Mappings of Spaces with Families of Cones and Space-Time Transformations (*).

A. D. ALEXANDROV (Novosibirsk)

To B. SEGRE on his seventieth birthday.

Summary. — Let A, A' be affine spaces with $\dim A \geq 3$, $\dim A = \infty$ being not excluded. Let C, C' be cones in A and A' ; C_x, C'_y denote the cones with the vertices x, y got from C, C' by translation. Such one-to-one mappings $f: A \rightarrow A'$ are considered that for every $x \in A$ there exists such an $y \in A'$ that $f(C_x) = C'_y$. It is shown that f is affine provided C, C' belong to one of certain classes of cones: those of the second order and « strictly convex » ones, which are suitably defined, especially in infinitely dimensional space. Cones of the second order in n -space are determined by conditions of the following types: (1) $\varphi(x) \equiv x_1^2 + \dots + x_m^2 - x_{m+1}^2 - \dots - x_n^2 = 0$, (2) $\varphi(x) \geq 0$, (3) $\varphi(x) > 0$ plus the vertex $(0, \dots, 0)$. Strictly convex cones are of 6 types, the simplest representatives of which in 4-spaces are 3 « double cones »: (1) $x^2 \equiv x_0^2 - \sum_{i=1}^3 x_i^2 = 0$, (II) $x^2 \geq 0$, (III) $x^2 > 0$ plus the vertex $(0, \dots, 0)$, and three « ordinary » ones got from (I)-(III) by adding the condition $x_0 \geq 0$. In space-time these cones correspond to different kinds of connection of events, e.g. (II) consists of points x causally connected with $0 = (0, \dots, 0)$, an event at x influences that at 0 or vice versa. Thus, the above result implies that a 1—1 mapping of space-time onto itself preserving causal connection is affine. Therefore mere preservation of causal connection implies Lorentz group.

1. — Introduction.

1.1. — Let A be an affine space $\dim A = n + 1 \geq 3$ and $x_0, x_1, \dots, x_n$ cartesian co-ordinates in A . « Lorentz transformation with translation and dilatation » is a linear transformation preserving the form $x_0^2 - \sum_{i=1}^n x_i^2$ plus parallel translation and multiplication of all x_i by any number $\neq 0$. There takes place

THEOREM A₁. — Every one-to-one mapping of the space A onto itself, which maps each cone

$$(1) \quad (x_0 - a_0)^2 = \sum_{i=1}^n (x_i - a_i)^2$$

onto a cone of the same kind, is a Lorentz transformation with translation and dilatation.

(*) Entrata in Redazione il 9 luglio 1974.

(The essence of the result consists, of course, in the absence of any continuity hypotheses).

This theorem was established by me in 1949 [1]. It was formulated for four-dimensional space when (1) expressed the law of light propagation, and thus, as it was pointed out in [1], it follows from Theorem A₁ that constancy of light velocity alone implies Lorentz group with translations and dilatations.

In [1] there was given no proof but an indication of its basic point: « the linearity of the mapping is proved quite elementary by means of the well-known properties of the ruled surfaces of the second order (ordinary surfaces in three-dimensional space). The linearity being proved, the formulas of Lorentz transformation are deduced by an ordinary procedure ». In the present paper we apply this elementary method to more general problems.

1.2. – Somewhat later the two other results have been obtained by me.

THEOREM A₂. – *Every one-to-one mapping of the space A onto itself which maps every cone*

$$(2) \quad x_0 - a_0 = \sqrt{\sum_{i=1}^n (x_i - a_i)^2}$$

onto a cone of the same type is a Lorentz transformation with translation and dilatation.

THEOREM A₃. – *The same is true if we replace the cones (2) by solid cones*

$$(3) \quad x_0 - a_0 \geq \sqrt{\sum_{i=1}^n (x_i - a_i)^2}.$$

(The roots being non-negative, (2) and (3) imply $x_0 \geq a_0$).

The conditions of Theorems A₁-A₃ do not imply that vertices of cones are mapped into vertices. If C_a denotes a cone indicated in one of our theorems, $a = (a_0, \dots, a_n)$ being its vertex, and f denotes the mapping, then our condition reads: for every $a \in A$ there exists such $b = (b_0, \dots, b_n)$ that $f(C_a) = C_b$, but we do not suppose $b = f(a)$.

Theorems A₁-A₃ were published in [2] with proofs (somewhat simplified by V. V. OVCHINNIKOVA) and an explanation of their meaning for Relativity theory. In this theory $n = 3$ and the cone (2) is the set of « world points » reached by light propagated from the world point $a = (a_0, \dots, a_3)$; while the cone (3) is the set of all those world points which are reachable by any influence (transfer of the energy-impulse) from an event at a ; in short, it is the domain of influence of an event at a . Thus, Theorem A₃ implies that every one-to-one mapping of space-time onto itself, which maps domains of influence onto domains of influence is Lorentz transformation with translations and dilatation.

1.3. – Note a corollary of Theorem A₃. Let Q_a denote the interior of the cone (3), i.e. the set defined by the inequality

$$(4) \quad x_0 - a_0 > \sqrt{\sum_{i=1}^n (x_i - a_i)^2}.$$

THEOREM A₄. – If a mapping $f: A \rightarrow A$ and its convers f^{-1} map every Q_a onto a Q_b , then f is a Lorents transformation with translation and dilatation ⁽¹⁾.

Let K_a denote the cone (3). It is evident that $K_a = \cap Q_c$ where all Q_c with $a \in Q_c$ are involved. Hence

$$f(K_a) = \bigcap_{a \in Q_c} f(Q_c) = \bigcap_{f(a) \in Q_b} Q_b,$$

where all Q_b with $f(a) \in Q_b$ are involved (as it follows from the condition that f^{-1} maps every Q_b onto a Q_a). Therefore, the set $\cap Q_b$ is $K_{f(a)}$, and thus, $f(K_a) = K_{f(a)}$ q.e.d.

1.4. – Let C_a denote one of four sets (1)-(4); the relation $x \in C_a$ means a connection between the points x, a , defined by one of the formulas (1)-(4). According to the physical meaning of these formulas in Relativity theory we may say in the case of formula (1) that the points a, x are connected by light, in the case of (2) a acts upon x by means of light (direct, non-reflected); in the case of inequality (3) the point a causally precedes x , the event in a acts (can act through other events) upon the event in x ; in the case (4) an event in a acts (can act) upon an event in x « mechanically » (the action being transferred from a to x by a particle with non-zero mass m_0) or by reflected light.

From the view point of this physical interpretation it is quite natural to suppose that the mapping f preserves the indicated connections between the points, i.e. f preserves the relation $x \in C_a$ so that $x \in C_a$ implies $f(x) \in C_{f(a)}$. It is equivalent to the condition that the respective relation (1), (2), (3) or (4) for pairs of points a, x is preserved under f and f^{-1} . Therefore, Theorems A₁-A₄ imply

THEOREM A. – One-to-one mapping f of the space A onto itself proves to be a Lorents transformation with translation and dilatation, as soon as f and f^{-1} preserve one of the relations (1)-(4); physically speaking, when f and f^{-1} preserve one of the following connections between the points (events):

- (1) connection by direct light,
- (2) influence by direct light,
- (3) causal precedence, influence (transfer of energy-impulse) of any kind,
- (4) mechanical influence and the influence by indirect (reflected) light.

Case (3) seems to be most important, for it means, in short, that causality implies Lorents group. In 1912 A. ROBB [8] gave an axiomatic of space-time based on the concept of causal order, but theorem A(3) shows a fundamental role of the causality in the determination of the space-time structure in the most direct and short way.

⁽¹⁾ The condition on f^{-1} may be omitted, as in Theorems A₁-A₃, but such a stronger form of Theorem A₄ does not follow from Theorem A₃ (or A₁) immediately.

This meaning of theorem A(3) and its consequences were expanded, in particular, in [3-6].

I.5. – Theorem A(1) or, what is the same, Theorem A₁ with the additional condition that the vertices of cones are mapped into the vertices was generalized by YU. BORISOV [8] in 1960; he proved:

THEOREM B. – *One-to-one mapping f of the space A onto itself is affine as soon as f and f^{-1} preserve for every two points x, a , the relation*

$$(4) \quad \sum_0^{m-1} (x_i - a_i)^2 - \sum_m^n (x_i - a_i)^2 = 0,$$

for a given m , $0 < m < n$; or in other words, if f maps every cone with the eq. (4) onto a cone of the same type and the vertex into the vertex.

I.6. – Further in § 2 we prove a theorem which includes Theorem A and gives new physical results.

In § 3 we give a summary of our further results which generalize Theorems A₁-A₃, B in various aspects, in particular, for more general cones and for infinitely dimensional affine spaces. The proofs of these results are given in §§ 4-7; in particular, § 4 contains an exposition of the method indicated in 1.1, based upon an application of ruled quadrics.

I.7. – During last ten years there appeared rather many papers on mappings of cones and analogous problems, *e.g.* [10-21]. However, the above quoted results obtained already long ago, as well as their physical consequences for relativity theory, have been left unnoticed and were partly discovered anew by some authors. Thus, it seemed useful to quote these old results the more so that, in their main part, as far as I knew, they were not rediscovered yet. In addition, their old proofs which we reproduce further are very simple, more simple than newly discovered ones.

For instance, in 1964 the paper by E. ZEEMAN [10] «Causality implies Lorentz group» was published which contained the cases (2), (4) of Theorem A but not Theorem A₃ or A(3), i.e. there was no proof that the preservation of the causal precedence implies Lorentz group, so the statement in the title of the paper was left unproved though it has been known since 1953 [2].

