Partial Regularity for the Solutions to Nonlinear Parabolic Systems (*).

M. GIAQUINTA (**) - E. GIUSTI (***)

Sunto. – *Si estendono a sistemi non lineari di tipo parabolico alcuni risultati di regolarità parziale delle soluzioni di sistemi ellittici.*

1. – Introduction.

The aim of this paper is to extend the results and the methods of [6] to nonlinear parabolic systems of partial differential equations:

$$\sum_{\alpha=1}^{N}\int_A u^\alpha \varphi_t^\alpha\, dz = \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\int_A a_{ij}^{\alpha\beta}(z,u)\, u_{x_j}^\beta \varphi_{x_i}^\alpha\, dz\,, \qquad \forall \varphi\in C_0^\infty(A:\mathbf{R}^N)\,, \tag{1.1}$$

where $A=\Omega\times(0,T)$ is an open set in $\boldsymbol{R}^{n+1}$, $z=(x,t)$, and the coefficients $a_{ij}^{\alpha\beta}(z,u)$ are supposed to be continuous in $\bar{A}\times\boldsymbol{R}^N$ and satisfy the ellipticity conditions:

$$|a_{ij}^{\alpha\beta}|\leqslant L\,,$$
$$\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N} a_{ij}^{\beta\alpha}\xi_\alpha^i\xi_\beta^j\geqslant|\xi|^2\,, \qquad \forall\xi\in\boldsymbol{R}^{nN}\,.$$

In these hypotheses, we shall prove that every weak solution of (1.1) is regular in A, with the possible exception of a singular set Σ, closed in A. If in addition the solution u belongs to the space $W_{p,\mathrm{loc}}^{1,\frac12}(A:\boldsymbol{R}^N)$ for some $p\geqslant 2$, one has

$$\mathcal{H}_{n+2-p+\sigma}(\Sigma,\delta)=0$$

for every $\sigma>0$, δ being a suitable metric in $\boldsymbol{R}^{n+1}$ and $\mathcal{H}_\alpha(x,\delta)$ being the α-dimensional Hausdorff measure relative to the metric δ.

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(***) University of Pisa and Stanford University. Research partially supported by NSF Grant GP 16115.

2. – Preliminaries.

We shall consider open sets $A = \Omega \times (0, T)$ in $\boldsymbol{R}^{n+1}$, where Ω is open in $\boldsymbol{R}^n$ and $T > 0$. A point in $\boldsymbol{R}^{n+1}$ will be denoted by $z = (x, t)$, $x \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}$. If $z_0 = (x_0, t_0)$ is in $\boldsymbol{R}^{n+1}$, and if $R > 0$, we define

$$B(x_0, R) = \{x \in \boldsymbol{R}^n : |x - x_0| < R\}, \tag{2.1}$$

$$\Lambda(t_0, R) = \{t \in \boldsymbol{R} : |t - t_0| < R^2\}, \tag{2.2}$$

$$Q(z_0, R) = B(x_0, R) \times \Lambda(t_0, R) . \tag{2.3}$$

If we introduce in $\boldsymbol{R}^{n+1}$ the metric

$$\delta(z_1, z_2) = \max\{|x_1 - x_2|, |t_1 - t_2|^{\frac{1}{2}}\}, \tag{2.4}$$

then the set $Q(z_0, R)$ is an open ball of radius R, centered at z_0. We shall write Q, Q_R instead of $Q(z_0, R)$ (and likewise for B and Λ) whenever it can be done without confusion.

In addition to the usual Sobolev spaces $W^1_p(A)$, we need the following spaces:

DEFINITION 1. – *$V^{1,0}_p(A)$ is the completion of $C^1(A)$ with respect to the norm:*

$$\|u\|_{p:A} = \left\{\int_A |u|^p dz + \sum_{i=1}^n \int_A \left|\frac{\partial u}{\partial x_i}\right|^p dz\right\}^{1/p} \equiv \left\{\|u\|^p_{p,A} + \sum_{i=1}^n \|u_{x_i}\|^p_{p,A}\right\}^{1/p}. \tag{2.5}$$

DEFINITION 2. – *$W^{1,\alpha}_p(A)$, $0 < \alpha < 1$, is the completion of $C^1(A)$ with respect to the norm:*

$$|||u|||_{\alpha,p:A} = \left\{\|u\|^p_{p,A} + \int_\Omega dx \int_{(0,T)\times(0,T)} \frac{|u(x,t) - u(x,s)|^p}{|t-s|^{1+\alpha p}} dt\,ds\right\}^{1/p} \equiv \{\|u\|^p_{p,A} + |u|^p_{\alpha,p:A}\}^{1/p}.$$

The following propositions are well-known:

LEMMA 1. – *Let $A = \Omega \times (0, T)$ be bounded and convex, and let $u \in W^{1,\frac{1}{2}}_p(A)$. Then:*

$$\int_A |u - u_A|^p dz \leqslant c_1 \delta(A)^p \left\{\sum_{i=1}^n \|u_{x_i}\|^p_{p,A} + |u|^p_{\frac{1}{2},p:A}\right\} \tag{2.7}$$

where

$$u_A = \frac{1}{\operatorname{meas} A} \int_A u\,dz,$$

and $\delta(A)$ is the diameter of A with respect to the metric δ: $\delta(A) = \max\{\operatorname{diam} \Omega, \sqrt{T}\}$.

