, \xi) = f(b, \xi)$ for all $\xi \in V$, then (C4)$'$-(C6)$'$ are valid; (b) if $E = \{x \in \mathcal{C}(J, F) : x(a) = x(b) = \theta\}$, $\theta \in K$, and $f(a, \theta) = f(b, \theta)$, then (C4)$'$-(C6)$'$ are valid; and (c) if $E = \{x \in \mathcal{C}(J, F) : x(a) = \theta\}$, $\theta \in K$, and $f(a, \theta) = \theta$, then (C4)$'$-(C6)$'$ are valid. To establish (a), for example, one may choose the family μ is the proof of Proposition 4 so that $\mu(a) = \mu(b)$. Then if y is as constructed in the proof, we have that $y(a) = y(b)$ and hence $y \in \Omega$.

REMARK 7. – The assumption that K is convex in Proposition 4 can be relaxed somewhat. Note that the convexity of K was used in order to be able to « connect » the points $\{\mu(s_i) : i = 0, \ldots, n\}$ in an appropriate manner. Instead of assuming K is convex, one could assume that there are numbers M, $\beta > 0$ with the property that if $\xi, \eta \in K$ with $|\xi - \eta| \leqslant \beta$, there is a continuous function $\psi : [0, 1] \to K$ such that $\psi(0) = \xi$, $\psi(1) = \eta$, and $|\psi(s) - \psi(0)| \leqslant M|\xi - \eta|$ for all $s \in [0, 1]$. This property is valid, for example, if $0 < r_1 \leqslant r_2$ and

$$K = \{\xi \in F : r_1 \leqslant \|\xi\| \leqslant r_2\} \quad \text{or} \quad K = \{\xi \in F : r_1 \leqslant \|\xi\|\}.$$

If $p \in [1, \infty)$ we let $\mathcal{L}^p(J, F)$ denote the space of all measurable functions $x : J \to F$ such that $|x| = \left[\int_J \|x\|^p \, ds\right]^{1/p} < \infty$. In the case that F is infinite dimensional, the integral is that of Bochner. Of course, in $\mathfrak{L}^p(J, F)$, we identify functions equal almost everywhere on J.

PROPOSITION 5. – Suppose that $p \in [1, \infty)$, $V = F$, and there is a continuous function $\psi: J \to [0, \infty)$ and a number $N > 0$ such that

$$\text{(4.4)} \qquad \begin{cases} \|f(s, \xi)\| \leqslant N\|\xi\| + \psi(s) & \text{for all } (s, \xi) \in J \times F \text{ and} \\ \int_J \psi(s)^p\, ds < \infty \end{cases}$$

If J is not compact, assume also that $\theta \in K$. Then, with $E = \mathfrak{L}^p(J, F)$, the mapping C defined on $\Lambda = E$ by (4.2) satisfies conditions (C4)′-(C6)′, and Ω is nonempty.

PROOF. – Assumption (4.4) is easily seen to imply that C maps E into E and is continuous. If $x, y \in E$ and $h > 0$ is such that $h\alpha < 1$, then we have from (P1) that

$$\|x(s) - y(s) - h[f(s, x(s)) - f(s, y(s))]\|^p \geqslant (1 - h\alpha)^p \|x(s) - y(s)\|^p \quad \text{for all } s \in J\,,$$

and integrating each side of this inequality over J shows that condition (C6)′ holds. It is also easy to see that Ω is nonempty and closed. Now let $x \in \Omega$ and let $\varepsilon > 0$. Define $w(s) = f(s, x(s))$ for all $s \in J$. Since $\int_J \|w\|^p ds < \infty$, there is a compact interval $J_0 \subset J$ such that $\int_{J - J_0} \|w\|^p ds \leqslant \varepsilon^p/3$. Also, by absolute continuity, there is a $\delta > 0$ such that if $S \subset J_0$ is measurable and $m(S) \leqslant \delta$ (where m is Lebesgue measure), then $\int_S \|w\|^p ds \leqslant \varepsilon^p/3$. By Lusin's Theorem there is a measurable (open) subset J_1 of J_0 such that $m(J_1) \leqslant \delta$ and x is continuous on $J_0 - J_1$. Set $J_2 = J_0 - J_1$. Since J_2 is compact and $x|J_2$ is continuous we have that $\{x(s): s \in J_2\}$ is compact. By $(P2)'$ there is an $h \in (0, \in)$ and a family $\{\mu(s): s \in J_2\}$ of members of K such that

$$\|x(s) + hf(s, x(s)) - \mu(s)\| \leqslant h\varepsilon 2^{-1} 3^{-1/p} (1 + m(J_2))^{-1/p} \quad \text{for } s \in J_2\,.$$

