On convolution operators leaving $L^{p,\lambda}$ spaces invariant.

by JAAK PEETRE (Lund, Sweden)

Summary. - *It is shown that certain convolution operators, including the Hilbert transform in the one dimensional case, leave invariant the $L^{p,\lambda}$ spaces studied by Campanato, Stampacchia and others.*

0. - Introduction.

The theory of $L^{p,\lambda}$ spaces and various generalizations of these spaces has been developped by several authors (see JOHN-NIRENBERG [9], MEYERS [10], STAMPACCHIA [16], CAMPANATO [1], [2], [3], [4], CAMPANATO-MURTHY [5], SPANNE [15], [15^bis] etc.). In particular in [4], CAMPANATO gives in the case $p=2$ interesting applications to homogeneous second order elliptic partial differential equations. The present paper arose from the following question which we posed to ourselves while studying CAMPANATO'S work. Is it true that the HILBERT transform, in one variable, or more general transforms in one or several variables, leaves $L^{p,\lambda}$ spaces invariant, $1<p<\infty$. We remark that this question is of some interest also from the following point of view. It is not known yet whether $L^{p,\lambda}$ spaces are stable by interpolation or not. Therefore it is of importance to find as many operators having the interpolation property with respect to $L^{p,\lambda}$ spaces as possible. Below (Section 1) we show that the answer to our above question is affirmative for a large class of convolution operators of a type first considesed by COTLAR [6], [7] (see also [12], [13], [14] for other applications of these operators). We give also (Section 2) applications of our results to homogeneous elliptic partial differential equations. They are of the same nature as those of CAMPANATO [4], now with $1<p<\infty$, and operators of arbitrary order, but, on the other hand, since we restrict ourselves to functions with compact supports only, they are considerably weaker.

1. - A general class of convolution operators and $L^{p,\Phi}$ spaces.

Let $\Phi=\Phi(r)$, $r>0$, be a positive increasing function. Let us assume also that $\Phi(2r)\leqq C\Phi(r)$. Let $1\leqq p\leqq\infty$. We denote by $L^{p,\Phi}$ the space of locally integrable functions $f=f(x)$, $x=(x_1, \ldots, x_n)\in R^n$, such that there exists a constant C such that for every $x_0\in R^n$ and every $r\geqq 0$ there is a number σ such that

$$\int_{|x-x_0|\leq r} |f(x)-\sigma|^p dx \leq C^p\,\Phi(r). \tag{1.1}$$

with the obvious modification in the special case $p=\infty$. We provide $L^{p,\Phi}$ with the semi-norm

$$\|f\|_{L^{p,\Phi}} = \inf C$$

where if is taken over all C that satisfy (1.1). If we identify functions that differ by a constant we get a norm and the corresponding quotient spaces becomes a BANACH space, thus complete.

The most important special case is $\Phi(r)=r^\lambda$, $0\leq\lambda\leq n+p$. We shall write $L^{p,\lambda}$ in place of L^{p,r^λ}. It is known that

$$L^{p,\lambda} = \begin{cases} L^p \text{ if } \lambda=0 \\ \text{MORREY space if } 0<\lambda<n \\ \text{JOHN-NIRENBERG space if } \lambda=n \\ \mathrm{Lip}_\alpha,\ \alpha=\dfrac{\lambda-n}{p} \text{ if } n<\lambda\leq n+p \end{cases}$$

see CAMPANATO [1], [2]; for the general case see SPANNE [15]. The main contribution of this paper is the following

THEOREM 1.1. - *Consider convolution operators of the form*

$$g(x)=Tf(x)=a*f(x)=\int a(x-y)f(y)\,dy$$

where we assume that $a=a(x)$ admits a decomposition of the form

$$a=\sum_{\nu=-\infty}^{\infty} a_\nu \tag{1.2}$$

with

$$\int a_\nu(x)\,dx=0, \tag{1.3}$$

$$a_\nu(x)=0 \quad \text{if} \quad |x|\geq 2^\nu, \tag{1.4}$$

$$\|\sum_{\nu'\leq\nu\leq\nu''} a_\nu * f\|_{L^p} \leq C\|f\|_{L^p} \tag{1.5}$$

whenever $f \in L^p$, *with* C *independent of* f, ν', ν'',

$$\| \text{grad } a_\nu \|_{L^q} \leqq C 2^{-\nu\left(\frac{n}{p}+1\right)}, \quad \frac{1}{p}+\frac{1}{q}=1, \tag{1.6}$$

with C *independent of* ν.

