The Maximum Principle for Control Distributions.

GEORGE K. FRANCIS (Champaign-Urbana, Illinois, U.S.A.) (*)

Summary. — *Pontrjagin's maximum principle holds for a switchable distribution of C^1 -regular nonautonomous vector fields on a finite dimensional C^2 -manifold. The flow-theoretic proof does not require that the distribution contains those autonomous fields that correspond to constant controllers in the classical formulation.*

1. — Introduction.

Many problems in the theory of optimal control have been advantageously reformulated in a geometrical setting. We follow the exposition of F. ALBRECHT [1, p. 27], but the reader may wish to refer also to [6, p. 36; 4, p. 6]. Let X be a differentiable manifold of class C^2 , let Ω be a topological space, and let $\xi: X \times \Omega \rightarrow TX$ be a controllable vector field. That is, for each $\omega \in \Omega$, let $x \rightarrow \xi(x, \omega)$ be a C^1 -vector field on the manifold (taking values in its tangent bundle TX) such that ξ , together with its partial differential with respect to x varies continuously in (x, ω) . The reader may think of X as realizing the constraints on the state variables, and Ω as realizing the restraints on the control variables of a parametrized system ξ of first order ordinary, not necessarily linear, differential equations. Let $\underline{U}$ be a class of not necessarily continuous paths $u: I_u \rightarrow \Omega$, I_u a compact real interval proper to the control u . For a given control $u \in \underline{U}$ with $I_u = [t_0, t_1]$, a response at $x_0 \in X$ is a curve $\alpha: [t_0, t^*] \rightarrow X$, $t_0 < t^* \leq t_1$, such that $\alpha(t_0) = x_0$ and $\alpha'(t) = \xi(\alpha(t), u(t))$ for almost every $t \in [t_0, t^*]$. Accordingly, we impose certain regularity conditions on the elements of $\underline{U}$ in order for responses to exist, be unique, and to vary smoothly with the initial data, we define the attainable set $A(x_0, \xi, \underline{U})$ to be the entirety of all endpoints $\alpha(t^*)$ of responses to $\underline{U}$ at x_0 . If X is finite dimensional and $\underline{U}$ is closed under certain operations on its elements, a maximum principle, in the sense of Pontrjagin, gives necessary conditions for a response to terminate on the topological boundary of the attainable set. The relation of such a response to an optimal solution of an associated problem of minimizing a certain cost functional is explained in [1, p. 64; 6, p. 309].

Since a response at x_0 is in any case an integral curve of a nonautonomous dynamical system on X , $\dot{x} = \varphi(x, t)$ where $\varphi(x, t) = \xi(x, u(t))$, it is the philosophy of this paper to prove a maximum principle for a distribution Φ of time dependent

(*) Entrata in Redazione il 12 marzo 1972.

vector fields $\varphi(x, t)$ on X in terms of the flows generated by them. For this, a field $\varphi \in \Phi$ must satisfy certain regularity conditions (Section 2) for the flow to exist, and the family Φ must be closed under the switching operation (Section 3) for the maximum principle to hold.

The principal reason for dispensing with the notion of a control or « strategy » space Ω in this paper is notational economy. In a subsequent study [9], however, we shall show how a large class of control data $(X, \Omega, \xi, \underline{U})$ and control distributions (X, Φ) may be canonically « embedded » in a « universal » control problem $(X, \Gamma X, ev, \underline{V})$, where $\underline{V}$ is a switchable family of regulated paths in a Fréchet space ΓX of vector fields on X and ev is the evaluation map $X \times \Gamma X \rightarrow TX$.

Section 4 is devoted to our « flow theoretic » version of F. Albrecht's more « field theoretic » proof of the main theorem [1, pp. 56-63]. Our modification, and hopefully, simplification of his ingenious approach gains a modicum of generality by lowering the necessary class of X from C^3 to C^2 , and by avoiding the use of constant controllers entirely. Our exposition is cast in terms of the coordinate free calculus of Banach manifolds [2, 5, 8] in order to pinpoint exactly the two places that finite dimensionality is required in the proof. (It is known that the maximum principle fails in infinite dimensional manifolds.)

2. – Regular vector fields.

This section serves to summarize known sufficient conditions for a time dependent vector field to generate local flows, and to introduce the notation that will be used in the remainder of this paper. Lemma A summarizes the context of regular vector fields of [1], which is based on Chapter X of [2].

An abstract *local flow* on X at x_0 may be defined as a smooth map $\mu: I \times J \rightarrow C^1(U, X)$, where $I \times J \times U$ is open in $R \times R \times X$ and contains (t_0, t_0, x_0) , such that $\mu[t, t](x) = x$ for all $(t, x) \in (I \cap J) \times U$, and $\mu[s, t] \circ \mu[r, s] = \mu[r, t]$ wherever this expression is well defined. Note that if both (s, t) and (t, s) are in $I \times J$, then $\mu[s, t]$ is a diffeomorphism on a subneighborhood of x_0 inside U .