Moreover, if $z(s) = x(s) + hf(s, x(s))$ for all $s \in J_2$; there is a mutually disjoint collection $\{S_i\}_1^n$ of measurable subsets of J_2 and an $s_i \in S_i$ for $i = 1, \ldots, n$ such that $\bigcup_{i=1}^n S_i = J_2$ and

$$\int_{J_2} \|z - \sigma\|^p ds \leqslant h^p \varepsilon^p 2^{-p} 3^{-2/p}\,,$$

where $\sigma: J_2 \to F$ is defined by $\sigma(s) = z(s_i)$ if $s \in S_i$. Hence if $\varrho: J_2 \to K$ is defined by $\varrho(s) = \mu(s_i)$ whenever $s \in S_i$, then ϱ is measurable and

$$\begin{aligned} \Big[\int_{J_2} \|z - \varrho\|^p ds\Big]^{1/p} &\leqslant \Big[\int_{J_2} \|\sigma - \varrho\|^p ds\Big]^{1/p} + \Big[\int_{J_2} \|z - \sigma\|^p ds\Big]^{1/p} \\ &\leqslant m(J_2)^{1/p} \sup\{\|\sigma(s) - \varrho(s)\|: s \in J_2\} + h\varepsilon 2^{-1} 3^{-1/p} \\ &\leqslant h\varepsilon 2^{-1} 3^{-1/p} = h\varepsilon 3^{-1/p}\,. \end{aligned}$$

Therefore,

$$\int_{J_2} \|x + hCx - \varrho\|^p ds \leqslant h^p \varepsilon^p / 3 .$$

Thus if $y(s) = x(s)$ if $s \in J - J_2$ and $y(s) = \varrho(s)$ if $s \in J_2$, then $y \in \Omega$ and

$$\begin{aligned} d(x + hCx, \Omega)^p &\leqslant \int_J \|x + hCx - y\|^p ds \\ &= \int_{J-J_2} h^p \|Cx\|^p ds + \int_{J_2} \|x + hCx - \varrho\|^p ds \\ &\leqslant \int_{J-J_0} h^p \|w\|^p ds + \int_{J_1} h^p \|w\|^p ds + h^p \varepsilon^p/3 \\ &\leqslant h^p \varepsilon^p/3 + h^p \varepsilon^p/3 + h^p \varepsilon^p/3 = h^p \varepsilon^p . \end{aligned}$$

This shows that (C5)′ holds and the proof of Proposition 5 is complete.

As a specific illustration of these results let $J = [0, 2\pi]$ and suppose that E is $\mathfrak{L}^p([0, 2\pi], F)$ where $p \in [1, \infty)$ or E is the space of all $x \in \mathfrak{C}([0, 2\pi], F)$ such that $x(0) = x(2\pi)$. Let

$$D(A) = \{x \in E : x, x' \text{ are absolutely continuous, } x'' \in E, x(0) = x(2\pi), \text{ and } x'(0) = x'(2\pi)\} ,$$

and define $Ax = x''$ for all $x \in D(A)$. Then A is the generator of a semigroup of linear contractions T on E (see BUTZER and BERENS [2, pp. 59-64]). Moreover, if $x \in E$ and $\lambda > 0$ then

$$(4.5) \qquad [(I - \lambda A)^{-1} x](s) = (2\pi)^{-1} \int_0^{2\pi} r(\lambda; s - r) x(\tau) \, d\tau \qquad \text{for all } s \in [0, 2\pi] ,$$

where $r(\lambda, \cdot)$ is the 2π-periodic function on $(-\infty, \infty)$ defined by

$$r(\lambda, t) = \pi\sqrt{\lambda}[\cosh \sqrt{\lambda}(\tau - \pi)][\sinh \sqrt{\lambda}\pi]^{-1} \qquad \text{for } \tau \in [0, 2\pi] .$$