In 1972 H. BORCHERS and G. HEGERFELDT [19] proved Theorem A(1). To stress the importance of this result they exclaim (written in italics with exclamation mark): «Thus constancy of light velocity c alone implies the Poincare group⁽¹⁾ (up to dilatations)!» Though, as was noted above, this result was published in [1] twenty two years earlier, then proved in [2] and after that was generalized by YU. BORISOV in his paper [8] which was reviewed in [8a] with a quotation of [2]. A related theorem on solid cones was proved in [7].

2. – Space-time transformations.

2.1. – Let again A be an affine space, $\dim A \geq 3$, with co-ordinates $x_0, \dots, x_n$. We denote by x^2 the quadratic form

$$x^2 \equiv x_0^2 - \sum_{i=1}^n x_i^2.$$

For a pair of points x, y we consider three symmetric relations:

$$(I) (x - y)^2 = 0, \quad (II) (x - y)^2 \geq 0, \quad (III) (x - y)^2 > 0$$

and three corresponding antisymmetric relations $(I^+)-(III^+)$ got from $(I)-(III)$ by adding the condition $x_0 \leq y_0$.

THEOREM 1. – *If a one-to-one mapping f of the space A onto itself and its converse f^{-1} preserves one of the six relations $(I)-(III^+)$, then f is a Lorentz transformation with translation and dilatation.*

Thus, Theorem 1 comprises six cases and includes Theorem A, the relations (I) , $(I^+)-(III^+)$ being equivalent to (1)-(4) of § 1. Case (II) is contained in a result got in [7], and case (III) — in [19]. Thus, all parts of Theorem 1 are not new, but we give them here a very simple proof.

Preservation of the antisymmetric relations $(I^+)-(III^+)$ immediately implies that of symmetric relations $(I)-(III)$ so that Theorem 1 for the relations $(I^+)-(III^+)$ is but a simple corollary of it for the relations $(I)-(III)$. We do not mention the relations $(x - y)^2 < 0$, $(x - y)^2 \leq 0$ for, being opposite to (II) and (III) , their preservation by f and f^{-1} is equivalent to that of (II) and (III) .

2.2. – In order to prove Theorem 1 we reformulate it in terms of cones. We define the sets

- (1) $C_x = \{y: (x - y)^2 = 0\}$: a « superficial » cone with the vertex x ;
- (2) $K_x = \{y: (x - y)^2 \geq 0\}$: a « solid » cone;
- (3) $Q_x = \{y: (x - y)^2 > 0\}$: the interior of K_x .

Adding the condition $x_0 \leq y_0$ we define the sets $(1^+)-(3^+)$: C_x^+ , K_x^+ , Q_x^+ — the halves of the sets C_x , K_x , Q_x .

The condition of Theorem 1 that f and f^{-1} preserve one of the relations $(1)-(III^+)$ is equivalent to the condition that for the corresponding set M_x (i.e. C_x or K_x etc.), for every x , $f(M_x) = M_{f(x)}$. Thus, Theorem 1 is equivalent to the following.

THEOREM 1a. – *If a one-to-one mapping f of the space A onto itself maps every set M_x of one of the above six types $(1)-(3^+)$ onto the set $M_{f(x)}$, then f is a Lorentz transformation with translation and dilatation.*

Further each time when a case of Theorem 1a is considered f denotes a mapping subject to the corresponding condition of Theorem 1a.

2.3. – Proof for the case (I)—cone C .

2.3.1. – If l is a generator of the cone C_a , $f(l)$ is that of the cone $f(C_a) = C_{f(a)}$.

We observe that if $b \in C_a$ and $b \neq a$, then $C_a \cap C_b$ is a generator ab of C_a . Hence, owing to the conditions imposed on f , there follows our statement.

2.3.2. – If a two-plane P contains two generators of a cone C_a , $f(P)$ is a two-plane too.

Proof. – Let P be a plane of the indicated type. Then, for every $x \in P$ the set $P \cap C_x$ consists of two generators of C_x . Thus, P is covered with two families of parallel generators of cones C_x .

As it is proved, f maps generators onto generators. Therefore, the set $f(P)$ is a ruled surface with two families of straight generators, i.e. one of three surfaces: (1) a plane, (2) a hyperboloid, (3) a hyperbolic paraboloid.

But on the plane P each generator crosses all the generators of the other family, while on a hyperboloid it does not take place. Therefore, $f(P)$ is not a hyperboloid.

On a paraboloid all generators of the same family are parallel to a two-plane. In our case they are generators of the cones C_x . But each cone has only two generators parallel to a given two-plane, and these generators belonging to different cones are pairwise parallel. Thus, there is only two directions of generators parallel to a two-plane, and, therefore, $f(P)$ cannot be a paraboloid.

Hence, $f(P)$ is a plane, q.e.d.

2.3.3. – The mapping f is affine.

PROOF. – Let l be a straight line. Take a point $x \in l$ and two lines l_1, l_2 passing through x within the cone C_x in such a way that l does not lie in the plane (l_1, l_2) , then either plane $P_1 = (l, l_1)$, $P_2 = (l, l_2)$ contains two generators of the cone C_x .

Hence, owing to 2.3.2, $f(P_1)$, $f(P_2)$ are planes, and as $l = P_1 \cap P_2$, so $f(l) = f(P_1) \cap f(P_2)$ is a line.

Thus, f maps every straight line onto a straight line, i.e. f is affine.

2.3.4. – The mapping f preserves the equality $(x - y)^2 = 0$ and is affine. Therefore, it differs from the transformation preserving the form $x^2 = x_0^2 - \sum x_i^2$, i.e. from a Lorentz transformation, by a translation and dilatation, q.e.d.

2.3.5. REMARK. – The proof of Theorem A₁ is almost the same. We observe that $C_a \cap C_b = C_a \cap C_c$ ($a \neq b \neq c$) if and only if $C_a \cap C_b$ is a generator of C_a . Therefore, if for every x $f(C_x) = C_y$, then generators are mapped onto generators, and it remains but to repeat the above deductions 2.3.2-2.3.4.

2.4. – Now Case (I) of Theorem 1 being considered, the proof for the remaining cases is very simple.

2.4.1. – Proof for the Case (II⁺).

The following two observations are evident. If a point x lies within a cone K_a^+ , then there exist such points $y, z \in K_a^+$ that $x \in K_y^+ \cap K_z^+$ and $K_y^+ \not\subset K_z^+$, $K_z^+ \not\subset K_y^+$. But if $x \neq a$ lies on the boundary of K_a^+ , i.e. $x \in C_a^+$, and $x \in K_y^+ \cap K_z^+$, $y, z \in K_a^+$, then either $K_y^+ \subset K_z^+$ or $K_z^+ \subset K_y^+$.

This difference of points x lying within K_a^+ or on the boundary is, evidently, preserved by a mapping f which maps every K_a^+ onto $K_{f(a)}^+$. Therefore, it maps a boundary onto boundary, i.e. $f(C_a^+) = C_{f(a)}^+$. Thus, we have the Case (I⁺), and since it immediately follows from Case (I), Theorem 1 for the Case (II⁺) is proved.

2.4.2. – Proof for the Case (III⁺). The following observation is obvious. For any given point a the intersection of all Q_x^+ containing a is a cone K_a^+ .

Therefore, if f maps every Q_a^+ onto $Q_{f(a)}^+$, it maps every K_a^+ onto $K_{f(a)}^+$. I.e. we have the above Case (II⁺), and Theorem 1 in the Case (III⁺) is proved.

2.5. – Proof for the Cases (II), (III). We make the following evident observation. If two points x, y lie in different « halves » of a cone K_a (e.g. $x \in K_a^+$, $y \in K_a^-$), then $K_x \cup K_y \supset K_a$. But if x, y lie in the same half (e.g. $x, y \in K_a^+$ and $x, y \neq a$), then $K_x \cup K_y \not\supset K_a$.

Hence, a mapping f which maps every K_a onto $K_{f(a)}$ maps halves onto halves, i.e. $f(K_a^+)$ is either $K_{f(a)}^+$ or $K_{f(a)}^-$.

And if for a given a $f(K_a^+) = K_{f(a)}^+$ (or $K_{f(a)}^-$), so it is for all a (since for every K_a^+ , K_b^+ there always exists $K_c^+ \supset K_a^+ \cup K_b^+$, but for K_a^+ , K_b^- it is not so).

Thus, for every x $f(K_x^+) = K_{f(x)}^+$ (or $K_{f(x)}^-$), i.e. we have Case (II⁺) and Theorem 1 for the Case (II) is proved.

The above observation holds for the sets Q_a : if $x \in Q_a^+$, $y \in Q_a^-$ then $Q_x \cup Q_y \supset Q_a$, but if $x, y \in Q_a^+$ (or Q_a^-), then $Q_x \cup Q_y \not\supset Q_a$. Therefore, above argument leads to the proof for the Case (III).

Thus, Theorem 1 is wholly proved.

2.6. – *The physical meaning of Theorem 1* is clear from the interpretation given in 1.4. According to this interpretation the relations (I⁺)-(III⁺) and (I)-(III) mean, respectively, influence and connection of events: (I) by direct light, (II) of any kind, (III) a mechanical one or by indirect light ⁽²⁾.

⁽²⁾ The preservation of a relation being equivalent to that of an opposite one, we can replace $(x-y)^2 \geq 0$ by $(x-y)^2 < 0$ which means that x, y are not causally connected or are « possibly simultaneous ». Thus, we may consider three pairwise disjunctive relations: (II) $(x-y)^2 < 0$ —no connection, (I) $(x-y)^2 = 0$ —direct light connection, (III) $(x-y)^2 > 0$ —other kind of connection.