Recall [2, p. 145] that a *regulated path* in a topological space is a function $p(t)$ from a real interval I such that right (left) limits $\lim_{s \rightarrow 0^+} p(t \pm s) = p(t^\pm)$ exist whenever $t \pm s \in I$ for some small $s > 0$. Recall [1, p. 35] that a regulated path in a Banach space has at most countably many discontinuities.

LEMMA A. – Let E be a Banach space, I a real interval, U open in E , and $f: U \times I \rightarrow E$ a function satisfying [1, p. 21]:

- (H1) For every $t \in I$, $f_t(x) = f(x, t)$ is in $C^1(U, E)$,
- (H2) $D_x f(x, t)$ is locally bounded on $U \times I$, and
- (H3) for every $c \in C^0(I, U)$, $f_c(t) = (f(c(t), t), D_x f(c(t), t))$ is a regulated path in $E \times L(E)$, where $D_x f$ is the x -partial derivative of f and $L(E)$ is the Banach space of continuous linear selfmaps of E .

Then [1, pp. 7-12], on some (relatively) open neighborhood of (x_0, t_0, t_0) in $U \times I \times I$ there exists the local flow $\mu = \int f$, where $\int f[s, t](x) = \alpha(t)$ satisfies $\alpha(s) = x$ and $\alpha'(t) = f(\alpha(t), t)$ almost everywhere. The notation « $\int f$ » for the flow is suggestive of the integral equation

$$(2.1) \quad \alpha(t) = x + \int_s^t f(\alpha(\tau), \tau) d\tau.$$

Moreover, the differential $\int f[s, t](x)_* = D_x(\int f[s, t](x))$ of the map $\int f[s, t]$ at x satisfies the variational equation. That is, $\alpha_*(t) = \int f[s, t](x)_*$ satisfies $\alpha_*(s) = I_x$, the identity on $L(E)$, and for almost every t , the equation

$$(2.2) \quad \alpha'_*(t) = D_x f(\alpha(t), t) \cdot \alpha_*(t).$$

A field satisfying (H1-3) is called a *regular vector field*. The next lemma summarizes the process, described in Chapter IV of [5] for instance, of continuing solutions of an ordinary differential equation on a manifold along a particular integral curve. See also [1, pp. 12-26].

LEMMA B. — Let X be a second countable Hausdorff manifold of class C^2 modelled on the separable Banach space E . Let $\varphi: X \times I_\varphi \rightarrow TX$ be a vector field for which Lemma A holds locally. That is, $h_* \circ \varphi \circ (h^{-1} \times id): h(U) \times I \rightarrow E$ is regular, where $(h, h_*): TU \rightarrow h(U) \times E$ is a coordinate chart of TX . (If $\sigma \in C^2(R, U)$ has $\sigma(0) = x$ and $\sigma'(0) = v \in T_x X$, recall that $h_*(v) = dh(\sigma(s))/ds|_{s=0}$ is an isomorphism of $T_x X$ with E). Let $x_0 \in X$ and $[t_0, t_1] \subset I$, be such that $\alpha(t) = \int \varphi[t_0, t](x_0)$ exists for all $t \in [t_0, t_1]$. Then, there exists an open neighborhood U_0 of x_0 , depending on (φ, t_0, t_1) , such that $\mu(t) = \int \varphi[t_0, t]$ is a C^1 diffeomorphism of U_0 into X for each $t \in [t_0, t_1]$. Moreover, $\mu(t_0)(x) = x$ for all $x \in U_0$ and for almost every t , $D_t \mu(t)(x) = \varphi(\mu(t)(x), t)$. The continuous map $[t_0, t] \times TU \rightarrow TX$ given by $\mu(t)(x)_* \cdot v$, where $v \in T_x X$ and $\mu(t)(x)_*$ is the differential of $\mu(t)$ at x , reduces to the identity on $T_x X$ when $t = t_0$.