Then T *maps* $L^{p,\Phi}$ *continuously into itself, provided*

$$\int_r^\infty \rho^{-\frac{n}{p}-1} (\Phi(\rho))^{\frac{1}{p}} \frac{d\rho}{\rho} \leqq C_r{}^{-\frac{n}{p}-1} (\Phi(r))^{\frac{1}{p}}, \quad r \geqq 0, \tag{1.7}$$

with C *independent of* r.

Before we enter into the details of the proof we give some remarks about the various conditions entering in the theorem.

REMARK 1.1. - Since a_ν has compact support by (1.4), (1.3) can be complemented by

$$\int \text{grad } a_\nu \, dx = 0 \tag{1.8}$$

REMARK 1.2. - COTLAR [6], [7] (see also [13]) has shown that (1.5) holds with $p=2$ if (1.3) and (1 4) hold and moreover

$$\| a_\nu \|_{L_1} \leqq C, \tag{1.9}$$

$$\| \text{grad } a_\nu \|_{L_1} \leqq C 2^{-\nu}. \tag{1.10}$$

(Note that (1.10) is (1.6) with $p=\infty$!) It follows then, for instance from HÖRMANDER [8], theorem 2.1, that (1.5) holds also with $1<p<\infty$ (see also [13]). For somewhat different aspects, see [12], [14].

REMARK 1.3. - It is clear that a decomposition of the form (1.2), with (1.3), (1.4), (1.9), (1.10), (1.6) being satisfied, can be found, simply by a suitable partition of unity, when a is homogeneous of degree-n, with vanising mean value and bounded gradient on the unit sphere (CALDERÒN-ZYGMUND transform). If $n=1$ we can take $a(x)=\frac{1}{x}$ (HILBERT transform).

REMARK 1.4. - If $\Phi(r)=r^\lambda$ then (1.7) holds provided $0 \leqq \lambda < n+p$. Thus our theorem can be applied to $L^{p,\lambda}$ spaces, $0 \leqq \lambda < n+p$.

After these remarks we proceed to give the

PROOF OF THEOREM 1.1:

We write

$$g_\nu(x) = T_\nu f(x) = a_\nu * f(x) = \int a_\nu(x-y)\, f(y) dy.$$

Let $f \in L^{p,\Phi}$. We want to show that $g = \sum_{\nu=-\infty}^{\infty} g_\nu \in L^{p,\Phi}$.

To this end, given r we shall give an estimate of

$$\left(\int_{|x| \leqq r} | g(x) - \tau |^p dx \right)^{\frac{1}{p}}$$

for a suitable τ.

Assume first $2^\nu \leqq r$. Then we may write, by (1.3) and (1.4),

$$g_\nu(x) = \int_{|y| \leqq 2r} a_\nu(x-y)\, (f(y) - \sigma)\, dy, \quad |x| \leqq r$$

with any σ. (Note that $|y| \leqq 2r$ if $|x| \leqq r$ and $|x-y| \leqq 2^\nu (\leqq r)$!) Applying (1.5) we get

$$(1.11) \qquad \left(\int_{|x| \leqq r} | \sum_{2^\nu \leqq r} g_\nu(x) |^p \, dx \right)^{\frac{1}{p}} \leqq C \left(\int_{|y| \leqq 2r} | f(y) - \sigma |^p \, dy \right)^{\frac{1}{p}}$$

$$\leqq C (\Phi(2r))^{\frac{1}{p}} \leqq C (\Phi(r))^{\frac{1}{p}}$$

if we choose σ conveniently and note that $\Phi(2r) \leq C\Phi(r)$ by assumption.

Next assume $2^\nu > r$. Then, by (1.8),

$$\operatorname{grad} g_\nu(x) = \int_{|x-y| \leqq 2^\nu} \operatorname{grad} a_\nu(x-y)\, (f(y) - \sigma_\nu)\, dy$$

for any σ_ν. It follows, by (1.6) and HÖLDER's inequality, that

$$| \operatorname{grad} g_\nu(x) | \leqq \left(\int | \operatorname{grad} a_\nu(x-y) |^q dy \right)^{\frac{1}{q}} \times$$

$$\times \left(\int_{|y| \leq 2.2^\nu} | f(y) - \sigma_\nu |^p dy \right)^{\frac{1}{p}} \leqq C\, 2^{-\nu\left(\frac{n}{p}+1\right)} (\Phi(2.2^\nu))^{\frac{1}{p}}$$