Lorentz group with translations and dilatations is transitive on each of three sets of pairs (x, y) , defined by these relations.

Theorem 1 asserts that the preservation of each of these six physical relations implies Lorentz group (with preservation of time direction in Cases (I⁺)-(III⁺)).

Case (II) is the most important; it means that *the preservation of a general causal connection alone implies Lorentz group and thus determines the structure of space-time*. The preservation of mechanical or light connection, as well as that of influence, is a physically more special and mathematically stronger condition.

We may say that the spatial-temporal structure of the world is its causal structure abstracted from its physical properties, except the mere relation of causal connection, in general. Thus, space-time may be defined as the set of all events with abstraction made of all properties with the above exception. The relation of influence (II⁺) determines, in addition, a direction of time.

This includes an answer to the problem set still by B. RIEMANN: what is the basis of metric relations in space? We see that these metric relations are based upon causal connection of events.

2.7. – In [10] Theorem 1 is proved for Cases (II⁺), (III⁺), and the latter is interpreted as «causality implies Lorentz group», i.e. causal precedence $x < y$ («an event at x can influence an event at y ») is defined by the relations $(x - y)^2 > 0$, $x_0 < y_0$ and the group preserving it is called «causality group». Thus, the influence by direct light is excluded as if an event cannot influence the other one by direct light. This is the more strange and unsatisfactory that, if $(x - y)^2 > 0$, $x_0 < y_0$ then an event at x can influence an event at y by the light scattered at some z . Then, according to the given definition we have to accept that $x \prec z$, though the influence from x to y is transferred through z . The only way to avoid such absurdities is to define causal precedence $x < y$ by the relations $(x - y)^2 \geq 0$, $x_0 < y_0$, that is by any kind of influence, i.e. by transfer of energy-impulse by mechanical particles, by light or in any some other way.

Thus, the statement that causality implies Lorentz group is proved in [10] at the cost of the very concept of causality in its ordinary general sense. But the statement that causality understood in the ordinary sense, without any restrictions, implies Lorentz group was proved in [2]. If instead of causal order we consider a symmetric relation of «undirected causality» implies Lorentz group too (with the reversal of time, of course).

2.8. – The fact that Cases (II), (II⁺) were left without consideration and, respectively, the restricted concept of causality with the exclusion of influence by light was introduced, is due to some difficulty that provide Cases (II), (II⁺) in comparison with the other ones.

We have shown in 1.3 that Case (III⁺) of Theorem 1 is a simple corollary of Case (II⁺).

2.8.1. – Another reduction of (III⁺) to (II⁺) and (I⁺) is based on a definition of topology by the sets Q^+ . These sets define the sets $Q_x^- = \{y: x \in Q_y^+\}$ and the in-

tersections $Q_x^+ \cap Q_y^-$ (provided $y \in Q_x^+$) make up an open basis in the space A . Therefore, if $f(Q_x^+) = Q_{f(x)}^+$ for every x , then f is continuous; hence

$$f(K_x^+) = K_{f(x)}^+, \quad f(C_x^+) = C_{f(x)}^+.$$

2.8.2. — The cones K_x^+ (and the open cones $Q_x^+ \cup (x)$) determine an order: $y \geq x$ if and only if $y \in K_x^+ (y \in Q_x^+)$. Previous considerations can be expressed in terms of these orders. In particular, 2.8.1 offers a definition of topology by an order. If on a set an (partial) order $x \leq y$ is given, the sets $P_x^+ = \{y: y \geq x\}$, $P_x^- = \{y: y \leq x\}$ are determined, and the simplest definition of topology is that of interval-topology: intervals $P_x^+ \cap P_y^-$ ($y \in P_x^+$) form an open basis. Thus, it is more convenient to suppose the sets $P_x \setminus (x)$ being open. Such a supposition is made, e.g. in a theory of «timelike spaces» [13] or «cinematics» [22]. Physically it means that the order $x \leq y$ is determined by a mechanical influence, influence by direct light being excluded. But in a theory that would correspond to a general concept of influence or causality the sets P_x must be closed.

3. — Further results.

3.1. — Further we consider an affine space A with $\dim A \geq 3$ not excepting $\dim A = \infty$. In this sense an affine space is a linear space over the field of real numbers, in which the translations $x \rightarrow x + a$ are introduced, and the points are considered as being «equivalent». Dimensionality $\dim A$ is understood as the greatest power of a set of independent vectors. An affine mapping $f: A \rightarrow A'$ is a linear mapping up to a translation. A mapping $f: A \rightarrow A'$ is affine provided it is one-to-one and maps straight lines onto straight lines.

3.2. — A cone is a set consisting of rays issued from a common point—the vertex of the cone. If the vertex a is a centre of symmetry of the cone C , then C consists of straight lines passing through a ; in this case we call these lines generators of C and C a double cone. Otherwise, generators of a cone are rays issuing from the vertex, and the cone is called an ordinary one.

We call an ordinary cone C strictly convex, *SC-cone*, provided it has the following properties.

(1) C contains more than one generator;

(2) P being a two-plane containing at least one generator of C , the set $V = P \cap C$ is an angle; a zero angle, i.e. a ray, not being excluded;

(3) the sides of the angle V are «extremal generators», i.e. they are contained within no angle $V' = C \cap P'$, P' being a 2-plane (if V is a ray, it is «the side» of V). *SC-cone*, thus defined, is convex.

We define a «double SC-cone» and its extremal generators by the same conditions with the only difference that the set $V = P \cap C$ is a double angle, i.e. two vertical angles or a line. It is equivalent to the definition that a double SC-cone consists of two «halves»—ordinary SC-cone having a common vertex and being symmetric with respect to it.

SC-cone is contained in no plane (Suppose the contrary: $C \subset P$. Let l be a non-extremal generator of C , and Q is a two-plane: $l \subset Q \not\subset P$. Then $C \cap Q = l$ and, according to the definition of a SC-cone, l must be an extremal generator).

Extremal generators of an SC-cone K form a «superficial SC-cone» ∂K ; all other generators form an «open SC-cone». If C is a double cone, C^+ , C^- denote its halves. In order to distinguish an SC-cone from a superficial or an open one we call it a closed (solid) SC-cone.

3.3. — Thus, we consider six types of SC-cones, three double ones: (1) closed, (2) superficial, (3) open, and the same three types of ordinary cones. Six types of cones considered in § 2 are the simplest instances of SC-cones.

If K is a cone, K_x denotes the cone with the vertex x , got from K by translation.

There takes place the following theorem which generalizes Theorem 1 of § 2.

THEOREM 2. — *Let A , A' be affine spaces and $K \subset A$, $K' \subset A'$ cones of one of the above six types. Let f be such an one-to-one mapping of A onto A' that for every $x \in A$, $f(K_x) = K'_{f(x)}$. Then f is affine.*

The proof of this theorem is a mere repetition of that of Theorem 1.

Theorem 2 may be strengthened as follows.

THEOREM 3. — *Let in the conditions of Theorem 2 f be an one-to-one mapping of A into A' and for every $x \in A$, $f(K_x) = K'_y$. Then f is affine and $f(A) = A'$.*

The condition $f(K_x) = K'_y$ does not suppose $y = f(x)$; thus, Theorem 3 generalizes, in particular, Theorem A₁-A₃ of § 1. The proof of Theorem 3 is given in §§ 4, 5.

3.4. — Now we define cones of the second order in an affine space A which is not supposed to be finite-dimensional.

An origin 0 being fixed, we may consider A as a linear space; x , a etc. will denote points as well as vectors. Let $A = A_1 \oplus A_2$, so that every vector x allows an unique representation:

$$x = x_1 + x_2, \quad x_1 \in A_1, \quad x_2 \in A_2.$$

Let scalar products be defined in A_1 and A_2 . We define a pseudo-scalar product in A : $xy = x_1y_1 - x_2y_2$ and, respectively, $x^2 = x_1^2 - x_2^2$.

By full analogy with 2.2 we define the cones

- (I) $C_a = \{x: (x - a)^2 = 0\}$
- (II) $K_a = \{x: (x - a)^2 \geq 0\}$
- (III) $Q_a = \{x: (x - a)^2 > 0\} \cup (a)$.

THEOREM 4. — Let in affine spaces A, A' pseudoscalar products be defined and, therefore, the sets (I)-(III), too. Let $f: A \rightarrow A'$ be such an one-to-one mapping that for the sets $M \subset A, M' \subset A'$ of one of these three types for every $a \in A, f(M_a) = M'_a$. Then f is affine and $f(A) = A'$.

The proof is given in §§ 6, 7.

Theorem 4 includes, in particular, Borisov's Theorem B of § 1, being transformed into it if the following conditions are added: (1) $A = A'$, (2) $\dim A < \infty$, (3) $f(A) = A'$, (4) $b = f(a)$, i.e. for every a $f(K_a) = K_{f(a)}$.

3.5. Addendum to Theorem 4.

3.5.1. — Apart from the above three kinds of sets (I)-(III), one may consider more 9 types: (1) 3 types of cones complementary to (I)-(III), i.e. the complements of the cones (I)-(III) plus the vertices, e.g. $(A \setminus K) \cup (a)$; (2) 6 types of sets which are complements of the above 6 types of cones; these sets are the same cones without vertices, e.g. $C_a \setminus (a), A \setminus C_a$ etc. Theorem 4 implies the following.