It should be pointed out that Dieudonné's condition (H3) of Lemma A [2, p. 303] has one surprisingly rich aspect we use later on. Let E, G be a pair of normed linear spaces, U open in E , I a real interval and $f: U \times I \rightarrow G$ a function such that $f_c(t) = f(c(t), t)$ is a regulated path in G for all $c \in C_0(I, U)$. A *selection* on I shall be a function $e: I \rightarrow \{-1, 0, +1\}$ such that $e(t) \neq +1(-1)$ at a right (left) end point of I , but otherwise arbitrary. A *proper selection* is one for which $e(t) \neq 0$ for any t . For a selection, define

$$f^e(x, t) = \lim_{s \rightarrow 0^+} f(x, t + e(t)s) = \begin{cases} f(x, t^+) & \text{if } e(t) = +1, \\ f(x, t) & \text{if } e(t) = 0, \\ f(x, t^-) & \text{if } e(t) = -1, \end{cases}$$

where the limits are taken along the constant path $c = x$. It is a consequence of Proposition 1, below, that this limit is independent of the path chosen. The existence

of every $f(x, t^+)$ is, however, insufficient to guarantee the existence of $f_c(t^+)$ for every c . Take $f(x, t) = \text{sgn}(xt) = -1, 0, +1$ according as xt is negative, zero or positive. While $f(x, t^+)$ exists for all (x, t) , $f(t \sin(t^{-1}), t)$ has no limit as $t \rightarrow 0^+$.

PROPOSITION 1. - If f is as above and $e(t_0) = +1$, then

$$(2.5) \quad \lim f^e(x, t) = f(x_0, t_0^+) \quad \text{as } (x, t) \rightarrow (x_0, t_0) \text{ and } 0 \leq t - t_0.$$

(The analogous assertion holds with $+$ replaced by $-$.)

PROOF. - Assume that, on the contrary, $(x_n, t_n) \rightarrow (x_0, t_0)$, $t_n \geq t_0$, but that $|y_n - y_0| > \varepsilon > 0$, where $y_n = f^e(x_n, t_n)$ and $y_0 = f(x_0, t_0^+)$. Assume, without loss of generality, that the t_n are weakly monotone decreasing and that the x_n lie in a convex neighborhood of x_0 . Recursively, choose small positive δ_n so that

$$|f(x_n, s_n) - f^e(x_n, t_n)| < \frac{1}{2}\varepsilon, \quad \text{for } s_n = t_n + e(t_n)\delta_n,$$

and the s_n are strictly monotone decreasing to t_0 . Setting $r_n = \frac{1}{2}(s_n + s_{n+1})$, construct the polygonal path $c \in C_0([t_0, s_1], U)$ by

$$c(t) = \begin{cases} \frac{s_n - t}{s_n - r_n} x_0 + \frac{t - r_n}{s_n - r_n} x_n & \text{if } r_n \leq t \leq s_n, \\ \frac{r_n - t}{r_n - s_{n+1}} x_{n+1} + \frac{t - s_{n+1}}{r_n - s_{n+1}} x_0 & \text{if } s_n \leq t \leq r_n, \\ x_0 & \text{if } t = t_0. \end{cases}$$

Continuity at t_0 follows from the fact that $|c(t) - x_0| \leq \sup_{n > m} |x_0 - x_n|$ for all $t \leq s_m$. Set $y^* = f_c(t_0^+)$. Since $f_c(r_n) = f(x_0, r_n)$ and $r_n \rightarrow t_0^+$, it follows that $y^* = y_0$. Another subsequence is $f_c(s_n) = f(x_n, s_n)$, for which $|f_c(s_n) - y_0| > \frac{1}{2}\varepsilon$. Contradiction.

The hypothesis (H.1) in Lemma A, that the field be continuous let alone continuously differentiable in x , seems unnecessarily demanding, especially since the common assumption [5, p. 61] that f be C^1 in both variables has already been liberalized by (H.2-3). Indeed, in a future study [9] we shall show that for a proper selection e , Proposition 1 implies that f_c^e is again regulated for all c , and that $f^e(x, t)$ is continuous in x and locally bounded in (x, t) . Further, replacing $D_x f$ in (H.3) by some weaker linear approximation than the Fréchet partial derivative, we shall demonstrate Lemma A for this properly larger class containing fields that violate (H.2) and hence (H.1).

3. - Control distributions.

For two time dependent vector fields φ and ψ on a manifold X and real numbers, $r_0 \leq r_1$, $s_0 \leq s_1$, the field σ is said to *switch* (φ, r_0, r_1) to (ψ, s_0, s_1) , if $[r_0, r_1] \subset I_\varphi$,

$[s_0, s_1] \subset I_\varphi$, and for all $x \in X$,

$$(3.1) \quad \varphi(x, t) = \begin{cases} \varphi(x, s_0 + t - r_1) & \text{if } r_1 \leq t \leq s_1 + r_1 - s_0. \\ \varphi(x, t) & \text{if } r_0 \leq t < r_1, \end{cases}$$

Note that if $r_0 = r_1$, the first condition is vacuous. For a given $\varepsilon > 0$, the switch is said to be *limited*, *advanced*, *retarded* or *exact*, if $|s_0 - r_1| < \varepsilon$, $r_1 \leq s_0 < r_1 + \varepsilon$, $r_1 - \varepsilon < s_0 \leq r_1$ or $s_0 = r_1$ respectively. Obviously, switching regular fields yields an regular field. A family of regular fields Φ which is closed under switching will be known as a *control distribution* on X . The *attainable set* $A(x_0, \Phi)$ is the set of end points $\varphi[t_0, t_1](x_0)$, where $\varphi \in \Phi$ and $[t_0, t_1] \subset I_\varphi$.

