A Note on the Disturbed Equations of Motion in the Non-Holonomic Coordinates.

DJORDJE S. DJUKIĆ (Novi Sad, Yugoslavia) (*)

Summary. – *The aim of this article is to establish the disturbed equations of motion in the non-holonomic coordinates. The governing differential equations of non disturbed motion are the Hamel-Boltzman equations. The disturbed equations obtained in this article can be used for consideration in holonomic and non-holonomic dynamics systems.*

1. – Introduction.

In many physical cases it is more convenient to study the problem under consideration not by original coordinate but by non-holonomic coordinates (see for example [7], [8], [9], [10]). In the present literature there are not general approach about stability of motion in the non-holonomic coordinates (see for example [1], [2], [3], exspecially, for exaustive review, paper [1]).

In the paper the following conventions will be observed:

1) Greek indices α, β and γ unrepeated imply a range of values from $m+1$ to n , and, when repeated in a single term, summation over that range.

2) Small italic indices a, b and c unrepeated imply a range of values from 1 to m , and, when repeated in a single term, summation over that range.

3) All another indices, greek or italic, takes values from 1 to n . The summation convention is again employed throughout.

Let us consider a holonomic mechanical system of n degrees of freedom. If the q^i are regarded as the generalized coordinates, and if t is the time, for a large class of dynamical systems, the kinetic energy is given by a quadratic form of the generalized velocity $\dot{q}^i$

$$(1) \quad T = \frac{1}{2} a_{rs} \dot{q}^r \dot{q}^s,$$

where a_{rs} is the fundamental metric tensor of Riemannian configuration space V_n .

Let us study the problem not by original velocities $\dot{q}^i$, but by their non-integrable linear combinations

$$(2) \quad \omega^\sigma = \varphi_i^\sigma \dot{q}^i,$$

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where φ_i^σ are given functions of coordinates q^i . The inverse expression of (2) is

$$(3) \quad \dot{q}^i = \varphi_\sigma^i \omega^\sigma,$$

where $\varphi_\sigma^i \varphi_j^\sigma = \delta_j^i$ and $\varphi_\sigma^i \varphi_i^\sigma = \delta_\sigma^\sigma$. Putting (3) into the expression of kinetic energy we have

$$(4) \quad \tilde{T} = \frac{1}{2} g_{\sigma\mu} \omega^\sigma \omega^\mu,$$

where

$$(5) \quad g_{\sigma\mu} = a_{rs} \varphi_\sigma^r \varphi_\mu^s$$

is the metric tensor with respect to the non-holonomic coordinates. With (2) can be connected the n linear differential forms

$$(6) \quad d\pi^\sigma = \varphi_i^\sigma dq^i,$$

where $d\pi^\sigma$ are differentials of non-holonomic coordinates.

Explicit form of the Hamel-Boltzman's equations of motion, for the dynamical systems with the kinetic energy (4), are (see [7], p. 176)

$$(7) \quad \dot{\omega}^\kappa + A_{\sigma\varrho}^\kappa \omega^\sigma \omega^\varrho = H^\kappa.$$

Here,

$$(8) \quad A_{\sigma\varrho}^\kappa = I_{\sigma\varrho}^\kappa + g^{\kappa\nu} g_{\tau\varrho} \gamma_{\sigma\nu}^\tau,$$

are coefficients of connection in non-holonomic coordinates,

$$(9) \quad \gamma_{\sigma\nu}^\tau = \varphi_\sigma^p \varphi_\nu^r \left(\frac{\partial \varphi_p^\tau}{\partial q^r} - \frac{\partial \varphi_r^\tau}{\partial q^p} \right)$$

are so called objects of anholonomy,

$$(10) \quad \Gamma_{\nu\varrho}^i = g^{i\sigma} \Gamma_{\sigma\nu\varrho}$$

and

$$(11) \quad \Gamma_{\sigma,\nu\varrho} = \frac{1}{2} \left(\frac{\partial g_{\sigma\nu}}{\partial \pi^\varrho} + \frac{\partial g_{\sigma\varrho}}{\partial \pi^\nu} - \frac{\partial g_{\nu\varrho}}{\partial \pi^\sigma} \right),$$

are Christoffel's symbols of the second and first order with respect to the non-holonomic coordinates. If the generalized external forces, acting on the system, are $Q_i(t, q, \dot{q})$, than generalized forces with respect to the non-holonomic coordinates are

$$(12) \quad \Pi_\nu = \varphi_\nu^i Q_i.$$

Putting (3) into (12) we have that Π_v is the functions of t , q^i and ω^σ , i.e.

$$(13) \quad \Pi_v = \Pi_v(t, q, \omega).$$

It is known (see for example [7]), when the integrability conditions for system (6) are not filled, than it is impossible to found explicit functional dependence $\pi^i = \pi^i(q^j)$, or inverse $q^i = q^i(\pi^j)$. In such cases coefficients of the connections $A_{e\sigma}^*$ are the functions of non-holonomic coordinates π^i only formaly. So, in this paper, will be assumed that coefficients $A_{e\sigma}^*$ are the functions of generalized coordinates q^i .