LEMMA 2. – *Let A be as before. Then the natural imbedding of $W_p^{1,\frac{1}{2}}(A)$ into $L_p(A)$ is compact.*

Finally we introduce another family of functional spaces.

DEFINITION 3. – *Let $A = \Omega \times (0, T)$ be bounded and convex, and let δ be the metric* (2.4). *$\mathfrak{L}^{p,\mu}(A, \delta)$, $\mu > 0$, is the space of all functions in $L_p(A)$ such that:*

$$[f]^p_{p,\mu;A} = \sup_{\substack{z_0 \in A\\ R>0}} [\operatorname{meas}(A \cap Q(z_0, R))]^{-\mu} \int_{A \cap (z_0, R)} |f - f_{z_0,R}|^p dz < +\infty$$

where

$$f_{z_0,R} = \frac{1}{\operatorname{meas}(A \cap Q(z_0, R))} \int_{A \cap Q(z_0,R)} f(z)\, dz\,.$$

$\mathfrak{L}^{p,\mu}(A, \delta)$ is a Banach space with norm

$$\left\{\|f\|^p_{p,A} + [f]^p_{p,\mu;A}\right\}^{1/p}.$$

These spaces have been introduced in [1] for the euclidean metric and in [3] for a general class of metrics including (2.4). We have the following result ([3], Theor. [3.1]).

LEMMA 3. – *If $\mu > 1$, then $\mathfrak{L}^{p,\mu}(A, \delta)$ is isomorphic to $C^{0,\alpha}(\Omega, \delta)$, the space of α-hölder continuous functions with respect to the metric δ, with $\alpha = ((n+2)/p)(\mu - 1)$.*

In the following, we shall consider vector-valued functions; if $S(A)$ is a topological space of real functions in A, we will denote by $S(A; \boldsymbol{R}^N)$ the product of N copies of $S(A)$, with the natural topology. It is obvious that Lemmas 1, 2 and 3 remain valid for vector-valued functions. Finally, with $S_{\text{loc}}(A)$ we denote the space of all functions f in A which belong to $S(A')$ for every $A' \subset\subset A$.

3. – Linear parabolic systems.

In this Section we collect a number of results concerning linear parabolic systems. Results of this type are known and can be found in the literature, although sometimes in a slightly different form.

By weak solution of the parabolic system

$$\frac{\partial u^\alpha}{\partial t} = \frac{\partial}{\partial x_i}\left(a^{\alpha\beta}_{ij}(z)\frac{\partial u^\beta}{\partial x_j}\right), \qquad \alpha = 1, \ldots, N\,, \tag{3.1}$$

we mean a function $u \in V^{1,0}_{2,\text{loc}}(A; \boldsymbol{R}^N)$ such that

$$\int_A u^\alpha \varphi^\alpha_t\, dz = \int_A a^{\alpha\beta}_{ij}(z)\, u^\beta_{x_j} \varphi^\alpha_{x_i}\, dz \tag{3.2}$$

for every $\varphi \in C_0^\infty(A; R^N)$. Here and in the following, the summation over repeated indices is understood, the latin indices $i, j, \ldots$ running from 1 to n, and the greek indices $\alpha, \beta, \ldots$ from 1 to N.

The coefficients $a_{ij}^{\alpha\beta}$ are bounded measurable functions such that

$$a_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \geq |\xi|^2 = \xi_\alpha^i\xi_\alpha^i\,, \qquad \forall \xi \in \boldsymbol{R}^{nN}\,,\ \forall z \in A\,, \tag{3.3}$$
$$|a_{ij}^{\alpha\beta}(z)| \leq L\,, \qquad \forall z \in A\,.$$

The following Lemmas are proved using methods similar to those in [7], Ch. III, § 4:

LEMMA 4. – *Let $u(z)$ be a weak solution of the system* (3.1) *in Q_{R_0}. Then $u \in W^{1,\frac{1}{2}}_{2,\mathrm{loc}}(Q_{R_0}; \boldsymbol{R}^N)$, and for every ϱ, R $(0 < \varrho < R < R_0)$ we have:*

$$|||u|||_{\frac{1}{2},2;Q_\varrho} \leq c_2(\varrho, R)\|u\|_{2,Q_R}\,. \tag{3.4}$$

PROOF. – Let $\varphi^\alpha = \omega(t)\eta^\alpha(x,t)$, $\operatorname{supp}\omega \subset \Lambda_R$. We get from (3.2):

$$\int v^\alpha \eta_t^\alpha\,dz = \int F_i^\alpha \eta_{x_i}^\alpha\,dz - \int F^\alpha \eta^\alpha\,dz \tag{3.5}$$

where

$$v^\alpha = \omega u^\alpha\,; \qquad F_i^\alpha = a_{ij}^{\alpha\beta} v_{x_j}^\beta\,; \qquad F^\alpha = u^\alpha \omega\,. \tag{3.6}$$