We fix the symbols x_0 , φ , and $t_0 < t_1$, with

$$\alpha(t) = \int \varphi[t_0, t](x_0) \quad \text{and} \quad \alpha_*(t) = \int \varphi[t_0, t](x_0)_*.$$

There are two types of so called *infinitesimal Φ -alternative* tangent vectors at x associated with α ,

$$(\text{Type 1}) \quad v = \alpha_*(s)^{-1} \cdot \varphi(\alpha(s), s^+),$$

$$(\text{Type 2}) \quad v = -\alpha_*(s)^{-1} \cdot \varphi(\alpha(s), s^-),$$

where $s \in [t_0, t_1] \cap I_\varphi$ and $\varphi \in \Phi$ for Type 1, and $t_0 < s \leq t_1$ for Type 2.

THEOREM 1. – If every vector in the tangent space T_0X at x_0 is a finite non-negative linear combination of a fixed finite collection of infinitesimal alternatives associated to the data $(\Phi, x_0, \varphi, t_0, t_1, \alpha)$ as above, then $x_1 = \alpha(t_1)$ is an interior point of the attainable set $A(x_0, \Phi)$.

The integral path α is said to be *Φ -extremal* if a certain adjoint or Hamiltonian equation has a maximal solution. With F. ALBRECHT [1, p. 36] we express this in terms of a nontrivial continuous linear functional h on T_0X such that for almost every $t \in [t_0, t_1]$,

$$(3.2) \quad \sup_{\varphi \in \Phi} h(\alpha_*(t)^{-1} \cdot \varphi(\alpha(t), t)) = h(\alpha_*(t)^{-1} \cdot \varphi(\alpha(t), t)) = 0.$$

Precisely, if $e((t_0, t_1]) = -1 = -e(t_0)$, then for every t

$$\sup_{\varphi \in \Phi} h(\alpha_*(t)^{-1} \cdot \varphi(\alpha(t), t^+)) = h(\alpha_*(t)^{-1} \cdot \varphi^e(\alpha(t), t)) = 0.$$

Roxin's theorem [7], asserting that once an integral path dips into the interior of the attainable set it remains therein, follows easily from the flow Lemma B.

We shall call α a *Φ -boundary*, if $\alpha(t)$ lies in the topological boundary of $A(x_0, \Phi)$ for all $t \in [t_0, t_1]$.

THEOREM 2 (Pontrjagin's Maximum Principle). If X is a finite dimensional manifolds then a Φ -boundary is also a Φ -extremal.

PROOF. — Theorem 2 follows from the observation that if α is not extremal, then the closure of the cone in T_0X generated by the finite nonnegative linear combinations of the infinitesimal Φ -alternatives fills out the entire tangent space [1, p. 37]. If X is finite dimensional, the conditions of Theorem 1 may be met. Hence $\alpha(t_1)$ is not on $\text{Bdy } A(x_0, \Phi)$ and α is not a Φ -boundary.

4. — Proof of theorem 1.

Let $U = U(\varphi, t_0, t_1, x_0)$ be the neighborhood in Lemma B where the flow $\mu(t) = \int [t_0, t]$ takes its source. We shall assume that U is a convex coordinate chart on the manifold and henceforth identify it with its image in E . For an infinitesimal alternative vector v of Type 1, there exists an $\varepsilon_v > 0$ and a neighborhood $U_v \subset U$ of x_0 , both depending on s and φ , such that the one-parameter family of diffeomorphisms,

$$(4.1) \quad g(r) = \mu(s)^{-1} \circ \int \psi[s, s+r] \circ \mu(s),$$

is well defined from U_v into U for $r \in [0, \varepsilon_v]$ and $[s, s+r] \subset I_\varphi$. We shall call g the *alternative local flow* for v . For a Type 2 alternative, this flow is given by

$$(4.2) \quad g(r) = \mu(s)^{-1} \circ \mu(s-r),$$

where, in addition, $\varepsilon_v < s - t_0$ must be assured.

Evidently, $g(0)$ is the identity on U_v , and its differential $g(0)(x)_*$ is the identity on E , for each $x \in U_v$, under the identification $T_x X = E$ we have made.