2. – The disturbed equations of motion in the non-holonomic coordinates.

The equations of an undisturbed path C of the representative single particle in the configuration space, and with respect to non-holonomic coordinates are (7). Let q^σ be the coordinates of a point P of C and $*q^\sigma = q^\sigma + x^\sigma$ the coordinates of the corresponding point $*P$ of the disturbed path $*C$, x^σ being infinitesimal. The vector x^σ is the disturbance vector (see [2], p. 78). In a point P of C the representative particle have quasi-velocity ω^* . In the corresponding point $*P$ of the disturbed path $*C$, the quasi-velocity vector have components $*\omega^* = \omega^* + \Omega^*$. We shall call vector Ω^* the disturbance vector of the quasi-velocity vector. The condition for stability of motion is that x (the magnitude of the vector x^σ) and Ω (the magnitude of the vector Ω^*) should remain permanently small.

Let us substitute $*\omega^* = \omega^* + \Omega^*$ in (7), since $*C$ is also the path of the representative particle, and obtain

$$(14) \quad \dot{\omega}^* + \dot{\Omega}^* + *A_{e\sigma}^*(\omega^e + \Omega^e)(\omega^\sigma + \Omega^\sigma) = *II^* + \varepsilon\Phi^*,$$

where the asterisk indicates quantities to be calculated at $*P$. Here ε is a small parameter and $\Phi^*(t, q, \omega)$ contravariant components of the external forces which are acting on the system only during the disturbed motion. Expanding quantities $*A_{e\sigma}^*$ and $*II^*$, and retaining only first powers of small quantities we have

$$(15) \quad \begin{cases} *A_{e\sigma}^* = A_{e\sigma}^* + \frac{\partial A_{e\sigma}^*}{\partial q^\delta} x^\delta, \\ *II^* = II^* + \frac{\partial II^*}{\partial q^\delta} x^\delta + \frac{\partial II^*}{\partial \omega^\delta} \Omega^\delta. \end{cases}$$

Combining (15), (14) and (7) we have

$$(16) \quad \dot{\Omega}^* + x^\delta \omega^e \omega^\sigma \frac{\partial A_{e\sigma}^*}{\partial q^\delta} + A_{e\sigma}^*(\Omega^e \omega^\sigma + \omega^e \Omega^\sigma) = x^\delta \frac{\partial II^*}{\partial q^\delta} + \Omega^\delta \frac{\partial II^*}{\partial \omega^\delta} + \varepsilon\Phi^*.$$

The disturbed quasi-velocities must also be in the consistent with the relations (2)

$$(17) \quad {}^*\omega^\sigma = {}^*\varphi_i^\sigma {}^*\dot{q}^i,$$

or after expanding

$$(18) \quad \omega^\sigma + \Omega^\sigma = \left(\varphi_i^\sigma + \frac{\partial \varphi_i^\sigma}{\partial q^\delta} x^\delta \right) (\dot{q}^i + \dot{x}^i).$$

Taking in account only small quantities of the first order and using (2) we have

$$(19) \quad \Omega^\sigma = \varphi_i^\sigma \dot{x}^i + x^\delta \varphi_\delta^i \omega^e \frac{\partial \varphi_i^\sigma}{\partial q^\delta}.$$

This is the connections between the disturbances of the generalized coordinates x^i and the disturbances of the quassi-velocity's components Ω^σ . It is easy to get inverse relations of (19) (using $\varphi_\sigma^i \varphi_j^\sigma = \delta_j^i$)

$$(20) \quad \dot{x}^i = \varphi_\sigma^i \Omega^\sigma + x^\delta \omega^\sigma \frac{\partial \varphi_\sigma^i}{\partial q^\delta}.$$

The stability of motion can be estimated using n differential equations (16) together with n relations (19) or (20). As the solution of this systems we have $2n$ quantities ω^σ and x^i , i.e. the disturbances of the quassi-velocity vector and the disturbances of the generalized coordinates, as a functions of the time.