Now let $\beta_\varepsilon(t)$ be a mollifier, $\beta_\varepsilon(t) = \beta_\varepsilon(-t)$; if we put $\eta(x,t) = \beta_\varepsilon * \theta = \theta_\varepsilon(x,t)$, we obtain

$$\int v_\varepsilon^\alpha \theta_t^\alpha\,dz = \int F_{i,\varepsilon}^\alpha \theta_{x_i}^\alpha\,dz - \int F_\varepsilon^\alpha \theta^\alpha\,dz\,. \tag{3.7}$$

In particular, if $\theta^\alpha(x,t) = \gamma(t)\psi^\alpha(x)$, with $\gamma \in \mathfrak{D}(\Lambda_R)$ and $\psi^\alpha \in \mathring{W}_2^1(B_R)$, we get the equation

$$\int \langle v_\varepsilon, \psi\rangle \gamma'\,dt = \int \{\langle F_{i,\varepsilon}, \psi_{x_i}\rangle - \langle F_\varepsilon, \psi\rangle\}\gamma\,dt \tag{3.8}$$

where

$$\langle f, g\rangle = \int f^\alpha g^\alpha\,dx\,, \tag{3.9}$$

and hence, in the sense of distributions,

$$\frac{d}{dt}\langle v_\varepsilon, \psi\rangle = -\langle F_{i,\varepsilon}, \psi_{x_i}\rangle + \langle F_\varepsilon, \psi\rangle\,. \tag{3.10}$$

Now let $\hat{f}(\tau)$ denote the Fourier transform of $f(t)$. We have

$$i\tau\langle \hat{v}_\varepsilon, \psi\rangle = \langle \hat{F}_{i,\varepsilon}, \psi_{x_i}\rangle - \langle \hat{F}_\varepsilon, \psi\rangle \tag{3.11}$$

for every $\tau \in \boldsymbol{R}$ and every $\psi \in \mathring{W}_2^1(B_R)$. In particular, we can choose

$$\psi = -i \operatorname{sign} \tau \hat{v}_\varepsilon \sigma(x)^2$$

with $\sigma \in \mathfrak{D}(B_R)$; integrating with respect to τ, we easily get:

$$\int |\tau|\, |\sigma \hat{v}_\varepsilon|^2 dx\, d\tau \leqslant \|\sigma F_{i,\varepsilon}\|_2 \left\{ \|\sigma v_{\varepsilon,x}\|_2 + \|\sigma_x v_\varepsilon\|_2 \right\} + \|\sigma F_\varepsilon\|_2 \|\sigma v_\varepsilon\|_2 . \tag{3.12}$$

Finally, if we let $\varepsilon \to 0$ (remember that for every function $g(x, t)$ with compact support, we have

$$\int dx \int \frac{|g(x,t) - g(x,s)|^2}{|t-s|^2}\, dt\, ds = 2 \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt \int |\tau| |\hat{g}|^2 d\tau\, dx \Big),$$

and choose ω and σ in such a way that $\omega\sigma \equiv 1$ on Q_ϱ, we easily get the conclusion of the lemma. q.e.d.

LEMMA 5. – *With the hypotheses of Lemma 4, we have*

$$| u |_{2,Q_\varrho} \leqslant c_3(\varrho, R) \|u\|_{2,Q_R} \tag{3.13}$$

for every R, ϱ $(0 < \varrho < R < R_0)$.

PROOF. – If $\varphi = \beta_\varepsilon * \theta$, we have from (3.2)

$$\int_{Q_R} u_\varepsilon^\alpha \theta_t^\alpha dz = \int_{Q_R} (a_{ij}^{\alpha\beta} u_{x_j}^\beta)_\varepsilon \theta_{x_i}^\alpha dz . \tag{3.14}$$

In particular, if $\theta = \sigma^2(x)\omega^2(t) u_\varepsilon$, with $\sigma(x) \in \mathfrak{D}(B_R)$ and $\omega(t) \in \mathfrak{D}(\Lambda_R)$, we get

$$0 = \frac{1}{2} \int_{B_R} \sigma^2 dx \int_{\Lambda_R} \frac{d}{dt} |\omega u_\varepsilon|^2 dt = \int_{Q_R} (a_{ij}^{\alpha\beta} u_{x_j}^\beta)_\varepsilon \omega^2 (\sigma^2 u_\varepsilon^\alpha)_{x_i} dz - \int_{Q_R} |u_\varepsilon|^2 \omega \omega_t \sigma^2 dz ,$$

whence, if $\varepsilon \to 0$,

$$\int a_{ij}^{\alpha\beta} u_{x_j}^\beta u_{x_i}^\alpha \omega^2 \sigma^2 dz = \int |u|^2 \omega \omega_t \sigma^2 dz - 2 \int a_{ij}^{\alpha\beta} u_{x_j}^\beta \omega^2 \sigma^\alpha \sigma_{x_i} dz . \tag{3.15}$$

With the usual choice of σ and ω, it is not difficult to get (3.13) from (3.15). q.e.d.