PROPOSITION 2. — If g is an alternative local flow for v , then for each $x \in U_v$ the curve $g_x(r) = g(r)(x)$ has, for almost every r , a tangent vector given by

$$(4.3) \quad g'_x(r) = \begin{cases} \mu(s)(g_x(r))_*^{-1} \cdot \psi(\mu(s) \circ g(r)(x), s+r) & \text{(Type 1)}, \\ -\mu(s)(g_x(r))_*^{-1} \cdot \varphi(\mu(s) \circ g(r)(x), s-r) & \text{(Type 2)}. \end{cases}$$

Moreover, $g'(0^+)(x_0) = v$, and for $(x, w) \in U_v \times E$,

$$(4.5) \quad \lim_{(r, w) \rightarrow (0^+, x_0)} (g(r)(x), g(r)(x)_* \cdot w, g'_x(r)) = (x_0, w, v).$$

PROOF. — The computation of $g'_x(r)$ follows immediately from the definition of the symbols and the chain rule. The continuous variation with respect to initial data of the solutions of the differential equations in Lemma A and (H.3) suffice for the evaluation of $g'(0^+)(x_0)$.

Supplemented by the generosity of the limit in Proposition 1, this also suffices to establish (4.5).

Let $v_1, v_2, \dots, v_n$ be a finite number of infinitesimal alternatives, labelled in such a way that $s_1 \leq s_2 \leq \dots \leq s_n$ and $s_{i-1} < s_i$ if v_i is of Type 2. Choose common $\varepsilon_0 > 0$ and $U_0 \subset U$ sufficiently small, such that $\varepsilon_0 < s_i - s_{i-1}$ for all Type 2 v_i , and that the composition

$$G(r_1, r_2, \dots, r_n) = g_n(r_n) \circ \dots \circ g_2(r_2) \circ g_1(r_1)$$

is well defined on U_0 with values in U for all $\mathbf{r} = (r_1, r_2, \dots, r_n) \in B^n(\varepsilon_0) = [0, \varepsilon_0]^n$ in R^n . Having identified U with its E -image, we continue to denote the origin in E by x_0 to distinguish it from $\mathbf{0}$, the origin in R^n . Let $F(\mathbf{r}) = G(\mathbf{r})(x_0)$, then $F(\mathbf{0}) = x_0$. Associated with the same set of infinitesimal alternatives is the linear map $L: R^n \rightarrow E: L(\mathbf{r}) = \sum r_i v_i$. In Proposition 3 we show that L is a sufficiently good linear approximation of F so that (Proposition 4) x_0 is an interior point of $F(B^n(\varepsilon_0))$ when L maps the first orthant $R_+^n = \{\mathbf{r} | \mathbf{r} \geq 0\}$ onto E . (By abuse of notation, vector inequalities signify that the inequality holds for all components.) Finally, it remains to show (Proposition 5) that $\mu(t_1) \circ F(B^n(\varepsilon_0)) \subset A(x_0, \Phi)$ in order for $x_1 = \mu(t_1)(x_0)$ to be an interior point of the attainable set.

PROPOSITION 3. — For $\mathbf{r}'$ and $\mathbf{r}''$ in $B^n(\varepsilon_0)$,

$$(4.6) \quad |F(\mathbf{r}'') - F(\mathbf{r}') - L(\mathbf{r}'' - \mathbf{r}')| = |\mathbf{r}'' - \mathbf{r}'| \theta(\mathbf{r}', \mathbf{r}'') ,$$

where θ is a modulus of continuity vanishing as $(\mathbf{r}', \mathbf{r}'') \rightarrow (\mathbf{0}, \mathbf{0})$.

PROOF. — Adding and subtracting terms, we see that LHS(4.6) $\leq |S_1| + |S_2| + \dots + |S_{n+1}|$, where

$$S_i = F(r'_1, \dots, r'_{i-1}, r''_i, \dots, r''_n) - F(r'_1, \dots, r'_i, r''_{i+1}, \dots, r''_n) - (r''_i - r'_i) v_i .$$

For a fixed index i let us abbreviate

$$\mathbf{u} = (r'_1, \dots, r'_{i-1}, 0, \dots, 0) \quad \text{and} \quad \mathbf{w} = (0, \dots, 0, r''_{i+1}, \dots, r''_n) ,$$

and, without loss of clarity, drop the subscript i . Thus

$$\begin{aligned} S &= G(\mathbf{w}) \circ g(r'') \circ F(\mathbf{u}) - G(\mathbf{w}) \circ g(r') \circ F(\mathbf{u}) - (r'' - r') v = \\ &= \int_{r'}^{r''} \{G(\mathbf{w})(g(r) \circ F(\mathbf{u}))_* \cdot g'(r)(F(\mathbf{u}))\} dr - (r'' - r') g'(0^+)(x_0) = \\ &= \int_{r'}^{r''} \{G(\mathbf{w})(g(r) \circ F(\mathbf{u}))_* \cdot g'(r)(F(\mathbf{u})) - G(\mathbf{0})(g(0) \circ F(\mathbf{0}))_* \cdot g'(0^+)(F(\mathbf{0}))\} dr . \end{aligned}$$

Hence, by Proposition 2, $|S| = |r'' - r'| \theta(r', r'', u, w)$. But then, each S_i is surely $|r'' - r'| \theta(r', r'')$.