Let us consider a special case of the relations (2) defined with

$$(21) \quad \varphi_i^\sigma = \delta_i^\sigma; \quad \varphi_\sigma^i = \delta_\sigma^i.$$

Than, using (2), (9), (8), (19), (12), (5), (10) and (11) we have

$$(22) \quad \begin{cases} \omega^\sigma = \dot{q}^\sigma; & H_\nu = Q_\nu; & \Omega^\sigma = \dot{x}^\sigma \\ \gamma_{\sigma\nu}^\tau = 0; & g_{\sigma\mu} = a_{\sigma\mu}; & A_{\sigma\sigma}^\kappa = \left\{ \begin{matrix} \kappa \\ \sigma \end{matrix} \right\} \end{cases}$$

where $\left\{ \begin{matrix} \kappa \\ \sigma \end{matrix} \right\}$ are Christoffel's symbols of the second order formed with respect to the fundamental metric tensor $a_{\sigma\mu}$.

Combining (21), (22) and (16) we have

$$(23) \quad \ddot{x}^\kappa + 2 \left\{ \begin{matrix} \kappa \\ \sigma \end{matrix} \right\} \dot{x}^\sigma \dot{q}^\sigma + \dot{q}^\sigma \dot{q}^\sigma x^\delta \cdot \frac{\partial}{\partial q^\delta} \left\{ \begin{matrix} \kappa \\ \sigma \end{matrix} \right\} - x^\delta \frac{\partial Q^\kappa}{\partial q^\delta} = \dot{x}^\delta \frac{\partial Q^\kappa}{\partial \dot{q}^\delta} + \varepsilon \Phi^\kappa.$$

This differential equations are the original Synge's equations of the disturbances (compare (23) with [2], p. 78 and [8], p. 623).

3. – The non-holonomic dynamical systems.

Derived equations (16) and (19) can be used for the non-holonomic dynamical systems.

Let us suppose that a holonomic dynamical system is subjected to a $n - m$ non-holonomic constraints of the type

$$(24) \quad \varphi_i^\alpha \dot{q}^i = 0 ,$$

where φ_i^α are a known functions of the generalized coordinates q^i .

The non-holonomic coordinates we will chose in such a way that $d\pi^1, \dots, d\pi^m$ are independent, and another $n - m$ identically equal to zero (from the equations of the non-holonomic constraints)

$$(25) \quad \dot{\pi}^\alpha \equiv \omega^\alpha = \varphi_i^\alpha \dot{q}^i = 0 .$$

For the disturbed motion we have that the velocities must satisfy the disturbed non-holonomic constraints, i.e.

$$(26) \quad {}^*\omega^\alpha = {}^*\varphi_i^\alpha {}^*\dot{q}^i = 0 .$$

Using (25), and remembering that ${}^*\omega^\alpha = \omega^\alpha + \Omega^\alpha$, we have that the disturbances of $n - m$ quassi-velocities are also equal to zero

$$(27) \quad \Omega^\alpha = 0 .$$

Let be R_μ the generalized forces of constraints with respect to the non-holonomic coordinates. If this forces are ideal than we have that

$$(28) \quad R_a = 0 \quad \text{and} \quad R_\alpha \neq 0 .$$

In this case the explicit form of the first m equations of motion (7) is (see [7], p. 177)

$$(29) \quad \dot{\omega}^a + A_{bc}^a \omega^b \omega^c = II^a .$$

From the last $n - m$ equations of the system (7) we can found the forces of the non-holonomic constraints

$$(30) \quad R^\alpha = A_{cb}^\alpha \omega^c \omega^b - II^\alpha .$$

Taking in account (27) and (25), we have, from (16) m equations for the disturbances of the quassi-velocity's vector

$$(31) \quad \dot{\Omega}^a + x^\delta \omega^b \omega^c \frac{\partial A_{bc}^a}{\partial q^\delta} + A_{bc}^a (\Omega^b \omega^c + \omega^b \Omega^c) = x^\delta \frac{\partial II^a}{\partial q^\delta} + \Omega^b \frac{\partial II^a}{\partial \omega^b} + \varepsilon \Phi^a .$$

This m differential equations must be solved together with the non-holonomic constraints for the disturbances, which we get combining (19), (20), (25) and (27)

$$(32) \quad \Omega^a = \varphi_i^a \dot{x}^i + x^\delta \varphi_b^i \omega^b \frac{\partial \varphi_i^a}{\partial q^\delta},$$

or in the inverse form

$$(33) \quad \dot{x}^i = \varphi_a^i \Omega^a + x^\delta \omega^a \frac{\partial \varphi_a^i}{\partial q^\delta}.$$

The systems (31) and (33) gives $m + n$ differential equations of the first order, from which, as solution, we have m quantities Ω^a and n quantities x^i in the functional dependence of the time.