COROLLARY. — The conclusion of Theorem 4 remains valid if the sets M, M' belong to one of three types of complementary cones or to one of 6 types of complements of our cones provided, in the latter case, $f(A) = A'$.

In particular, a cone M_a complementary to Q_a is $M_a = \{x: (x-a)^2 \leq 0\}$. Thus it is the cone K_a determined by the pseudoscalar product $xy = x_2y_2 - x_1y_1 = -xy$. Therefore, the statement of Corollary concerning the cones complementary to Q_a coincides with the statement of Theorem 4 on the cones K_a : Analogously, the statement on cones complementary to K_a coincides with that on cones Q_a .

3.5.2. — The space A being infinite-dimensional, the question on the continuity of an affine mapping is not trivial one. Its solution for a mapping f in Theorem 4 is as follows.

Introduce in A a scalar product $(xy) = x_1y_1 + x_2y_2$ and therefore the norm $|x|: |x|^2 = x_1^2 + x_2^2$. The following proposition takes place.

If one of subspaces A_1, A_2 is finite-dimensional, then every mapping f subject to the conditions of Theorem 4 is continuous in the norm $|x|$. But if both A_1, A_2 are infinite-dimensional, then for every norm defined in A there exists such $f: A \rightarrow A$ subject to the condition of Theorem 4, which is not bounded in the given norm.

We shall leave this proposition as well as the above Corollary without proof which, by the way, is not difficult.

3.6. Addendum to Theorem 3.

3.6.1. — Along with six types of cones in Theorem 3 one may consider six types of complementary cones, i.e. the complement of a cone plus the vertex, e.g. the cone complementary to K_a is $(A \setminus K_a) \cup (a)$. Then we may consider 12 types of complements to the above 6 + 6 types of cones, i.e. the same cones without vertices.

Altogether it makes up 18 types of sets. The following proposition takes place for them.

The conclusion of Theorem 3 holds for these 18 types of sets, too.

We leave it here without proof.

3.6.2. — The problem on the continuity of a mapping f in Theorem 3 has the following solution.

Let Q_x^+ , Q_x^- denote the halves of an open double SC -cone (the vertex x being excluded).

Consider the sets $Q_x^+ \cap Q_y^-$, provided $y \in Q_x^+$, and define « Q -topology» in the space A by accepting the collection of these sets for an open basis. Every type of cones— C , $C^\pm$, K etc. — that appear in Theorem 3 determines the corresponding sets Q_x^+ , Q_x^- and, therefore, determines Q -topology in A . The following proposition takes place.

A mapping f in Theorem 3 is homeomorphic, in the sense of Q -topologies defined in the spaces A and A' by means of relative cones M , M' .

The proof is obvious.

4. — Lemmas.

In this section the terms plane and mapping always denote a 2-plane and a one-to-one mapping.

4.1. LEMMA. — *Let a plane P be covered with two families of parallel straight lines l and let $g: P \rightarrow A'$ be such a mapping that, for every l , $g(l)$ is a straight line. Then $g(P)$ is either a plane or a hyperbolic paraboloid.*

PROOF is contained in 2.3. — $g(P)$ is a ruled surface with two families of generators, i.e. it is either a plane or a paraboloid, or a hyperboloid. But hyperboloid is excluded, for on P every two lines from different families cross each other while on a hyperboloid it is not so.

4.2. — Let in spaces A , A' double cones C , C' be given with the following properties:

(C) Every line passing through the vertex of the cone C is contained in two planes, each of which crosses C along (at least) two generators.

(C') Generators of C' cover no plane angle.

It is easy to see that superficial SC -cones have these properties.

LEMMA. — *Let C , C' be the above cones and $f: A \rightarrow A'$ such a mapping that the image of every line parallel to a generator of C is a line parallel to a generator of C' . Then f is affine.*

The proof is contained in 2.3; at first, we note the following proposition.

4.2.1. – If the conditions of Lemma hold and a plane P contains two lines parallel to generators of the cone C , then $f(P)$ is a plane.

This statement is quite analogous to 2.3.2 and the proof is quite the same. The plane P is covered by two families of lines parallel to generators of C and their images are lines parallel to generators of C' . Thus, owing to Lemma 4.1, $f(P)$ is either a plane or a paraboloid. On a paraboloid the generators belonging to the same family are parallel to a plane. In our case these generators being parallel to those of the cone C' , it contradicts the condition (C') . Therefore, $f(P)$ is a plane.

4.2.2. – Now we finish the proof of Lemma. The condition (C) implies that every line l is the intersection of two planes, each of which contains two generators of the cone C . Hence, owing to 4.2.1 $f(l)$ is the intersection of two planes, i.e. $f(l)$ is a line and, therefore, f is affine.

4.3. LEMMA. – Let C, C' be ordinary cones with the properties 4.2(C), (C') and let a mapping $f: A \rightarrow A'$ be such that the image of every ray parallel to a generator of C is a ray parallel to a generator of C' . Then f is affine.

The proof is based on the following proposition which strengthens Lemma 4.1.

4.3.1. – Let V be a (convex) angle in a plane and $g: V \rightarrow A'$ such a mapping that every ray $l \subset V$ parallel to (or contained in) a side of V is mapped onto a ray. Then $g(V)$ is either an angle on a plane or a part of a hyperbolic paraboloid bounded by two semigenerators.

PROOF. – Let l_1, l_2 be the sides of the angle V . The said rays l form two families of rays parallel to l_1 and l_2 . Therefore, $g(V)$ proves to be a part of a ruled surface with two families of generators, i.e. $g(V)$ is a part either of a plane or of paraboloid, or of a hyperboloid. Since the origins of rays l lie on the rays l_1, l_2 , the origins of ray $g(l)$ lie on the rays $g(l_1), g(l_2)$; and every two rays with the origins on the different rays $g(l_1), g(l_2)$ cross each other.

This implies, at first, that $g(V)$ is a part of one of the above indicated surfaces bounded by the rays $g(l_1), g(l_2)$, i.e. by semigenerators. And, secondly, a part of a hyperboloid is excluded, since on it not all the semigenerators cross each other, or not every point of every semigenerator $g(l)$ is a point of intersection with another semigenerator $g(l')$. This observation proves our statement 4.3.1.

4.3.2. – Let the condition of Lemma 4.3 hold and let P be a plane parallel to two generators l_1, l_2 of the cone C . Then P is mapped into a plane and, for every angle $V \subset P$ with the sides parallel to l_1, l_2 , $f(V)$ is an angle.

PROOF. – According to the conditions of Lemma the rays parallel to l_1, l_2 are mapped onto rays. Hence, applying 4.3.1, we conclude that the image of an angle V

bounded by two such rays is an angle either on a plane or on a paraboloid. But the latter is excluded because of the condition (C') like the proof of Lemma 4.2. Therefore, $f(V)$ is a plane angle, and, since it is true for every such angle $V \subset P$, so $f(P)$ is contained in a plane.

4.3.3. – Now we prove Lemma 4.3. Let l be a line in A . According to the condition (C) it is contained in two planes P_1, P_2 , each of which is parallel to two generators of the cone C . According to 4.3.2 $f(P_1), f(P_2)$ are contained in some planes Q_1, Q_2 , so that $f(l) \subset Q_1 \cap Q_2$.

The planes Q_1, Q_2 do not coincide. For the proof we take a point $a \in l$. It is possible to construct on the planes P_1, P_2 such angles V_1, V_2 with the sides parallel to generators of the cone C that a lies within V_1 and V_2 : Then $f(V_1), f(V_2)$ are angles on the planes Q_1, Q_2 , and the point $f(a)$ lies within them. The rays parallel to the sides of V_1 and V_2 and covering these angles are mapped onto rays covering the angles $f(V_1), f(V_2)$.

Therefore, if the planes Q_1, Q_2 coincided, the angles $f(V_1), f(V_2)$ had common points which were not images of points belonging to the line $l = P_1 \cap P_2$: Thus, the planes Q_1, Q_2 do not coincide, and, therefore, $f(l)$ is contained in a line.

Thus, every line is mapped into a line; it implies that f is affine, and Lemma 4.3 is proved.

5. – Proof of Theorem 3.

Here we prove Theorem 3. It includes six statements concerning three types of both double and ordinary SC -cones defined in 3.2. We shall consider them successively starting with superficial cones. f will denote such an one-to-one mapping $A \rightarrow A'$ that for cones $M \in A, M' \subset A'$ of a type considered, for all $x \in A, f(M_x) = M'_x$.

5.1. Case I: double superficial SC -cone C .

5.1.1. – Generators of C are characterized by the following property.

If l is a generator of C_a , there exist such cones C_b, C_c ($b \neq c$) that $C_a \cap C_b = C_a \cap C_c = l$.

Conversely, if $b \neq c$ and $C_a \cap C_b = C_a \cap C_c$ then this intersection is a generator of C_a .

PROOF. – Let l be a generator of C_a . Then, as it is easy to see, $x \in l$ implies $C_a \cap C_x = l$. Therefore for every $b, c \in l$ we have $C_a \cap C_b = C_a \cap C_c = l$.

Let now $b \neq c$ and $C_a \cap C_b = C_a \cap C_c$: The vertex of a double cone being its centre of symmetry, the set $M = C_a \cap C_b = C_a \cap C_c$ is symmetric with respect to the middle points of the segments ab, ac .

The product of inversions at these points gives a translation which transforms M into itself. Therefore, M allows a group of translations. But, since M is contained in a strictly convex cone C_a , it is possible only if M is contained in a generator of C_a . But if $b \notin C_a$, $M = C_a \cap C_b$ is not contained in a generator. Therefore $b \in C_a$ and hence $M = C_a \cap C_b$ is a generator of C_a , q.e.d.