PROPOSITION 4. — Let $F: B^n(\varepsilon_0) \rightarrow E$ be any function for which $F(0) = x_0$ and L is any linear map for which $L(R_+^n) = E$. If (4.6) holds, then x_0 is an interior point of $F(B^n(\varepsilon_0))$.

PROOF. — Note that, by hypothesis, E is finite dimensional. Let $e = (1, 1, \dots, 1)$ and solve $L(r) = -L(e)$ with $r \geq 0$. Then $L(r + e) = x_0$ and $r + e > 0$. Hence the kernel of L is at least one dimensional, and so extends into R_+^n . If

$$k > \max_{|x|=1} \min \{ |r| : r \in L^{-1}(x) \cap R_+^n \},$$

then for every $x \in E$ and $x \neq x_0$ we can solve $L(r) = x$ such that $0 < r < k|x|$.

Now choose δ , $0 < \delta < \varepsilon_0$, sufficiently small so that θ is exceeded by $1/2k$ on $B^n(\delta)$. We propose to show that for every $x_* \in E$ with $|x_*| < \delta/2k$, $x_* = F(r_*)$ for some r_* . Recursively construct a sequence of points $r_n \in B^n(\delta)$ such that either the construction stops with some $F(r_n) = x_0$, or that

$$(i_n) \quad 0 < r_n - r_{n-1} < \delta/2^n,$$

and

$$(ii_n) \quad |x_n| < \delta/k2^{n+1},$$

where

$$(III_n) \quad x_n = x_* - F(r_n) \neq x_0.$$

For $n = 1$, either $x_* = x_0$, and the construction stops with $r_1 = r_* = 0$, or $x_* \neq x_0$. Set $r_0 = 0$ and solve $L(r_1) = x_*$ with $0 < r_1 < k|x_*| < \delta/2$. Hence (i₁) holds. For (iii₁), let $x_1 = x_* - F(r_1)$ and for (ii₁), compute

$$|F(r_1) - x_*| = |F(r_1) - F(r_0) - L(r_1 - r_0)| < (\delta/2)(1/2k).$$

Inductively, if $x_n = x_0$ then $F(r_n) = x_*$. Otherwise, $x_n \neq x_0$ and we solve $L(r) = x_n$ with $0 < r < k|x_n| < \delta/2^{n+1}$ by (ii_n). Set $r_{n+1} = r + r_n$ to obtain (i_{n+1}).

Furthermore, from (i_j), $j \leq n$, we have that

$$0 < r_n = (r_n - r_{n-1}) + \dots + (r_1 - r_0) < (\delta/2^n + \dots + \delta/2) < \delta,$$

$$0 < r_{n+1} = r + r_n < \delta/2^{n+1} + \delta/2^n + \dots + \delta/2 < \delta.$$

Hence

$$(4.7) \quad |F(r_{n+1}) - F(r_n) - L(r_{n+1} - r_n)| < |r_{n+1} - r_n|/2k.$$

Setting $x_{n+1} = x^* - F(\mathbf{r}_{n+1})$ and recalling (iii)_n, we have $\text{LHS}(4.7) = |-x_{n+1}|$. By (i)_n, $\text{RHS}(4.7) < \delta/k2^{n+1}$. Hence (ii)_{n+1} holds and the recursion is complete.

Unless the process ceases at $F(\mathbf{r}_n) = x_*$, we have by (i), a Cauchy sequence $\mathbf{r}_n$ in $B^n(\delta)$ that converges, say to $\mathbf{r}_*$. By (ii), the sequence x_n in E converges to the origin x_0 . Since F is continuous in $B^n(\delta)$ for sufficiently small δ , $F(\mathbf{r}_*) = x_*$ by (iii).

PROPOSITION 5. — Returning to the notation of the beginning of this section,

$$\mu(t_1)(F(\mathbf{r})) \in A(x_0, \Phi), \quad \text{for } \mathbf{r} \in B^n(\varepsilon_0).$$

PROOF. — Set $s_0 = t_0$ and $s_{n+1} = t_1$. We recursively show

$$\mu(s_i) \circ F(r_1, \dots, r_i, 0, \dots, 0) \in A(x_0, \Phi), \quad \text{for } i = 0, 1, \dots, n+1.$$