Combining last $n - m$ equations (16) with (27), (25), (28) and (30) we have the disturbances of the forces of the non-holonomic constraints in the form

$$(34) \quad \delta R^\alpha = x^\delta \omega^b \omega^c \frac{\partial A_{bc}^\alpha}{\partial q^\delta} + A_{bc}^\alpha (\Omega^b \omega^c + \omega^b \Omega^c) - x^\delta \frac{\partial II^\alpha}{\partial q^\delta} - \Omega^a \frac{\partial II^\alpha}{\partial \omega^a} - \varepsilon \Phi^\alpha.$$

4. - Equations for the magnitudes of the disturbance vector and the disturbance vector of the quasi-velocities.

Let us introduce the unit disturbance vector μ^r , with respect to the metric tensor a_{rs} of the configuration Riemannian space V_n , and co-directional with x^r

$$(35) \quad x^r = x \mu^r,$$

where

$$(36) \quad a_{rs} \mu^r \mu^s = 1; \quad x = \sqrt{a_{rs} x^r x^s},$$

and the unit disturbance vector of the quasi-velocities κ^σ , with respect to the metric tensor $g_{\sigma\mu}$ of the non-holonomic space, and co-directional with Ω^σ

$$(37) \quad \Omega^\sigma = \Omega \kappa^\sigma,$$

where

$$(38) \quad g_{\sigma\mu} \kappa^\sigma \kappa^\mu = 1, \quad \Omega = \sqrt{g_{\sigma\mu} \Omega^\sigma \Omega^\mu}.$$

Here x is the magnitude of the disturbance vector and Ω is the magnitude of the disturbance vector of the quasi-velocities. Using (35), (36), (37), (38) and (3)

it is easy to obtain next relations

$$(39) \quad \mu_r \dot{x}^r = \dot{x} - \frac{x}{2} \frac{\partial a_{rs}}{\partial q^k} \varphi_\sigma^k \omega^\sigma \mu^r \mu^s,$$

$$(40) \quad \kappa_\sigma \dot{\Omega}^\sigma = \dot{\Omega} - \frac{\Omega}{2} \frac{\partial g_{\sigma\mu}}{\partial q^k} \varphi_\nu^k \omega^\nu \kappa^\sigma \kappa^\mu,$$

where μ_r and κ_σ are covariant components of the vectors μ^r and κ^σ .

Combining (39), (40), (16) and (20) we have

$$(41) \quad \dot{\Omega} + I_1 \Omega + I_2 x = I_3,$$

$$(42) \quad \dot{x} + I_4 x = I_5 \Omega,$$

where invariants have the forms

$$(43) \quad I_1 = \left(A_{e\sigma}^\kappa \omega^\sigma \kappa^e + A_{e\sigma}^\kappa \omega^e \kappa^\sigma - \frac{\partial \Pi^\kappa}{\partial \omega^\delta} \kappa^\delta \right) \kappa_\kappa - \frac{1}{2} \frac{\partial g_{\sigma\mu}}{\partial q^k} \varphi_\nu^k \omega^\nu \kappa^\sigma \kappa^\mu,$$

$$(44) \quad I_2 = \left(\frac{\partial A_{e\sigma}^\kappa}{\partial q^\delta} \omega^e \omega^\sigma - \frac{\partial \Pi^\kappa}{\partial q^\delta} \right) \kappa_\kappa \mu^\delta,$$

$$(45) \quad I_3 = \varepsilon \kappa_\kappa \Phi^\kappa,$$

$$(46) \quad I_4 = -\frac{1}{2} \frac{\partial a_{rs}}{\partial q^k} \varphi_\sigma^k \omega^\sigma \mu^r \mu^s - \omega^e \mu_i \mu^\delta \frac{\partial \varphi_e^i}{\partial q^\delta},$$

$$(47) \quad I_5 = \varphi_\sigma^j \mu_j \kappa^\sigma.$$

Equations (41) and (42) may be called the invariant equations for the magnitudes of the disturbance vector and the disturbance of the quasi-velocity vector.

For the non-holonomic dynamical systems, in the presence of $n - m$ non-holonomic constraints (24), from (27) and (37), we have that

$$(48) \quad \kappa^\alpha = 0; \quad \kappa^a \neq 0.$$

Using (25), (27), (31), (33), (39), (40) and (48) we will get the invariant equations for non-holonomic dynamical system. These equations are of the same type as (41) and (42). The difference is only in the values of the invariants (43) ÷ (47). The invariants for non-holonomic dynamical system we can get from (43) ÷ (47) if we take in account (48), (25), (27) and if we instead of the index κ write index a .

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