5.1.2. – A mapping f that maps cones C_x onto cones C'_y preserves, evidently, the above property which characterizes generators. Therefore, f maps generators onto generators. Hence, owing to Lemma 4.2 f is affine.

5.2. – Case (I⁺): ordinary superficial SC -cone C^+ . We shall write C instead of C^+ .

5.2.1. – Generators of C are characterized by the following property. (1) If l is a generator of the cone C_a , there exist such cones C_b, C_c ($C_b \neq C_c$) that $C_a \cap C_b = C_a \cap C_c = l$. (2) Conversely, if $C_b \neq C_c$ and $C_a \cap C_b = C_a \cap C_c \neq \emptyset$, then this intersection is a generator of C .

The statement (1) follows from the following observation. For every point x lying on the prolongation of a generator l of C_a $C_x \cap C_a = l$.

Prove (2). Let $C_a \cap C_b = C_a \cap C_c = M \neq \emptyset$. Translating C_b by the vector ba , we see that C_a contains the set $M' = M + ba$. Likewise, $C_a \supset M'' = M + ca$.

Thus, x and y being two points of M , C_a contains two triplets of points $xx'x''$, $yy'y''$, with $xx' = yy' = ba$, $xx'' = yy'' = ca$ (and, therefore, lying in a 3-plane). But the cone C_a , being strictly convex, can contain such triplets if and only if they are contained in one of its generators. Therefore, the vectors ba, ca are parallel to this generator, hence, there follows the statement (2).

5.2.2. – A mapping f that transforms the cones C_x into the cones C'_y of the same type, preserves the property which, owing to 5.2.1, characterizes generators. Therefore it transforms generators into generators, and, owing to Lemma 4.3, f is affine.

5.3. – Case (II⁺): ordinary SC -cone K^+ . We shall write K instead of K^+ .

5.3.1. – A mapping f that maps every K_a onto K'_b , maps vertices into vertices, i.e. $f(K_a) = K'_{f(a)}$.

PROOF. – Let $f(K_a) = K'_b$.

Then we have such a point $c \in K_a$, that $f(c) = b$. Hence, $b \in f(K_c)$. But, b being the vertex of $K'_b = f(K_a)$, it implies that $f(K_a) \subset f(K_c)$. Thus, $K_a \subset K_c$. But, since $c \in K_a$, we have $K_c \subset K_a$. Therefore, $K_c = K_a$, i.e. $c = a$, i.e. $b = f(a)$.

5.3.2. – Now we may distinguish the interior and the boundary points of a cone K_a as it was done in 2.4.1. Thus, we reduce Case (II⁺) to Case (I⁺).

5.4. – Case (III⁺): ordinary open SC -cone Q^+ . We write Q, K instead of Q^+, K^+ .

5.4.1. – A mapping f that maps every Q_a onto Q'_b , maps vertices into vertices. The proof is quite the same as that of 5.3.1.

5.4.2. – For every a the cone K_a is the intersection of all Q_x which contain a and are contained in any given Q_b with $b \neq a \in Q_b$.

The proof is evident.

5.4.3. – For every a $f(K_a) = K'_{f(a)}$.

PROOF. – Let $a \in Q_b$, $b \neq a$. Then $K_a = \cap Q_x$ where all Q_x with $a \in Q_x \subset Q_b$ are involved. Therefore $f(K_a) = \cap f(Q_x) = \cap Q'_y$.

Here $f(a) \in Q'_y$, and, since $f(Q_b) = Q'_c$, all such Q'_y that $y \in Q'_y$ and $f(a) \in Q'_y$ are involved. But because of 5.4.2 it means that $\cap Q'_y = K'_{f(a)}$. Thus, $f(K_a) = K'_{f(a)}$, and the Case (III⁺) is reduced to Case (II⁺).

5.5. – Case (II): double closed SC -cone K .

5.5.1. – For any given point a

$$(a) = \cap K_x$$

where all such K_x are involved that $a \in K_x$, $x \neq a$.

The proof is evident.

5.2.2. – $a \in K_x$, $x \neq a$, if and only if there exist such cones K_u that $K_u \subset K_a \cup K_x$, $K_u \neq K_a$, K_x .

PROOF. – One can easily see that if $a \in K_x$, then for every point u of the segment ax $K_u \subset K_a \cup K_x$.

Let $a \notin K_x$. Take a point u and construct 2-plane $P \supset (a, x, u)$. The intersections $V_a = P \cap K_a$, $V_x = P \cap K_x$, $V_u = P \cap K_u$ are double angles with the parallel sides. Since $a \notin K_x$, the more $a \notin V_x$ and one can immediately see that $V_u \not\subset V_a \cup V_x$ for every position of u . Thus $K_u \not\subset K_a \cup K_x$ and 5.5.2 is proved.

5.5.3. – The vertex a of a cone K_a is the intersection of all such K_x that there exist $K_u \subset K_a \cup K_x$, $K_u \neq K_a$, K_x .

This is an immediate corollary of 5.5.1 and 5.5.2.

5.5.4. – A mapping f that maps every cone K_a onto a cone K'_b maps vertices into vertices.

PROOF. – Let a be a given point and $f(K_a) = K'_b$. Owing to 5.5.3

$$(a) = \cap K_x,$$

where all such K_x are involved that there exist $K_u \subset K_a \cup K_x$, $K_u \neq K_a$, K_x .

Therefore $f(a) = \cap f(K_x)$, $f(K_x) \cup f(K_a) \supset f(K_u) = K'_u$. On the other hand, if we take all such K'_y that $K'_y \cup f(K_a)$ contain K'_w , then $\cap K'_y \subset \cap f(K_x)$ and, because of 5.5.3 $\cap K'_y$ is the vertex of the cone $f(K_a)$, i.e. $\cap K'_y = b$. Therefore $b = f(a)$, q.e.d.

5.5.5. – Now we can prove, repeating the argument 2.5, that halves of the cones K_x are mapped onto halves of the cones K'_y , and Case (II) is reduced to (II⁺).

5.6. – Case (III): double open *SC*-cone Q . This Case can be reduced to the Case (III⁺) by the same argument that has been just applied to the Case (II).

Thus, Theorem 3 is proved for all Cases (I)-(III⁺).

5.7. – Both Cases (II), (III) may be reduced to the Case (I) as follows. It is not difficult to verify two following propositions.

5.7.1. – Let $x \in K_a$, then $x \in C_a$ if and only if there exist such points $u \neq a, x$, that $K_u \supset K_a \cap K_x$.

5.7.2. – Let $x \notin Q_a$, then $x \in C_a$ if and only if there exist such points $u \neq a, x$ that $Q_u \setminus (u) \subset Q_a \cup Q_x$.

These propositions imply that if a mapping f maps either cones K_a onto cones $K'_{f(a)}$ or cones Q_a onto cones $Q'_{f(a)}$, then in both cases it maps superficial cones C_a onto $C'_{f(a)}$, and, thus, the Cases (II), (III) are reduced to (I).

6. – Proof of Theorem 4, Case (I).

6.1. – Let the conditions of Theorem 4 hold. An origin being fixed in the space A , we equally speak on points x and vectors x . We have $A = A_1 \oplus A_2$, so for every vector x , $x = x_1 + x_2$, $x_1 \in A_1$, $x_2 \in A_2$.

Further, if a, x etc. are vectors, a_1, a_2, x_1, x_2 etc. are their components in A_1 and A_2 :

Scalar products being defined in A_1, A_2 , we have a pseudoscalar product in A : $xy = x_1y_1 - x_2y_2$, $x^2 = x_1^2 - x_2^2$.

We consider the cones $C_a = \{x: (x - a)^2 = 0\}$ and we put $\{x: x^2 = 0\} = C$.

Let in a space A' cones C'_b be defined in an analogous manner, and let a one-to-one mapping $f: A \rightarrow A'$ be such that for every $a \in A$

$$f(C_a) = C'_b.$$

Theorem 4 asserts that f is an affine mapping of A onto A' . In this paragraph we prove this statement.

6.2. – *On planes contained in a cone.* Planes of any dimensionality ≥ 1 are meant. When saying that a plane P is contained in a cone $C_a: P \subset C_a$, we imply that it passes through its vertex. One-dimensional plane $P \subset C$ is a generator of C .

6.2.1. – The plane (a, b) stretched on vectors a, b is contained in C if and only if

$$(1) \quad a^2 = b^2 = ab = 0.$$

PROOF. – The plane (a, b) being composed by vectors $(\alpha a + \beta b)$ it is contained in C if and only if $(\alpha a + \beta b)^2 = 0$ for every α, β . This is equivalent to (1).

6.2.2. – If A_1 or A_2 is one-dimensional, C contains no planes but generators. But if $\dim A_1$ and $\dim A_2 > 1$, C contains 2-planes.

PROOF. – Let $\dim A_1 = 1$, i.e. A_1 be a line. Then, if R is a three-plane and $A_1 \subset R$, the set $R \cap C$ is a cone of revolution. Therefore, C contains no planes but generators (for, otherwise, we could draw R through 2-plane $P \subset C$ and would have $P \subset R \cap C$).

Let now $\dim A_1, \dim A_2 > 1$. Take independent vectors $a_1, b_1 \in A_1$ and such vectors $a_2, b_2 \in A_2$ that

$$a_1^2 = a_2^2, \quad b_1^2 = b_2^2, \quad a_1 b_1 = a_2 b_2.$$

Then the vectors $a = a_1 + a_2, b = b_1 + b_2$ satisfy (1) and, therefore, $(a, b) \subset C$ ⁽³⁾.