An immediate consequence of the two previous lemmas is

COROLLARY 1. – *Let u be as before. Then for every ϱ, R $(0 < \varrho < R < R_0)$, we have*

$$|||u|||_{\frac{1}{2},2:Q_\varrho} \leqslant c_4(\varrho, R) \|u\|_{2,Q_R} . \tag{3.16}$$

Let us now recall a result concerning systems with constant coefficients. The proof can be found in [2].

Lemma 6. – *Let $u(t)$ be a weak solution of* (3.1) *in Q, and let the coefficients $a_{ij}^{\alpha\beta}$ be constant in Q_1. Suppose further that $u \in L_2(Q_1, \boldsymbol{R}^N)$. Then for every $\varrho < 1$, we have*

$$\frac{1}{\operatorname{meas} Q_\varrho}\int_{Q_\varrho}|u - u_{Q_\varrho}|^2 dz \leqslant c_5 \varrho^2 \frac{1}{\operatorname{meas} Q_1}\int_{Q_1}|u - u_{Q_1}|^2 dz . \tag{3.17}$$

We conclude this Section with a simple result concerning the convergence of solutions of parabolic systems.

Lemma 7. – *Let $a_{ij}^{\alpha\beta(\nu)}(z)$ $(\nu = 1, 2, \ldots)$ be a sequence of bounded measurable functions in Q_1, verifying* (3.2) *with L independent of ν, and such that*

$$\lim_{\nu\to\infty} a_{ij}^{\alpha\beta(\nu)}(z) = a_{ij}^{\alpha\beta}(z) \qquad a.e.\ in\ Q_1 . \tag{3.18}$$

Let $u^{(\nu)}$ be a sequence of function in $L_2(Q_1; \boldsymbol{R}^N)$, weak solutions in Q_1 of the system (3.1) *with coefficients $a_{ij}^{\alpha\beta(\nu)}$:*

$$\int_{Q_1} u^{(\nu)\alpha}\varphi_t^\alpha dz = \int_{Q_1} a_{ij}^{\alpha\beta(\nu)}(z)\, u_{x_j}^{(\nu)\beta}\varphi_{x_i}^\alpha dx , \qquad \forall \varphi \in C_0^\infty(Q_1, \boldsymbol{R}^N) . \tag{3.19}$$

Suppose further that

$$\lim_{\nu\to\infty} u^{(\nu)} = u \qquad weakly\ in\ L^2(Q_1; \boldsymbol{R}^N) . \tag{3.20}$$

Then $u \in V_{2,\mathrm{loc}}^{1,0}(Q_1, \boldsymbol{R}^N)$, and for every $R < 1$ we have

$$\lim_{\nu\to\infty} u^{(\nu)} = u \qquad strongly\ in\ L_2(Q_R, \boldsymbol{R}^N) , \tag{3.21}$$

$$\lim_{\nu\to\infty} u_{x_i}^{(\nu)} = u_{x_i} \qquad weakly\ in\ L_2(Q_R, \boldsymbol{R}^N) \tag{3.22}$$

$(i = 1, \ldots, n)$. *In addition, $u(z)$ is a weak solution of the system*

$$\int_{Q_1} u^\alpha \varphi_t^\alpha dz = \int_{Q_1} a_{ij}^{\alpha\beta}(z)\, u_{x_j}^\beta \varphi_{x_i}^\alpha dz , \qquad \forall \varphi \in C_0^\infty(Q_1, \boldsymbol{R}^N) . \tag{3.23}$$

Proof. – It is easily seen that (3.20) and Lemma 5 imply (3.22), while (3.21) follows from Corollary 1, Lemma 2, and (3.20). Finally, letting $\nu \to \infty$ in (3.19), one gets (3.23) at once. q.e.d.

4. – The regularity of solutions of parabolic systems.

We shall consider solutions of nonlinear parabolic systems of partial differential equations:

$$\int_A u^\alpha \varphi_t^\alpha dz = \int_A a_{ij}^{\alpha\beta}(z, u) u_{x_j}^\beta \varphi_{x_i}^\alpha dx , \qquad \forall \varphi \in C_0^\infty(A, \boldsymbol{R}^N) , \tag{4.1}$$

where, as usual, $A = \Omega \times (0, T)$ is an open set in $\boldsymbol{R}^{n+1}$, $z = (x, t)$, $x \in \Omega$, $0 < t < T$, and we sum over repeated indices, the latin indices running from 1 to n, and the greek indices from 1 to N.

The coefficients $a_{ij}^{\alpha\beta}(z, u)$ are supported to be continuous functions in $\bar{A} \times \boldsymbol{R}^N$, and to satisfy the ellipticity conditions:

i) $|a_{ij}^{\alpha\beta}(z, u)| \leqslant L$, $\forall z \in \bar{A}$, $u \in \boldsymbol{R}^N$,

ii) $a_{ij}^{\alpha\beta}(z, u)\, \xi_\alpha^i \xi_\beta^j = |\xi|^2$, $\forall \xi \in \boldsymbol{R}^{nN}$, $z \in \bar{A}$, $u \in \boldsymbol{R}^N$.