Recall that if $x \in E$ and $t \geqslant 0$ then

$$[T(t)x](s) = \lim_{n \to \infty} [(I - n^{-1} t A)^{-n} x](s) \qquad \text{for } s \in [0, 2\pi] .$$

Noting that $r(\lambda, \tau) \geqslant 0$ and $(2\pi)^{-1} \int_0^{2\pi} r(\lambda, \tau) \, d\tau = 1$, it follows easily that conditions (C1)-(C3) are fulfilled if K has any of the following forms: (a) K is a closed cone in F (b) K is a closed ball in F with center θ; (x) K is the intersection of a closed cone with a closed ball of center θ; or (*d*) P is a closed cone in F, $\eta \in P$, and $K = \{\xi \in F: \xi - \eta \in P\}$. In particular, these results give existence and uniqueness criteria for mild solutions with values in K to the equation

$$(4.6)\qquad \begin{cases} \varphi_t(t, s) = \varphi_{ss}(t, s) + f(s, \varphi(t, s)) & \text{for } (t, s) \in [0, \infty) \times [0, 2\pi] \\ \varphi(0, s) = z(s) \quad \text{for } s \in [0, 2\pi],\ z \in \Omega \\ \varphi(t, 0) = \varphi(t, 2\pi) \quad \text{and} \quad \varphi_t(t, 0) = \varphi_t(r, 2\pi) & \text{for } t \geqslant 0. \end{cases}$$

REMARK 8. – Note that if $\alpha < 0$ then by Theorem 5 there is a unique K-valued solution x^* to the periodic equation $x''(s) + f(s, x(s)) = \theta$ for all $s \in [0, 2\pi]$, $x(0) = x(2\pi)$ and $x'(0) = x'(2\pi)$. Moreover, each solution φ to (4.6) satisfies $\lim_{t\to\infty} \varphi(t, s) = x^*(s)$, with convergence in the E norm.

REMARK 9. – In the case that K is convex assumption (P2) may be expressed in terms of hyperplanes. In particular, (P2) holds only in case for each $\xi \in \partial K$ and each $\xi^* \in F^*$ such that $\xi^*(\xi) = \varrho \geqslant \varrho$ and $\mathrm{Re}[\xi^*(\eta)] \leqslant 0$ for all $\eta \in K$, it follows that $\mathrm{Re}\big[\xi^*(f(s, \xi))\big] \leqslant 0$ for all $s \in J$. Note also that if F is the space $\mathbb{R}^n$, $f(s, \xi) = (f_i(s, \xi))_{i=1}^n$ for all $(s, \xi) \in J \times V$, and $K = \{(\xi_i)_1^n: \xi_i \geqslant 0 \text{ for } i = 1, \ldots, n\}$, then (P2) holds only in case $f_j(s, \xi) \geqslant 0$ whenever $\xi = (\xi_i)_1^n \in K$ and $\xi_j = 0$.

Now we consider a boundary value problem on the right half line. Let E be either the space $\mathfrak{L}^1([0, \infty), F)$ or the space $\mathfrak{C}_0^0([0, \infty), F)$ consisting of all continuous functions $x: [0, \infty) \to F$ with $x(0) = \theta$, $\lim_{s\to\infty} x(s) = \theta$, and $|x| = \max\{\|x(s)\|: s \geqslant 0\}$. Let

$$D(A) = \{x \in E: x, x' \text{ are absolutely continuous, } x(0) = \theta \text{ and } x'' \in E\}$$

and define $Ax = x''$ for all $x \in D(A)$.

LEMMA 3. – Let E, $D(A)$ and A be as in the above paragraph. Then $D(A)$ is dense in E, the resolvent set of A contains $(0, \infty)$, and if $\lambda > 0$, $y \in E$, and $\eta = \lambda^{-1/2}$,

$$(4.7)\qquad [(I - \lambda A)^{-1} y](s) = (\eta/2) \int_s^\infty \big[\exp[\in(s - \tau)] - \exp[-\eta(s + \tau)]\big] y(\tau)\, d\tau + (\eta/2) \int_0^s \big[\exp[\eta(\tau - s)] - \exp[-\eta(s + \tau)]\big] y(\tau)\, d\tau$$

for all $s \geqslant 0$. Also, $|(I - \lambda A)^{-1} y| \leqslant |y|$ for all $\lambda > 0$ and $y \in E$. (The proof follows routine y and is omitted.)