6.2.3. – If vectors $a, b \in C$ are independent, there exists such a vector $c \in C$ that $(a, c) \subset C$ but $(b, c) \not\subset C$.

PROOF. – If $(a, b) \not\subset C$ the statement is obvious, for then every plane $P \subset C$ containing a does not contain b .

Let $(a, b) \subset C$, and therefore (1) takes place. Define a vector c with the components:

$$c_1 = \frac{2a_1 b_1}{a_1^2} a_1 - b_1, \quad c_2 = b_2.$$

Then, using (1), a simple calculation gives $c^2 = ac = 0$, and since $a^2 = 0$, we have $(a, c) \subset C$.

On the other hand, owing to $b_1^2 = b_2^2$, we have:

$$bc = b_1 c_1 - b_2 c_2 = \frac{2}{a_1^2} ((a_1 b_1)^2 - a_1^2 b_1^2).$$

Hence $bc \leq 0$. Suppose $bc = 0$, i.e. $(a_1 b_1)^2 = a_1^2 b_1^2$. Then (1) implies $(a_2 b_2)^2 = a_2^2 b_2^2$. Therefore, $b_1 = \lambda a_1, b_2 = \mu a_2$, and owing to (1) $\lambda = \mu$, i.e. $b = \lambda a$. Therefore, a, b being independent, $bc \neq 0$ and thus $(b, c) \not\subset C$, q.e.d.

⁽³⁾ REMARK. – Let P_1 be a subspace $P_1 \subset A_1$, and φ a mapping $P_1 \rightarrow A_2$ preserving scalar product. Then the set P of all x with $x_2 = \varphi(x_1), x_1 \in P_1$, is a plane contained in O . Conversely, let $P \subset C$ be a plane and P_1 a set of all components x_1 of vectors $x \in P$. Define a mapping $\varphi: P_1 \rightarrow A_2$ as follows, $\varphi(x_1) = x_2$, where $x_2 \in A_2$ and $x = x_1 + x_2 \in P$. This mapping φ preserves scalar product.

6.2.4. — We say that a plane $P \subset C$ is a maximal one, if there exists no plane $P' \subset C$ such that $P \subset P'$, $P \neq P'$. Every plane $P \subset C$ is contained in a maximal one.

LEMMA. — A generator of the cone C is the intersection of all maximal planes containing it.

PROOF. — Let l be a generator of C and M the intersection of all maximal planes $P \supset l$. In order to prove that $M = l$, it suffice to show that $b \in C \setminus l$ implies $b \notin M$.

Let $a \in l$, $b \in C \setminus l$ ($a, b \neq 0$). Then a, b are independent and owing to 6.2.3 there exists such $c \in C$ that $(a, c) \subset C$, $(b, c) \not\subset C$. Therefore, no plane $P \subset C$ containing (a, c) can contain b . Hence, $b \notin M$, and our Lemma is proved.

6.2.5 — Let a be such a vector that $a^2 \neq 0$. The mapping $x \rightarrow x'$:

$$(2) \quad x' = x - 2a \frac{ax}{a^2}$$

is isometric (i.e. $x'y' = xy$) and, therefore, transforms the cone C into itself. This mapping is the «reflection» in the plane $ax = 0$.

Every two generators l, l_1 which lie in no plane $P \subset C$, could be transformed one into another by means of such a reflection.

PROOF. — Let $b \in l$, $b_1 \in l_1$ ($b, b_1 \neq 0$). Then $(b - b_1)^2 \neq 0$ (since, if $b, b_1 \in C$, then $b^2 = b_1^2 = 0$, and $(b - b_1)^2 = 0$ implies $bb_1 = 0$, thus, owing to 6.2.3, b, b_1 lie in a plane $P \subset C$). Put in (2) $a = b - b_1$. Because of $b^2 = b_1^2 = 0$ we have $(b - b_1)^2 = -2bb_1$ and, therefore, (2) for $a = b - b_1$ and $x = b$ gives:

$$b' = b + 2(b - b_1) \frac{(b - b_1)b}{2bb_1} = b - (b \geq b_1) = b_1,$$

q.e.d.

6.3. — On intersections of cones.

6.3.1. — Let $a \in C$ ($a \neq 0$), then $C_a \cap C$ is the sum of all planes $P \subset C$ which contain a , i.e. which contain the generator $l \ni a$.

PROOF. — Let M be the said sum. If a plane $P \subset C$ and $a \in P$, then $P \subset C_a$. Thus, $P \subset C \cap C_a$, and therefore $M \subset C \cap C_a$.

Let now $b \in C \cap C_a$. Then $b^2 = (b - a)^2 = 0$ and, because of $a^2 = 0$, $ab = 0$. Therefore, owing to 6.2.1, $(a, b) \subset C$. Since that is true for every $b \in C \cap C_a$, we conclude that $C \cap C_a \subset M$. Thus $C \cap C_a = M$, q.e.d.

6.3.2. LEMMA. — Let a, b be two different points. Then $C \cap C_a = C \cap C_b$ if and only if a, b lie on a generator of the cone C (not in its vertex).

PROOF. — (1) If a, b lie on a generator of C (not in the vertex), then $C \cap C_a = C \cap C_b$. It immediately follows from 6.3.1.

Now we shall prove: (2) if $C \cap C_a = C \cap C_b$, then the points a, b belong to a generator of C .

Let one of the points a, b belong to C : e.g. $a \in C$. Then, owing to 6.3.1, $C \cap C_a$ is the sum M of all maximal planes $P: a \in P \in C$. On the other hand, owing to 6.2.7, the intersection of these planes is generator l_a of C , $a \in l_a$.

But $a \in C$ implies $0 \in C_a$ and, since $C \cap C_a = C \cap C_b$, $0 \in C_b$ too; hence $b \in C$. Therefore, the above observation is applied to b : the maximal planes P that form the set M contain b and their intersection is the generator l_b . Thus, $l_a = l_b$, i.e. a, b belong to one generator of C .

Let now $a, b \notin C$. The set $C \cap C_a$ is defined by the equations $x^2 = 0$, $(x - a)^2 = 0$, or $x^2 = 0$, $2ax = a^2$. It means that $C \cap C_a = C \cap P_a$, P_a being the plane $2ax = a^2$. Likewise, $C \cap C_b = C \cap P_b$, P_b being the plane $2bx = b^2$.

Thus, $C \cap C_a = C \cap C_b$ implies $C \cap P_a = C \cap P_b$. But, since $a \neq b$, $P_a \neq P_b$ and the above equality implies that $C \cap P_a$ is contained in the plane $Q = P_a \cap P_b$ with $\text{codim } Q = 2$. But it is impossible. In fact, let $c \in P \setminus Q$. Let R be a 3-plane containing components c_1, c_2 and a vector $d \in A_2$ independent with c_2 : Then the set $C' = C \cap R$ is a cone of revolution in R with the axis along c_1 ; the set $P' = P_a \cap R$ is a plane and $Q' = Q \cap R$ — a line. Therefore, $C' \cap P' \neq C' \cap Q'$, hence $C \cap P_a \neq C \cap P_b$.

Thus, if $a, b \notin C$, the equality $C \cap C_a = C \cap C_b$ is impossible, and our Lemma is proved.

6.3.3 — Lemma 6.3.2 implies:

A generator of C (the vertex being excluded) is the set of all such $a \in C$ for which there exist such $b \neq a$ that $C \cap C_a = C \cap C_b$.

6.3.4. — The vertex of the cone C is characterized by the following property. A point $z \in C$ is the vertex of C if and only if for every cone $C_a \neq C$ containing z there exist such cones $C_b \neq C_a$ that $C \cap C_a = C \cap C_b$.

PROOF. — Let z be the vertex of C and $z \in C_a$ ($z \neq a$). Then $a \in C$, too, and owing to 6.3.2, for every b lying on the same generator, $C \cap C_a = C \cap C_b$.

Let now z be different from the vertex of C . Take a point $a \in C_z \setminus C$. Then $z \in C_a$.

Suppose $C \cap C_a = C \cap C_b$: Then, owing to Lemma 6.3.2, the points a, b belong to a generator of C , i.e. $a \in C$ in contradiction with the choice of a . Therefore, for every $b \neq a$, $C \cap C_a \neq C \cap C_b$.

6.4. — Proof of Theorem 4. Case (I).

Now we turn to a mapping $f: A \rightarrow A'$, $f(C_a) = C'_b$ for every $a \in A$.

6.4.1. – For every a , $f(C_a) = C'_{f(a)}$. It is an immediate corollary of 6.3.4 for f preserves the property which, owing to 6.3.4, characterizes the vertex of a cone.

6.4.2. – $f(A) = A'$ and for every $b \in A'$, $f^{-1}(C'_b) = C_a$, $a = f^{-1}(b)$.

PROOF. – For every $y \in C' = f(C)$ there exists $x = f^{-1}(y) \in C$, and, thus, owing to 6.4.1, $C'_y = f(C_x)$. But the sum of all C'_y with $y \in C'$ is A' . Hence, $f(A) = A'$.

Therefore, for every $y \in A'$ we have $f^{-1}(y) = x$ and, owing to 6.4.1, $f(C_x) = C'_y$, i.e. $f^{-1}(C'_y) = C_x$.

6.4.3. – f maps generators onto generators.

It is an immediate corollary of 6.3.3 and 6.4.1, for f , mapping vertices into vertices, preserves the property that, by 6.3.3, characterizes generators.