$$\leqq C\, 2^{-\nu\left(\frac{n}{p}+1\right)} (\Phi(2^\nu))^{\frac{1}{p}}, \quad |x| \leq r,$$

as in the previous case, if we note also that $|y| \leq 2.2^{\nu}$ if $|x-y| \leq 2^{\nu}$ and $|x| \leq r \, (< 2^{\nu})$. Thus

$$(1.12) \qquad \Big(\int\limits_{|x| \leq r} |g_{\nu}(x) - g_{\nu}(0)|^{p} dx \Big)^{\frac{1}{p}} \leq C r^{\frac{n}{p}+1} 2^{-\nu\left(\frac{n}{p}+1\right)} (\Phi(2^{\nu}))^{\frac{1}{p}}, \quad 2^{\nu} > r.$$

By summation we get from (1.11) and (1.12)

$$\Big(\int\limits_{|x| \leq r} |g(x) - \tau|^{p} dx \Big)^{\frac{1}{p}} \leq C(\Phi(r))^{\frac{1}{p}} + r^{\frac{n}{p}+1} \sum_{2^{\nu}>r} 2^{-\nu\left(\frac{n}{p}+1\right)} (\Phi(2^{\nu}))^{\frac{1}{p}},$$

with $\tau = \sum_{2^{\nu}>r} g_{\nu}(0)$, or if we use finally (1.7)

$$(1.13) \qquad \Big(\int\limits_{|x| \leq r} |g(x) - \tau|^{p} dx \Big)^{\frac{1}{p}} \leq C(\Phi(r))^{\frac{1}{p}}.$$

If we apply (1.13) to $f(x - x_0)$ we obtain

$$\Big(\int\limits_{|x-x_0| \leq r} |g(x) - \tau|^{p} dx \Big)^{\frac{1}{p}} \leq C(\Phi(r))^{\frac{1}{p}}$$

with C independent of x_0, r, τ. Thus by (1.1) $g \in L^{p,\Phi}$ and the proof is complete.

We conclude this Section by indicating still a generalisation of the preceding result.

REMARK 1.5. - Instead of $L^{p,\Phi}$ we can consider following CAMPANATO [2], [3], [5] the more general spaces $L_k^{p,\Phi}$ where k is a integer ≥ 0. These are obtained by replacing (1.1) by

$$\Big(\int\limits_{|x-x_0| \leq r} |f(x) - \sum_{j \leq k} \sigma_j x^j|^{p} dx \Big)^{\frac{1}{p}} \leq C(\Phi(r))^{\frac{1}{p}},$$

where $\sum_{j \leq k} \sigma_j x^j$ stands for a general polynomial of degree $\leq k$. It is easy to see how to extend theorem 1.1 to this case. The main changes occur in conditions (1.3), (1.6) and (1.7); for instance (1.6) has to be replaced by

$$\| \operatorname{grad}^{k+1} a_{\nu} \|_{L^q} \leq C 2^{-\nu\left(\frac{n}{p}+k\right)}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

where $\operatorname{grad}^{k+1}$ stands for a $(k+1)th$ gradient. We leave the details to the reader.

2. - Application to elliptic partial differential operators.

We shall only consider the simplest case.

Let $P_0(D)$ be any homogeneous elliptic partial differential operator of order m and denote by D^m any pure derivation of order m,

$$D^m = \left(\frac{\partial}{\partial x_1}\right)^{m_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{m_n}, \quad m = m_1 + \dots + m_n .$$

We define a by

$$\hat{a}(\xi) = \frac{\xi^m}{P_0(\xi)}$$

where $\hat{a}$ denotes the FOURIER transform; thus formally we have

$$a(x) = (2\pi)^{-n} \int e^{ix\xi} \frac{\xi^m}{P_0(\xi)} d\xi .$$

By remark 1.3 we can apply theorem 1.1 (provided (1.7) is fulfilled). Taking $f = P_0(D)u$ where u is any function m times continuously differentiable and with compact support, we obtain

$$\| D^m u \|_{L^{p,\Phi}} \leqq C \| P_0(D)u \|_{L^{p,\Phi}} . \tag{2.1}$$

Next we consider homogeneous partial differential operators of the form

$$P(x, D) = P_0(D) + \Sigma a_m(x) D^m . \tag{2.2}$$

Before we make precise the assumptions on the coefficients $a_m(x)$, we shall say a few words about multipliers in $L^{p,\Phi}$ spaces.