We have the following

LEMMA 8. – *For every τ, $0 < \tau < 1$, and for every $M > 0$, there exist two constants ε_0 and R_0 such that if $u(z)$ is a weak solution of (4.1), and if for some $z_0 \in A$ and some $R < R_0 \cap \delta(z_0, \partial A)$ $(a \cap b = \min\{a, b\})$, we have*

$$U(z_0, R) \equiv \frac{1}{\operatorname{meas} Q_R} \int_{Q(z_0, R)} |u - u_{Q(z_0, R)}|^2 dz < \varepsilon_0^2 \tag{4.2}$$

and

$$|u_{Q(z_0, R)}| \leqslant M ; \tag{4.3}$$

then:

$$U(z_0, \tau R) \leqslant 2 c_5 \tau^2 U(z_0, R) \tag{4.4}$$

where c_5 is the constant appearing in Lemma 5.

PROOF. – Suppose the lemma is false. Then for some τ and M, there exist sequences $z_\nu \in A$, $\varepsilon_\nu \to 0$, $R_\nu \to 0$, and a sequence $u^{(\nu)}$ of weak solutions of (4.1) such that

$$|u^{(\nu)}_{Q(z_\nu, R_\nu)}| \leqslant M , \tag{4.5}$$

$$U^{(\nu)}(z_\nu, R_\nu) = \varepsilon_\nu^2 , \tag{4.6}$$

and

$$U(z_\nu, \tau R_\nu) > 2 c_5 \tau^2 \varepsilon_2^\nu . \tag{4.7}$$

If we define

$$v^{(\nu)}(z) = v^{(\nu)}(x, t) = \varepsilon_\nu^{-1}\big(u^{(\nu)}(x_\nu + R_\nu x, t_\nu + R_\nu^2 t) - u^{(\nu)}_{Q(z_\nu, R_\nu)}\big\}, \tag{4.8}$$

we have $(Q = Q(0, 1))$:

$$\int_Q v^{(\nu)\alpha} \varphi_t^\alpha dz = \int_Q a_{ij}^{\alpha\beta}(x_\nu + R_\nu x, t_\nu + R_\nu^2 t;\ \varepsilon_\nu v^{(\nu)} + u^{(\nu)}_{Q(x_\nu, R_\nu)}) v_{x_j}^{(\nu)\beta} \varphi_{x_i}^\alpha dz, \qquad \forall \varphi \in C_0^\infty(Q; \boldsymbol{R}^N); \tag{4.9}$$

and, from (4.6), (4.7):

$$V^{(\nu)}(0, 1) = \frac{1}{\operatorname{meas} Q}\int_Q |v^{(\nu)}|^2 dz = 1, \tag{4.10}$$

$$V^{(\nu)}(0, \tau) > 2c_5 \tau^2. \tag{4.11}$$

From (4.10), passing in case to a subsequence, we get

$$v^{(\nu)} \to v \quad \text{weakly in } L_2(Q; \boldsymbol{R}^N), \tag{4.12}$$

$$\varepsilon_\nu v^{(\nu)} \to 0 \quad \text{a.e. in } Q, \tag{4.13}$$

$$z_\nu \to \bar{z}; \quad u^{(\nu)}_{Q(x_\nu, R_\nu)} \to \lambda, \tag{4.14}$$

whence

$$a_{ij}^{\alpha\beta}(x_\nu + R_\nu x, t_\nu + R_\nu^2 t;\ \varepsilon_\nu v^{(\nu)} + u^{(\nu)}_{Q(x_\nu, R_\nu)}) \to a_{ij}^{\alpha\beta}(\bar{z}, \lambda) \quad \text{a.e. in } Q. \tag{4.15}$$

It follows from Lemma 6 that the function v is a weak solution in Q of the system:

$$\int_Q v^\alpha \varphi_t^\alpha dz = \int_Q a_{ij}^{\alpha\beta}(\bar{z}, \lambda) v_{x_j}^\beta \varphi_{x_i}^\alpha dz, \qquad \forall \varphi \in C_0^\infty(Q; \boldsymbol{R}^N),$$

whence, from Lemma 5,

$$V(0, \tau) \leqslant c_5 \tau^2 V(0, 1). \tag{4.16}$$

On the other hand, passing to the limit in (4.10), (4.11), we get

$$V(0, 1) \leqslant \liminf_{\nu\to\infty} V^{(\nu)}(0, 1) = 1, \tag{4.17}$$

$$V(0, \tau) = \lim_{\nu\to\infty} V^{(\nu)}(0, \tau) \geqslant 2c_5 \tau^2, \tag{4.18}$$

and these inequalities contradict (4.16). q.e.d.

The constants ε_0 and R_0 in the preceding lemma depend on τ, M, n, N, and on the coefficients $a_{ij}^{\alpha\beta}$. In the following, we shall emphasize the dependence on M by writing $\varepsilon_0(M)$, $R_0(M)$. The following lemma is essentially an iteration of the preceding.