6.4.4. – For every plane $P \subset C$, $f(P)$ is a plane and, P being a maximal one, $f(P)$ is either; moreover, if P is not a line, f is affine on P .

PROOF. – Every line l in a plane $P \subset C$ is a generator of every cone C_a with $a \in l$. Thus, since f maps generators onto generators, $f(P)$ is a plane and f is affine on P , provided P is not a line.

Let a plane $P' \subset C'$ contain $f(P)$. Then $f^{-1}(P')$ is a plane and $f^{-1}(P') \supset P$. Therefore, if P is a maximal plane $f^{-1}(P') = P$, i.e. $P' = f(P)$. Hence $f(P)$ is a maximal plane, too.

6.4.5. – If $\dim A_1 = 1$ (or $\dim A_2 = 1$), f is affine.

PROOF. – If $\dim A_1 = 1$, generators of C are maximal planes (by 6.2.2) and 6.4.3, 6.4.4 imply that $f(l)$ are generators and maximal planes of the cone $C' = f(C)$. Hence, C' does not contain planes but generators. Thus, since generators are mapped onto generators, Lemma 4.2 is applicable, and f is affine.

6.5. – Further we suppose that C contains 2-planes.

6.5.1. – Let $P_0, P_1 \subset C$ be such maximal planes, that the plane $Q_0 = P_0 \cap P_1$ has codimensionality 1 in them. Then R being the plane stretched on P_0 and P_1 , $f(R)$ is one stretched on $f(P_0)$ and $f(P_1)$.

Before proving this statement we make two remarks.

(1) The said pairs of planes P_0, P_1 exist: for every maximal plane $P_0 \subset C$ there exist such maximal planes $P_1 \subset C$ that $P_0 \cap P_1$ has in P_0 and P_1 codimensionality 1.

Such P_1 may be got by means of pseudoreflexion of P_0 at any plane $ax = 0$ (provided it does not contain P_0).

(2) If C contains no planes but generators, our statement 6.5.1 is reduced to that a 2-plane containing two generators of C is transformed into a 2-plane. Thus, 6.5.1 is a generalization of 4.2.1.

6.5.2. – Proof of 6.5.1. Let P, P', Q denote planes contained in R and parallel to P_0, P_1, Q_0 , respectively. For every P and P' the intersection $P \cap P'$ is Q .

Each plane (P') is contained in every cone C_x with $x \in P(x \in P')$. Hence, owing to 6.4.4, f is affine on each P and P' . In particular, $f(P), f(P'), f(Q)$ are planes $f(Q) = f(P) \cap f(P')$ and $f(Q)$ has codimensionality 1 in $f(P)$ and $f(P')$.

We prove that all $f(Q)$ are parallel to each other. Let Q_1, Q_2 be two of the planes Q . If they are contained in P (or P'), $f(Q_1)$ and $f(Q_2)$ are parallel owing to affinity of f on P . Suppose Q_1, Q_2 are contained neither in P , nor in P' . Take such P, P' that $P \supset Q_1, P' \supset Q_2$. Then $P \cap P' = Q_3$, and, by previous argument, $f(Q_1) \parallel f(Q_3), f(Q_2) \parallel f(Q_3)$, hence $f(Q_1) \parallel f(Q_2)$.

Thus, $f(R)$ consists of parallel planes $f(Q)$ as R consists of planes Q .

6.5.3. – Now we shall prove that $f(R)$ is a plane.

Let S be such a 2-plane that $S \subset R$ and $S \cap Q_0 = (0)$, 0 being the vertex of C . Then R is stretched on S and Q_0 , and every plane Q crosses S at a single point. The intersections $P_0 \cap S = l_0, P_1 \cap S = l_1$ are generators of C , which are not contained in Q_0 . Respectively, for every P and P' the intersections $P \cap S = l, P' \cap S = l'$ are generators, $l \parallel l_0, l' \parallel l_1$ (of every cone C_x with $x \in l$ or $x \in l'$). Thus, S is covered with two families of parallel generators l, l' .

Hence, by Lemma 4.1, $f(S)$ is either a plane or a hyperbolic paraboloid.

In the first case, $f(R)$ consists of parallel planes $f(Q)$ which pass through the points of the plane $f(S)$, and therefore $f(R)$ is a plane.

Let now $f(S)$ be a paraboloid. Its generators are the lines $f(l)$ and $f(l')$, the first belonging to one family and the second to the other one. Hence, the lines $f(l)$ are parallel to a 2-plane T . We take T containing the vertex of C , so that $T \supset f(l_0)$.

Since the generator l_0 is not contained in the plane Q_0 , $f(l_0) \notin f(Q_0)$. And since $T \supset f(l_0)$, $f(Q_0) \not\subset T$. Therefore, there are two possible cases: (1) $T \cap f(Q_0)$ is a line, (2) $T \cap f(Q_0)$ is a point.

Let V be the plane stretched on $f(Q_0)$ and T . Since $T \supset f(l_0)$ and the plane $f(P_0)$ is stretched on $f(Q_0)$ and $f(l_0)$, $V \supset f(P_0)$. In the case (1) $V = f(P_0)$, in the case (2) $V \neq f(P_0)$.

Every plane P consists of planes Q crossing the line $l \subset P$. Respectively, $f(P)$ consists of planes $f(Q)$ crossing the line $f(l)$. And since $f(Q) \parallel f(Q_0)$ and $f(l) \parallel T$, so $f(P) \parallel V$.

Let $V = f(P_0)$. Then $f(P) \parallel V$ implies that the planes $f(P)$ are parallel to $f(P_0)$. They cross the line $f(l_0)$ and through each point $x \in f(l_0)$ passes one of them. Therefore, their sum $\cup f(P)$ is a plane. But, since the plane R consists of the planes P , i.e. $R = \cup P$, so $\cup f(P) = f(R)$ and, therefore, $f(R)$ is a plane.

Suppose now $V \neq f(P_0)$. As it is proved, the planes $f(P)$ are parallel to V . We translate them so they pass through the point 0 . With them the lines $f(l)$, i.e. gener-

ators of paraboloid $f(S)$ are translated twoo. Being parallel to the plane T , they will lie in it, covering it with the exception of a line. The plane V being stretched on T and P_0 , the translated planes $f(P)$ fill the plane V with the exception of a plane (which is stretched on the said line and $f(Q_0)$).

Every plane $f(P)$ is contained in a cone C'_x . Therefore, being translated, they are contained in C' and, since they cover V with the exception of a plane, so $V \subset C$.

But $V \supset f(P_0)$ and $f(P_0)$ being a maximal plane contained in C' , $V = f(P_0)$. This contradicts the supposition.

Thus, $V = f(P_0)$, $f(R)$ is a plane, and the statement 6.5.1 is proved.

6.6. – Now we prove that f is affine.

6.6.1. – Q being a 2-plane containing two generators of C , $f(Q)$ is a 2-plane either.

PROOF. – If $Q \subset C$, then f is affine on Q by 6.4. Suppose $Q \not\subset C$ and let l, l_1 be generators of C contained in Q .

Let P be a maximal plane containing l and $R(P, l_1)$ the plane stretched on P and l_1 .

Owing to 6.2.4, the generator l is the intersection of all maximal planes P containing it. To every such P there corresponds the plane $R_{\mathcal{F}}$ stretched on P and l_1 . The intersection of all P being l , that of all $R_{\mathcal{F}}$ is the plane $Q \supset l, l_1$:

$$(3) \quad \cap R_{\mathcal{F}} = Q.$$

As it is shown in 6.2.5, it is possible to transform l into l_1 by means of reflection in a plane T , preserving the cone C . Then the plane P is transformed into another maximal plane $P_1 \supset l_1$. The plane $P \cap P_1 = P \cap T = P_1 \cap T$ has, in P and P_1 , co-dimensionality 1. Hence $R(P, l_1)$ is stretched on P and P_1 , and, owing to 6.5.1, $f(R)$ is a plane.

Now, (3) implies that $f(Q) = \cap f(R_{\mathcal{F}})$, and, therefore, $f(Q)$ is a plane, q.e.d.

6.6.2. – It is easy to see that every line l is the intersection of 2-planes Q each of which contains 2 generators of any cone C_a with $a \in l$.

The cones C_x being equal and parallel to each other, proposition 6.6.1 is applied to every C_x ; therefore $f(Q)$ are 2-planes and $f(l)$ being their intersection, is a line. This being true for every line l , f is affine.

7. – Proof of Theorem 4, Cases (II), (III).

7.1. – Let again $A = A_1 \oplus A_2$ and a pseudoscalar product be introduced in A . Now we consider the cones $K_a = \{x: (x-a)^2 \geq 0\}$ and $Q_a = \mathring{K}_a \cup (a)$, where $\mathring{K}_a = \{x: (x-a)^2 > 0\}$; $\mathring{K} = \{x: x^2 > 0\}$, $Q = \mathring{K} \cup (0)$.

Let in a space A' the cones K'_b, Q'_b be analogously defined, and let a one-to-one mapping $f: A \rightarrow A'$ be such that for every $a \in A$ $f(K_a) = K'_b$ or for every $a \in A$ $f(Q_a) = Q'_b$. Then Theorem 4 (II), (III) asserts that f is affine and $f(A) = A'$.

We shall prove this statement by reducing to the result of § 6, i.e. to Theorem 4 (I); it will be proved that in both cases f maps the cones $C_a: (x-a)^2 = 0$ onto cones $C'_b: (x-b)^2 = 0$, and, therefore, is affine by the result of § 6.