Denote by M^Ω the space of locally bounded measurable functions g such that

$$| g(x) - g(x_0) | \leqq C\Omega(| x - x_0 |), \quad | x - x_0 | \leqq \frac{1}{2}$$

with C independent of x and x_0. Here and in the sequel

$$\Omega(r) = \frac{(\Phi(r))^{\frac{1}{p}}}{r^{\frac{n}{p}}} \cdot \frac{1}{\int_r^1 \rho^{-\frac{n}{p}} (\Phi(\rho))^{\frac{1}{p}} \frac{d\rho}{\rho}}$$

Note that if $\Phi(r) = r^{\lambda}$ then

$$\Omega(r) \sim \begin{cases} 1 & \text{if } 0 \leqq \lambda < n \\ \dfrac{1}{|\log r|} & \text{if } \lambda = n \\ r^{\alpha}, \quad \alpha = \dfrac{\lambda - n}{p} & \text{if } n < \lambda \leqq n + p \end{cases}$$

where $\sim$ stands for equivalence.

We have the following

LEMMA 2.1. - Let $g \in L^{\infty} \cap M^{\Omega}$ *and* let $f \in L^p \cap L^{p,\Phi}$. *Then* $gf \in L^p \cap L^{p,\Phi}$ *and we have for any* R, $0 < R \leqq 1$ *the following inequality*

$$\| gf \|_{L^{p,\Phi}} \leqq \tag{2.3}$$

$$\leqq \left(C_1 \sup | g(x) | + C_2 \sup_{|x - x_0| \leqq R} \frac{| g(x) - g(x_0) |}{\Omega(| x - x_0 |)} \right) \| f \|_{L^{p,\Phi}}$$

$$+ \left(C_3 \sup | g(x) | + C_4 \sup_{|x - x_0| \leqq R} \frac{| g(x) - g(x_0) |}{\Omega(| x - x_0 |)} \right) \| f \|_{L^p}$$

where, C_2, C_3, C_4 *are constants that depend on* R *while as* C_1 *is independent of* R. *Moreover if*

$$\int_0^1 \rho^{-\frac{n}{p}} (\Phi(\rho))^{-\frac{1}{p}} \frac{d\rho}{\rho} < \infty \tag{2.4}$$

then $C_2 \to 0$ *as* $R \to 0$.

If $\Phi(r) = r^{\lambda}$ then (2.4) holds if $n \leqq \lambda \leqq + p$.

We prostpone the somewht tedious proof of the lemma for a moment and proceed to its consequences.

We have the following

THEOREM 2.1. - *Assume that the coefficients* $a_m(x)$ *of the operator* $P(x, D)$ *given by* (2.2) *belong to* M^{Ω}. *Assume* Φ *satisfies* (1.7). *Let*

$$\delta = \Sigma \sup \frac{| a_m(x) - a_m(x_0) |}{\Omega_1(| x - x_0 |)},$$

where

$$\Omega_1(r) = 1 \quad \textit{if (2.4) holds,}$$

$$\Omega_1(r) = \max(\Omega(r), 1) \quad \textit{otherwise.}$$

There exists a number $\delta_0 > 0$ *(dedending on* $P_0(D)$ *and* Φ *only) such that if* $\delta < \delta_0$ *then we have the inequality*

$$\| D^m u \|_{L^{p,\Phi}} \leqq C(\| P(x, D)u \|_{L^{p,\Phi}} + \Sigma \| D^m u \|_{L^p})$$

for any function u which is m times continuously differentiable and with compact support, with C independent of u.

This follows at once from lemma 2.1 combined with (2.1), by the usual KORN type armument. See MIRANDA [11], p. 120, or many other places. We omit details.

REMARK 2.1. - Inequality (2.5) clearly is of the CAMPANATO type (see [4]). However this author allows functions with no particular restriction on the support. We could as well have treated functions with arbitrary support and even lower order terms in (2.2) but then would at once appear rather unpleasant remainder terms of the type $\| D^k f \|_{L^{p,\Phi}}$ with $k < m$ and the whole thing becomes not any more so simple. Maybe CAMPANATO'S approach combined with our results would give a possibility to get rid of these terms.