Lemma 3 shows that A is the generator of a semigroup T of linear contractions on E, and hence

$$[T(t)x](s) = \lim_{n\to\infty} [(I - tn^{-1}A)^{-n} x](s) \qquad \text{for } x \in E \text{ and } t, s \geqslant 0\,.$$

Using (4.7) and the fact that

$$\left[\exp[\eta(s-\tau)]-\exp[-\eta(s+\tau)]\right], \quad \left[\exp[\epsilon(\tau-s)]-\exp[-\eta(s+\tau)]\right]\geqslant 0$$
$$\text{for all } s,\ \tau\geqslant 0,$$

it follows that (C1)-(C3) are fulfilled if K has any of the following forms: (a) K is a closed cone in F; (b) K is the closed ball of center θ and radius ϱ; or (c) K is the intersection of a closed cone with the closed ball of center θ and radius ϱ. Thus our results apply to the equation

$$(4.8)\qquad \begin{cases} \varphi_t(t,s) = \varphi_{ss}(t,s)+f(s,\varphi(t,s)) & \text{for } t,\ s\geqslant 0\,, \\ \varphi(0,s) = z(s) \quad \text{for } s\geqslant 0\,,\ z\in\Omega\,,\ \varphi(t,0)=\theta\,, & \\ \text{and either} & \\ \lim_{s\to\infty}\varphi(t,s)=\theta \quad \text{for } t>0 \qquad \text{or} \qquad \int_0^\infty \|\varphi(t,s)\,ds<\infty \quad \text{for } t>0\,. & \end{cases}$$

REMARK 10. – Again note that if $\alpha<0$ we have from Theorem 5 that is a unique K-valued solution x^* to the equation $x''(s)+f(s,x(s))=\theta$ for all $s\in[0,\infty)$ such that $x(0)=\theta$ and either $\lim_{s\to\infty}x(s)=\theta$ or $\int_0^\infty\|x(s)\|\,ds<\infty$. Moreover, each solution φ to (4.8) satisfies $\lim_{t\to\infty}\varphi(t,s)=x^*(s)$, with convergence in the E norm.

REFERENCES

[1] V. BARBU, *Continuous perturbations of nonlinear m-accretive operators in Banach spaces*, Boll. Un. Mat. Italiana, **6** (1972), pp. 270-278.

[2] P. L. BUTZER - H. BERENS, *Semi-Groups of Operators and Approximation*, Springer-Verlag, New York, 1967.

[3] H. CARTAN, *Calcul Differential*, Paris, 1967.

[4] M. G. CRANDALL - T. LIGGETT, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math., **113** (1971), pp. 265-298.

[5] D. L. LOVELADY, *A Hammerstein-Volterra integral equation with a linear semigroup convolution kernal*, Indiana Univ. Math. J. (to appear).

[6] R. H. MARTIN Jr., *Differential equations on closed subsets of a Banach space*, Trans. Amer. Math. Soc., **179** (1973), pp. 399-414.

[7] R. H. MARTIN Jr., *Approximation and existence of solutions to ordinary differential equations in Banach space*, Funk. Ekvac., **16** (1973), pp. 195-211.

[8] M. NAGUMO, *Über die Laga der Integralkuruen gewohnlicher Differentialglerchugen*, Proc. Phys.-Math. Soc. Japan, **24** (1942), pp. 551-559.

[9] N. PAVEL, *Approximate solutions of Cauchy problems for some differential equations on Banach spaces* (to appear).

[10] I. SEGAL, *Nonlinear semi-groups*, Ann. of Math., **78** (1963), pp. 339-364.

[11] G. F. WEBB, *Continuous nonlinear perturbations of linear accretive operators in Banach spaces*, J. Funct. Anal., **10** (1972), pp. 191-203.