For $i = 0$,

$$\mu(s_0) \circ F(0, \dots, 0) = \int \varphi[t_0, t_0](x_0) = x_0.$$

Inductively, suppose that

$$\mu(s_{i-1}) \circ F(\dots, r_{i-1}, 0, \dots) = \int \sigma_{i-1}[t_0, q_{i-1}](x_0)$$

for some $\sigma_{i-1} \in \Phi$ and $q_{i-1} \geq t_0$. For v_i of Type 1,

$$\begin{aligned} \mu(s_i) \circ F(\dots, r_i, 0, \dots) &= \int \psi[s_i, s_i + r_i] \circ \mu(s_i) \circ F(\dots, r_{i-1}, 0, \dots) \\ &= \int \psi[s_i, s_i + r_i] \circ \int \varphi[s_{i-1}, s_i] \circ \int \sigma_{i-1}[t_0, q_{i-1}](x_0) = \int \sigma_i[t_0, q_i](x_0), \end{aligned}$$

where σ_i is obtained by switching σ_{i-1} to φ and then to ψ_i , and $q_i = q_{i-1} + r_i + s_i - s_{i-1}$. If v_i is of Type 2, we have that

$$\begin{aligned} \mu(s_i) \circ F(\dots, r_i, 0, \dots) &= \mu(s_i - r_i) \circ F(\dots, r_{i-1}, 0, \dots) \\ &= \int \varphi[s_{i-1}, s_i - r_i] \circ \int \sigma_{i-1}[t_0, q_{i-1}](x_0) = \int \sigma_i[t_0, q_i](x_0), \end{aligned}$$

where σ_i is σ_{i-1} switched to φ and $q_i = q_{i-1} - r_i + s_i - s_{i-1}$.

In the last step,

$$\mu(t_1) \circ F(r_1, \dots, r_n) = \int \varphi[s_n, t_1] \circ \int \sigma_n[t_0, q_n](x_0) = \int \sigma[t_0, q](x_0),$$

where σ is σ_n switched to φ and

$$(4.8) \quad q = t_1 - \sum (-1)^{T(i)} r_i, \quad \text{where } v_i \text{ has type } T(i).$$

Note that the switching required to construct σ is $n\varepsilon_0$ -limited.

5. - Application.

It is clear from (4.8) that $I_\sigma = [t_0, q]$, but, in general, we cannot expect that $q = t_1$: Thus, to prove a maximum principle for the time-dependent or «small» attainable set

$$A(x_0, \Phi, t_0, t_1) = \left\{ \int \varphi[t_0, t_1](x_0) : \varphi \in \Phi \text{ and } [t_0, t_1] \subset I_\varphi \right\},$$

the argument may be modified as follows.

PROPOSITION 6. - If $x_1 = \alpha(t_1) = \int \varphi[t_0, t_1](x_0)$ lies on the boundary of $A(x_0, \Phi, t_0, t_1)$ then there is a nontrivial linear functional h on T_0X such that for all $t \in (t_0, t_1)$

$$(5.1) \quad \sup_{\varphi \in I} h(\alpha_*(t)^{-1} \cdot \varphi(\alpha(t), t^+)) = h(\alpha_*(t)^{-1} \cdot \varphi(\alpha(t), t^-)).$$

PROOF. - Consider a set of $n = 2m$ infinitesimal alternatives v_i such that $s_{2j} = s_{2j+1}$, v_{2j} is of Type 2 and v_{2j+1} of Type 1, $j = 1, \dots, m$. Let us write

$$s_{2j} = s_{2j+1} = s'_j \quad \text{and} \quad A: R^m \rightarrow R^n: A(\mathbf{r}') = \mathbf{r},$$

for the linear embedding $r_{2j} = r_{2j+1} = r'_j$. Note that $A(B^m(\varepsilon_0)) \subset B^n(\varepsilon_0)$. Hence $L' = L \circ A$ and $F' = F \circ A$ are well defined, and Propositions 3 and 4 hold with m replacing n . Applying Proposition 5 to this data yields $q = t_1$ because $T(2j) = 1 + T(2j+1)$. We note that if $v_{2j+1} = \alpha_*(s'_j)^{-1} \cdot \psi_j(\alpha(s'_j), s'^+_j)$, the switching step becomes

$$(5.2) \quad \int \sigma_j[t_0, s'_j] = \int \psi_j[s'_j, s'_j + r'_j] \circ \int \varphi[s'_{j-1}, s'_j - r'_j] \circ \int \sigma_{j-1}[t_0, s'_{j-1}](x_0).$$

Now consider the cone K in the T_0X generated by vectors

$$(Type\ 3) \quad w = \alpha_*(s)^{-1} \cdot \psi(\alpha(s), s^+) - \alpha_*(s)^{-1} \cdot \varphi(\alpha(s), s^-),$$

where $s \in (t_0, t_1) \cap I_\varphi$ and $\varphi \in \Phi$. If $K \neq T_0X$ then the functional for a supporting hyperplane has $\sup h(w) = 0$, and hence (5.1) follows. On the other hand, if $K = T_0X$, it remains to find a finite set of generators w_j of Type 3 with distinct s_j . If we let the vector w defined above vary with s , then $\lim_{\sigma \rightarrow s} w(\sigma) = w(s)$. (This follows from the continuity of α and α_* in s and the limit (2.5)). Consequently, if the w_j span T_0X but some $s_j = s_{j+1}$, we may replace w_{j+1} by a nearby w'_{j+1} with $s'_{j+1} > s_j$:

Recall the closure operations on the family of controllers of [1, p. 28], which we translate into the notation of fields:

(C.1) For $(\varphi, s) \in \Phi \times I_\varphi$, the autonomous vector field $\psi(x, t) = \varphi(x, s)$ for all $t \in I_\varphi$, is in Φ .

- (C.2) For $\varphi \in \Phi$, the restriction of φ to a subinterval of I_φ is in Φ .
- (C.3) For $(\varphi, s) \in \Phi \times R$, the s -shift, $\varphi_s(x, t) = \varphi(x, t - s)$ for all $t \in I_\varphi + s$, is in Φ .
- (C.4) The exact switch of two Φ fields is in Φ .
- (C.5) The zero field belongs to Φ .

Note that closure under (C.2-4) implies closure under switching (3.1). Conversely, trivial switching ($r_1 = s_0 = s_1$) restricts φ to $[r_0, r_1]$. Advanced switching can be interpreted as an intermediate switch to the zero field on $[r_1, s_0]$, and retarded switching is the exact switch of φ to the $(r_1 - s_0)$ -shift of φ . Thus the only essential difference is the absence of (C.1), which corresponds to constant controllers. This omission is motivated by a control situation in which every attempt at holding a controller constant fails due to «drifting» of the equipment.

On the other hand, we have had to keep limited inexact switching ((C.3) and $|s| \leq n\epsilon_0$) for Proposition 5, and arbitrary shifting is required for transitivity: $x_1 \in A(x_0) \Rightarrow A(x_1) \subset A(x_0)$. Indeed, exact switching ((C.2) and (C.4)) alone does not guarantee the maximum principle. For example, let Φ_0 be the distribution generated by the single field in R^2 , $\varphi(x_1, x_2, t) = (x_1, 6x_1t)$, $I_\varphi = [0, 1]$, under exact switching only. Then Φ_0 consists only in φ and its restrictions. Hence $\alpha(t) = (t, 2t^2)$ is a Φ_0 -boundary, x_0 the origin. It is not an extremal, because $\alpha_*(t)^{-1} \cdot \varphi(\alpha(t), t) = (1, 3t^2)$. Hence, the Type 1 alternatives at $t = 0$ and 1 and the Type 2 alternative at $t = \frac{1}{2}$ span the plane under nonnegative linear combination.

From the flow theoretic point of view, an autonomous (in fact, a piecewise autonomous) field is freely shiftable in the sense that the integral depends on the length but not location of the interval. Thus we observe that (5.2) can be achieved by exact switching if ψ_i is piecewise autonomous. Hence Proposition 6 holds also if we exchange (C.1) for (C.3) in our notion of switching. This is a partial answer to the question raised by H. HALKIN [3] whether shifting can be avoided. We shall go deeper into this question in a subsequent paper.

We wish to thank Prof. F. ALBRECHT for his indispensable criticism and patient reading of several drafts of this paper, which began as a scholium to this masterful work [1].

REFERENCES

- [1] F. ALBRECHT, *Topics in Control Theory*, Springer, New York, 1968.
- [2] J. DIEUDONNÉ, *Foundations in Modern Analysis*, 2nd Edition, Academic Press, New York, 1969.
- [3] G. FRANCIS, *The Pontrjagin-Albrecht maximum principle*, presented to the Control Theory Conference of the Society for Natural Philosophy, Ann Arbor, 1969.

- [4] H. HALKIN, *On the necessary condition for optimal control of nonlinear systems*, J. Anal. Math., **12** (1963), pp. 1-82.
 - [5] S. LANG, *Introduction to Differentiable Manifolds*, Interscience Publishers, New York, 1962.
 - [6] E. LEE - L. MARKUS, *Foundations of Optimal Control Theory*, John Wiley and Sons, New York, 1967.
 - [7] E. ROXIN, *A geometric interpretation of Pontrjagin's maximum principle*, in: *Nonlinear Differential Equations and Nonlinear Mechanics*, Academic Press, New York, 1963.
 - [8] J. SCHWARTZ, *Nonlinear Functional Analysis*, notes by H. FATTORINI, R. NIRENBERG and H. PORTA, Courant Institute, New York, 1965.
 - [9] G. FRANCIS, *Caonical representation of certain optimal control problems*, to appear in J. Math. Anal. Applic. (1973).
-