LEMMA 9. – *Let* $\tau < (2c_5)^{-\frac{1}{2}}$, *and let* u *be a weak solution of* (4.1) *such that for some* $z_0 \in A$ *and for some* $R < R_0(2M) \cap \delta(z_0, \partial A)$, *we have*

$$|u_{Q(z_0,R)}| \leqslant M\,, \tag{4.19}$$

$$U(z_0, R) < \eta_0^2(M)\,, \tag{4.20}$$

where

$$\eta_0(M) = \min\{\varepsilon_0(2M),\ M\tau^{1+n/2}(1-\tau\sqrt{2c_5})\}\,. \tag{4.21}$$

Then for every integer $k \geqslant 0$, *we have*

$$U(z_0, \tau^k R) \leqslant (2c_5\tau^2)^k U(z_0, R)\,. \tag{4.22$_k$}$$

PROOF. – We have, for every $\varrho > 0$,

$$|u_{Q(z_0,\tau\varrho)} - u_{Q(z_0,\varrho)}| \leqslant \tau^{-1-n/2} U(z_0, \varrho)^{\frac{1}{2}}\,, \tag{4.23}$$

whence

$$|u_{Q(z_0,\tau^k R)}| \leqslant |u_{Q(z_0,R)}| + \tau^{-1-n/2} \sum_{j=0}^{k-1} U(z_0, \tau^j R)^{\frac{1}{2}}\,. \tag{4.24}$$

For $k = 0$, the inequality (4.22) is trivial. Suppose now that $(4.22)_k$ is true for every $h \leqslant k$. From (4.24) and (4.21), we easily get

$$|u_{Q(z_0,\tau^k R)}| \leqslant 2M\,, \tag{4.25}$$

while from $(4.22)_k$, recalling that $2c_5\tau^2 < 1$, we obtain at once

$$U(z_0, \tau^k R) < \varepsilon_0^2(2M)\,. \tag{4.26}$$

From (4.25), (4.26), and the preceding Lemma, we get $(4.22)_{k+1}$. q.e.d.

We can now prove the first part of the regularity theorem.

THEOREM 1. – *For every weak solution* $u(z)$ *of* (4.1), *there exists an open set* $A_0 \subset A$ *such that* $u(z)$ *is (locally)* α*-hölder continuous (with respect to the metric* δ*) in* A_0, *for every* $\alpha < 1$.

PROOF. – Let $z_0 \in A$ be such that

$$\sup_{0<R<\delta(z_0,\partial A)} |u_{Q(z_0,R)}| \leqslant M/2 < +\infty\,, \tag{4.27}$$

$$\liminf_{R\to 0^+} U(z_0, R) = 0\,. \tag{4.28}$$

Let $\alpha = 1-\sigma$, $0<\sigma<1$, and let $\tau = (2c_5)^{-\frac{1}{2}\sigma}$. There exists an $R < R_0(2M) \cap \cap\, \delta(z_0, \delta A)$ such that

$$U(z_0, R) < \eta_0^2(M)\,.$$

Since $U(z, r)$ and $u_{Q(z,r)}$ are continuous functions of z and r $(r>0)$, there exists an s, $0<s<R/2$, such that for every $z \in Q(z_0, s)$, we have

$$|u_{Q(z,R_z)}| \leqslant M\,, \tag{4.29}$$

$$U(z, R_z) < \eta_0^2(M)\,, \tag{4.30}$$

with $R_z = R - \delta(z, z_0) > R/2$.

From (4.29), (4.30), and Lemma 8, we get for every k:

$$U(z, \tau^k R_z) \leqslant (2c_5\tau^2)^k\, U(z, R_z) \leqslant \tau^{2k(1-\sigma)}\eta_0^2\,. \tag{4.31}$$

For every ϱ, $0<\varrho<R/2<R_z$, let h be the integer such that

$$\tau^{h+1}R_z < \varrho \leqslant \tau^h R_z\,.$$

We have

$$U(z, \varrho) \leqslant \left(\frac{\varrho}{\tau^{h+1}R_z}\right)^{n+2} U(z, \varrho) \leqslant \tau^{-n-2}\, U(z, \tau^h R_z) \leqslant \tau^{-n-4+2\sigma}\left(\frac{\varrho}{R_x}\right)^{2(1-\sigma)}\eta_0^2\,, \tag{4.32}$$

whence

$$U(z, \varrho) \leqslant \left(\frac{2}{R}\right)^{2-2\sigma}\eta_0^2 (2c_5)^{((n+4)/2\sigma)-1}\varrho^{2(1-\sigma)} \tag{4.33}$$

for every $z \in Q(z_0, s)$ and every $\varrho < R/2$.

It is easily seen that (4.33) implies that $u(x)$ belongs to $\mathfrak{L}^{2,\mu}\big(Q(z_0, s), \delta\big)$ with $\mu = 1 + \big(2/(n+2)\big)(1-\sigma)$, so that the theorem follows at once from Lemma 3. q.e.d.

REMARK 1. – The set $\Sigma = A - A_0$, which is closed in A, will be called the singular set of $u(z)$. A point z_0 is not in Σ if and only if (4.27) and (4.28) are satisfied.

With minor charges in the argument, one can show as in [6] that if the coefficients $a_{ij}^{\alpha\beta}$ are *uniformly continuous* in $\bar{A} \times \boldsymbol{R}^N$, then (4.28) alone is sufficient for the regularity of z_0.

5. – The singular set.

In this Section, we shall obtain an upper bound for the Hausdorff dimension of the singular set Σ.