Finally we indicate the

PROOF OF LEMMA 2.1:

Let x_0 and r be given. For any σ we have then

$$(2.6) \qquad J = \Big(\int\limits_{|x-x_0| \leqq r} | g(x) f(x) - g(x_0)\sigma |^p dx \Big)^{\frac{1}{p}} \leqq$$

$$\leqq \Big(\int\limits_{|x-x_0| \leqq r} | g(x) |^p \, | f(x) - \sigma |^p dx \Big)^{\frac{1}{p}} + |\sigma| \Big(\int\limits_{|x-x_0| \leqq r} | g(x) - g(x_0) |^p dx \Big)^{\frac{1}{p}} \leqq$$

$$\leqq \sup | g(x) | \Big(\int\limits_{|x-x_0| \leqq r} | f(x) - \sigma |^p dx \Big)^{\frac{1}{p}} + |\sigma| \, r^{\frac{n}{p}} \Omega(r) \sup_{|x-x_0| \leqq r} \frac{| g(x) - g(x_0) |}{\Omega(|x - x_0|)} .$$

Assume first $r > \frac{1}{2} R$. Then we take $\sigma = 0$ and get

$$J \leqq (\Phi(r))^{\frac{1}{p}} (\Phi(R))^{-\frac{1}{p}} \sup | g(x) | \; \| f \|_{L^p} .$$

This accounts for the term with coefficient C_2 in (2.3).

Assume next $r \leqq \frac{1}{2} R$. Then by (1.1) wo can find σ such that

$$\Big(\int\limits_{|x-x_0| \leqq r} | f(x) - \sigma |^p dx \Big)^{\frac{1}{p}} \leqq 2(\Phi(r))^{\frac{1}{p}} \| f \|_{L^{p,\Phi}} .$$

If we insert this estimate in the next but last term of (2.6) we get the term with coefficient C_2 which is thus independent of R. It remains to find an estimate for $|\sigma|$. To this end we consider $r_k = 2^k r$, $k = 0, 1, \ldots, N$, where $r_{N-1} \leqq \frac{1}{2} R < r_N$. We denote by σ_k, $k = 0, 1, \ldots, N$, the corresponding numbers constructed by the above procedure. (Note in particular that $\sigma_N = 0$!). Using now a well-known idea originating from MORREY (see e.g. CAMPANATO [1] or SPANNE [15]) we get

$$|\sigma_k - \sigma_{k+1}| \leqq C r_k^{-\frac{n}{p}} \left(\left(\int_{|x-x_0| \leq r_k} |f(x) - \sigma_k|^p \, dx \right)^{\frac{1}{p}} + \right.$$

$$\left. + \left(\int_{|x-x_0| < r_{k+1}} |f(x) - \sigma_{k+1}|^p dx \right)^{\frac{1}{p}} \right).$$

Using the above estimates for $\left(\int_{|x-x_0| \leqq r} |f(x) - \sigma|^p \, dx \right)^{\frac{1}{p}}$ we get

$$|\sigma| \leqq |\sigma_0 - \sigma_1| + |\sigma_1 - \sigma_2| + \ldots |\sigma_{N-1} - \sigma_N| \leqq$$

$$\leqq C \left(\int_r^R \rho^{-\frac{n}{p}} (\Phi(\rho))^{\frac{1}{p}} \frac{d\rho}{\rho} \| f \|_{L^{p,\Phi}} + R^{-\frac{n}{p}} \| f \|_{L^p} \right)$$

with C independent of R. If we insert this estimates in the last term of (2.6) we get the remaining therms of (2.3); indeed we find

$$C_2 \leq C \sup_{r \leqq \frac{1}{2} R} \frac{r^{\frac{n}{p}} \Omega(r)}{(\Phi(\rho))^{\frac{1}{p}}} \times \int_r^R \rho^{-\frac{n}{p}} (\Phi(\rho))^{\frac{1}{p}} \frac{d\rho}{\rho} \leqq$$

$$\leqq C \sup_{r \leqq \frac{1}{2} R} \frac{\int_r^R \rho^{-\frac{n}{p}} (\Phi(\rho))^{\frac{1}{p}} \frac{d\rho}{\rho}}{\int_r^1 \rho^{-\frac{n}{p}} (\Phi(\rho))^{\frac{1}{p}} \frac{d\rho}{\rho}} < \infty$$

and

$$C_4 \leqq C R^{-\frac{n}{p}} \sup_{r \leqq \frac{1}{2} R} \frac{r^{\frac{n}{p}} \Omega(r)}{(\Phi(r))^{\frac{1}{p}}} \leqq C R^{-\frac{n}{p}} \frac{1}{\int_{\frac{R}{2}}^1 \rho^{-\frac{n}{p}} (\Phi(\rho))^{\frac{1}{p}} \frac{d\rho}{\rho}} < \infty .$$

It is also plain that $C_2 \to 0$ as $R \to 0$, provided (2.4) holds true.

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Note (addead in proof). SPANNE (unpublished) has extended the results ef Section 1 to potential transforms. Se also his paper [15bis].