LEMMA 10. – *Let $u(z)$ be a function in $L_2(A)$, and let $z_0 \in A$ be such that*

$$U(z_0, R) \leqslant c_6 R^{2\varepsilon} \tag{5.1}$$

for some $\varepsilon > 0$ and some c_6 depending on u and z_0. Then there exists, and is finite, the limit

$$\lim_{\varrho \to 0^+} u_{Q(z_0, \varrho)} .$$

PROOF. – Let $f(r) = u_{Q(z_0, r)}$. We have, for $z \in A$ and $0 < \sigma < \varrho < \delta(z_0, \partial A)$,

$$|f(\varrho) - f(\sigma)|^2 \leqslant 2\left\{|u(z) - f(\varrho)|^2 + |u(z) - f(\sigma)|^2\right\}.$$

Integrating over $Q(z_0, \sigma)$,

$$\sigma^{n+2}|f(\varrho) - f(\sigma)|^2 \leqslant c_7\left\{\varrho^{n+2}\, U(z_0, \varrho) + \sigma^{n+2}\, U(z_0, \sigma)\right\},$$

whence, using (5.1),

$$|f(\varrho) - f(\sigma)| \leqslant c_8 \sigma^{-1-n/2}\{\varrho^{n+2+2\varepsilon} + \sigma^{n+2+2\varepsilon}\}^{\frac{1}{2}}. \tag{5.2}$$

In particular, we have

$$|f(2^{-i}\varrho) - f(2^{-i-1}\varrho)| \leqslant c_9 2^{-i\varepsilon}\varrho^{\varepsilon} ,$$

and, if $h < k$,

$$|f(2^{-h}\varrho) - f(2^{-k}\varrho)| \leqslant c_9 \varrho^{\varepsilon} \sum_{j=h}^{k-1} 2^{-j\varepsilon}, \tag{5.3}$$

so that the sequence $f(2^{-h}\varrho)$ converges, as $h \to \infty$, to some real number λ. It is easily seen that λ does not depend on ϱ; in fact, from (5.2) we get

$$|f(2^{-i}\varrho) - f(2^{-i}r)| \leqslant c_8 \left\{\frac{\varrho^{n+2+2\varepsilon} + r^{n+2+2\varepsilon}}{\max[\varrho, r]^{n+2}}\right\}^{\frac{1}{2}} 2^{-i\varepsilon}, \tag{5.4}$$

and the right-hand side of (5.4) tends to zero as $i \to \infty$. Finally, choosing $h = 0$ in (5.3) and letting $k \to \infty$, we get

$$|f(\varrho) - \lambda| \leqslant c_{10} \varrho^{\varepsilon} ,$$

which proves the lemma. q.e.d.

REMARK 2. – The preceding lemma and Remark 1 show that a point z_0 is regular if there exist constants $\varepsilon > 0$ and c_5 such that

$$U(z_0, R) \leqslant c_6 R^{2\varepsilon}$$

for every $R < \delta(z_0, \partial A)$; or, what is the same, if there exists a $\sigma > 0$ such that

$$\lim_{R\to 0^+} R^{-\sigma} U(z_0, R) = 0 . \tag{5.5}$$

Let us observe that if the coefficients are uniformly continuous, then it is sufficient to require (5.5) with $\sigma = 0$. This will give a slight difference between the two cases.

Let us now recall the Hausdorff measure.

DEFINITION 4. – *Let δ be the metric* (2.4), *and let α be a non-negative real number, The α-dimensional Hausdorff measure of a set $X \subset \mathbf{R}^{n+1}$, with respect to the metric δ. is given by*

$$\mathcal{H}_\alpha(X;\,\delta) = \lim_{\varepsilon\to 0^+} \inf\left\{\sum_i \delta(x_i)^\alpha;\ \bigcup_i X_i \supset X;\ \delta(X_i) < \varepsilon\right\}.$$

The δ-Hausdorff dimension of a set X is the infimum of the numbers α such that $\mathcal{H}_\alpha(X;\,\delta) = 0$.

The next lemma is a simple modification of Theorem 1 of [5].

LEMMA 11. – *Let $f \in L_{1,\mathrm{loc}}(A)$, and, for $0 < \alpha < n + 2$, let*

$$F_\alpha = \left\{z_0 \in A : \limsup_{\varrho\to 0^+} \varrho^{-\alpha} \int_{Q(z_0,\varrho)} |f|\,dz > 0\right\}.$$

Then we have

$$\mathcal{H}_\alpha(F_\alpha,\, \delta) = 0 .$$

The following result concludes the proof of the regularity theorem.

THEOREM 2. – *Let $u \in W^{1,\frac{1}{2}}_{p,\mathrm{loc}}(A;\, \mathbf{R}^n)$, $p \geqslant 2$, be a weak solution of* (4.1). *Then for every $\sigma > 0$, we have*

$$\mathcal{H}_{n+2-p+\sigma}(\Sigma,\, \delta) = 0 . \tag{5.6}$$

If the coefficients are uniformly continuous, we can take $\sigma = 0$ in (5.6).

REMARK 3. – From Lemma 4, it follows at once that for every weak solution $u(z)$ of (4.1), we have $\mathcal{H}_{n+\sigma}(\Sigma) = 0$, $\forall \sigma > 0$.

PROOF. OF THEOREM 2. – According to Remark 2, it is sufficient to prove that $\mathcal{H}_{n+2-p+\sigma}(E_\sigma,\, \delta) = 0$, where

$$E_\sigma = \left\{z_0 \in A : \limsup_{R\to 0^+} R^{-2\sigma/p}\, U(z_0, R) > 0\right\}.$$

We have

$$R^{-2\sigma/p}\, U(z_0, R) \leqslant c_{11}\left\{R^{-n-2-\sigma} \int_{Q(z_0,R)} |u - u_{Q(z_0,R)}|^p\, dz\right\}^{2/p},$$

and, by Lemma 1,

$$R^{-2\sigma/p}\,U(z_0, R) \leqslant c_{12}\left\{\left[R^{p-2-n-\sigma}\int\limits_{Q(z_0,R)}\sum_{i=1}^{n}|u_{x_i}|^p\,dz\right]^{2/p}+\right.$$

$$\left.+\left[R^{p-n-2-\sigma}\int\limits_{B(x_0,R)}dx\int\limits_{A(t_0,R)\times(A(t_0,R)}\frac{|u(x,t)-u(x,s)|^p}{|t-s|^{1+p/2}}\,dt\,ds\right]^{2/p}\right\}.$$

The functions $|u_{x_i}|^p$ belong to $L_{1,\text{loc}}(A)$, whence, by Lemma 11, the first integral at the right-hand side tends to zero $\mathcal{H}_{n+2-p+\sigma}$ almost everywhere.

For the second integral, we observe that the function

$$f(x, s, t) = \frac{|u(x,t)-u(x,s)|^p}{|t-s|^{1+p/2}}$$

belongs to $L_{1,\text{loc}}(\tilde{A})$, $\tilde{A} = \Omega\times(0, T)\times(0, T)$. If we denote by $\xi = (x, t, s)$ a point in $\tilde{A}$, by $\tilde{\delta}$ the metric

$$\tilde{\delta}(\xi_1, \xi_2) = \max\left\{|x_1-x_2|,\ |t_1-t_2|^{\frac12},\ |s_1-s_2|^{\frac12}\right\},$$

and by $\tilde{Q}$ the ball in the metric $\tilde{\delta}$, we get easily that

$$R^{p-n-2-\sigma}\int\limits_{\tilde{Q}(\xi_0,R)} f(\xi)\,d\xi$$

tends to zero $\tilde{\mathcal{H}}_{n+2-p+\sigma}$ almost everywhere. From that remark, it follows at once that $\mathcal{H}_{n+2-p+\sigma}(E_\sigma, \delta) = 0$. q.e.d.

In the case of elliptic systems in $\boldsymbol{R}^n$, one was able to prove (see [6] and [5]) that

$$H_{n-p+\sigma}(\Sigma) = 0\,, \qquad \forall\sigma > 0\,, \tag{5.8}$$

for continuous coefficients, and

$$H_{n-p}(\Sigma) = 0 \tag{5.9}$$

with the assumption of uniform continuity, H_α being the α-dimensional Hausdorff measure with respect to the usual metric in $\boldsymbol{R}^n$. Comparing (5.8) and (5.9) with the conclusion of Theorem 2, it seems at first that the introduction of the time variable gives a jump of two in the dimension of the singular set. The following argument shows that this is not the case. Let us first observe that the restriction of the metric δ to the t-axis gives the metric $d(t_1, t_2) = |t_1-t_2|^{\frac12}$ so that the Hausdorff measure $\mathcal{H}_\alpha(X, d)$ verifies

$$\mathcal{H}_\alpha(X, d) = H_{\alpha/2}(X)\,, \qquad \forall X\subset \boldsymbol{R}_t\,,$$

H_β being the usual Hausdorff measure in $\boldsymbol{R}^1$.

On the other hand, δ coincides with the usual metric when restricted to $\boldsymbol{R}^n$, so that the Hausdorff measure is also the same:

$$\mathcal{H}_\alpha(X, \delta) = X_\alpha(X)\,, \qquad \forall X \subset \boldsymbol{R}^n_x\,.$$

Finally, if π_1 and π_2 are the projection operators on $\boldsymbol{R}$ and $\boldsymbol{R}^n$, respectively, we have the following inequality ([4], Th. 2.10.25):

$$\int_{\mathbf{R}}^{*} H_\alpha\{\pi_2(\Sigma \cap \pi_1^{-1}(t))\}\, dH_{\beta/2}(t) \leqslant \mathcal{H}_{\alpha+\beta}(\Sigma, \delta) \tag{5.10}$$

for every non-negative α, β.

In particular, if we choose $\beta = 2$ and $\alpha = n - p + \sigma$, we conclude from (5.10) and (5.6) that

$$H_{n+\sigma-p}(\Sigma \cap \pi_1^{-1}(t)) = 0 \qquad \text{for almost every } t\,. \tag{5.11}$$

A similar result, with $\sigma = 0$, holds for uniformly continuous coefficients.

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