7.2. – We start with some properties of K and $\mathring{K}$.

7.2.1. – If $a^2 \geq 0$, i.e. $a \in K$, then for every $\lambda \in (0, 1)$

$$K_{\lambda a} \subset K \cup K_a, \quad \mathring{K}_{\lambda a} \subset \mathring{K} \cup \mathring{K}_a.$$

PROOF. – Let $a^2 \geq 0$, $\lambda \in (0, 1)$. The set $K_{\lambda a} \setminus K$ is determined by the inequalities: $(x - \lambda a)^2 \geq 0$, $x^2 < 0$. Hence, because of $a^2 \geq 0$ and $0 < \lambda < 1$, we have:

$$0 \leq (x - \lambda a)^2 = x^2 - 2\lambda ax + \lambda^2 a^2 < \lambda x^2 - 2\lambda ax + \lambda a^2,$$

i.e. $(x - a)^2 \geq 0$, i.e. $x \in K_a$.

Therefore, $K_{\lambda a} \setminus K \subset K_a$, i.e. $K_{\lambda a} \subset K \cup K_a$.

The set $\mathring{K}_{\lambda a} \setminus \mathring{K}$ is determined by the inequalities $(x - \lambda a)^2 > 0$, $x^2 \leq 0$ and the same deduction leads to the inclusion $\mathring{K}_{\lambda a} \setminus \mathring{K} \subset \mathring{K}_a$, i.e. $\mathring{K}_{\lambda a} \subset \mathring{K} \cup \mathring{K}_a$.

7.2.2. – If $a^2 < 0$, i.e. $a \notin K$, there exists no such point b ($b \neq a, 0$) that $\mathring{K}_b \subset K \cup K_a$, and the more, no such b that $\mathring{K}_b \subset \mathring{K} \cup \mathring{K}_a$ or $K_b \subset K \cup K_a$.

PROOF. – Let points a, b be given: $a^2 < 0$, $a \neq b \neq 0$. We shall prove that $\mathring{K}_b \not\subset K \cup K_a$.

Suppose $b \notin K \cup K_a$. Then for any given c and sufficiently small λ , $(b + \lambda c) \notin K \cup K_a$: But if $c^2 > 0$, then $(b + \lambda c) \in \mathring{K}_b$. Hence $\mathring{K}_b \not\subset K \cup K_a$.

Suppose now that $b \in K \cup K_a$, e.g. $b \in K$, i.e. $b^2 \geq 0$ (otherwise we can take a for the origin).

Let $b^2 > 0$. Then for any given c and sufficiently small λ , we have $(\lambda c - b)^2 > 0$, i.e. $\lambda c \in \mathring{K}_b$. But if we take such c that $c^2 < 0$, then $\lambda^2 c^2 < 0$, i.e. $\lambda c \notin K$. And, since $a^2 < 0$, we have, for small λ , $(\lambda c - a)^2 < 0$, i.e. $\lambda c \notin K_a$. Thus, $\lambda c \notin K \cup K_a$, and since $\lambda c \in \mathring{K}_b$, so $K_b \not\subset K \cup K_a$.

Let now $b^2 = 0$, i.e. $b_1^2 = b_2^2$. Take the vector $c = b_2$, i.e. $c_1 = 0$, $c_2 = b_2$. Then $c^2 = bc = -b_2^2$. Using these equalities and $b^2 = 0$, we have

$$(\lambda c - b)^2 = -\lambda^2 b_2^2 + 2\lambda b_2^2 = \lambda(2 - \lambda)b_2^2.$$

Hence, for $0 < \lambda < 2$, $(\lambda c - b)^2 \geq 0$, i.e. $\lambda c \in \mathring{K}_b$.

On the other hand, $c^2 = -b_2^2 < 0$ implies that $\lambda c \notin K$. Besides,

$$(\lambda c - a)^2 = -\lambda^2 b_2^2 + 2\lambda b_2 a_2 + a^2$$

and, since $a^2 < 0$, so for small λ , $(\lambda c - a)^2 < 0$, i.e. $\lambda c \notin K_a$.

Thus, for small λ , $\lambda c \in \overset{\circ}{K}_b$ but $\lambda c \notin K \cup K_a$, i.e. $\overset{\circ}{K}_b \not\subset K \cup K_a$, and the statement 7.2.2 is proved.

7.3. – There take place two following statements in a sense reciprocal to the above statements 7.2.1, 7.2.2.

7.3.1. – If $a^2 \leq 0$, i.e. $a \notin \overset{\circ}{K}$, then for every $\lambda \in (0, 1)$

$$K_{\lambda a} \supset K \cap K_a, \quad \overset{\circ}{K}_{\lambda a} \supset \overset{\circ}{K} \cap \overset{\circ}{K}_a.$$

7.3.2. – If $a^2 > 0$, i.e. $a \in \overset{\circ}{K}$, there exists no such point b ($b \neq a, 0$) that $K_b \supset K \cap K_a$ or $\overset{\circ}{K}_b \supset \overset{\circ}{K} \cap \overset{\circ}{K}_a$.

PROOF. – Along with the given pseudoscalar product $xy = x_1 y_1 - x_2 y_2$, we may consider $\widetilde{xy} = x_2 y_2 - x_1 y_1 = -xy$, in particular, $\widetilde{x^2} = -x^2$. Respectively, we define the sets $\overset{\circ}{K}_a: \widetilde{(x-a)^2} \geq 0$, $\overset{\circ}{K}_a: \widetilde{(x-a)^2} > 0$, $(\overset{\circ}{K}: \widetilde{x^2} \geq 0, \overset{\circ}{K}: \widetilde{x^2} > 0)$.

Applying the statements 7.2.1, 7.2.2 to these sets, we have (7.2.1*). If $\widetilde{a^2} \geq 0$, and $\lambda \in (0, 1)$, then $\overset{\circ}{K}_{\lambda a} \subset \overset{\circ}{K} \cup \overset{\circ}{K}_a$, $\overset{\circ}{K}_{\lambda a} \subset \overset{\circ}{K} \cup \overset{\circ}{K}_a$ (7.2.2*). If $\widetilde{a^2} < 0$, there exist no such points b ($b \neq a, 0$) that $\overset{\circ}{K}_b \subset \overset{\circ}{K} \cup \overset{\circ}{K}_a$ or $\overset{\circ}{K}_b \subset \overset{\circ}{K} \cup \overset{\circ}{K}_a$.

Now we observe the following remarks.

- (1) $\widetilde{a^2} \geq 0$ is equivalent to $a^2 \leq 0$, i.e. $a \notin \overset{\circ}{K}$.
- (2) $\widetilde{a^2} < 0$ is equivalent to $a^2 > 0$, i.e. $a \in \overset{\circ}{K}$.
- (3) $\overset{\circ}{K} = \{x: x^2 < 0\} = A \setminus K$, $\overset{\circ}{K} = \{x: x^2 \leq 0\} = A \setminus \overset{\circ}{K}$.

When the given sets are replaced by their complements, then sums and intersections exchange their roles with the conversion of inclusions. Thus, e.g. $\overset{\circ}{K}_b \subset \overset{\circ}{K} \cup \overset{\circ}{K}_a$ is equivalent to $K_b \supset K \cap K_a$.

Using these remarks we reformulate (7.2.1*) and (7.2.2*). Then we get just the statements 7.3.1, 7.3.2, and thus they are proved.

7.4. – Now we formulate three immediate corollaries of the above four statements 7.2.1-7.3.2. The cone K can be replaced in them by a cone K_c . With this change of denotations 7.2.1 and 7.2.2 immediately imply:

7.4.1. – $a \in K_c$, $a \neq c$, if and only if there exist such cones $K_b \neq K_a, K_c$ that $K_b \subset K_a \cup K_c$.

Further, all four propositions 7.2.1-7.3.2, evidently, imply two following statements.

7.4.2. - $a \in K_c \setminus \overset{\circ}{K}_c = C_c$, $a \neq c$ if and only if there exist such cones $K_b \neq K_a, K_c$ that $K_a \cap K_c \subset K_b \subset K_a \cup K_c$.

7.4.3. - $a \in K_c \setminus \overset{\circ}{K}_c = C_c$, $a \neq c$, if and only if there exist such cones $K_b \neq K_a, K_c$ that

$$\overset{\circ}{K}_a \cap \overset{\circ}{K}_c \subset \overset{\circ}{K}_b \subset \overset{\circ}{K}_a \cup \overset{\circ}{K}_c.$$

7.5. - Now we characterize the vertex of a cone K_a in terms of cones K_x .

7.5.1. - For every point a , $(a) = \cap K_x$, where all K_x with $x \neq a$, $x \in K_a$ are involved.

PROOF. - If $x \in K_a$, then $a \in K_x$ and therefore $a \in \cap K_x$.

On the other hand, for every point $b \neq a$ such $x \in K_a$, $x \neq a$, exist that $b \notin K_x$. Hence, $\cap K_x = (a)$.

7.5.2. - The vertex a of a cone K_a is such a point that $(a) = \cap K_c$, where all such K_c are involved that there exist cones $K_b \subset K_a \cup K_c$, $K_b \neq K_a, K_c$.

PROOF. - By 7.4.1, the existence of such cones K_b is equivalent to $a \in K_c$, and thus 7.5.2 follows from 7.5.1.

7.5.3. - Let f be such a mapping $A \rightarrow A'$ that for every $x \in A$ $f(K_x) = K'_y$. Then $f(K_x) = K'_{f(x)}$.

PROOF. - Let a be a given point and $f(K_a) = K'_d$. Then owing to 7.5.2

$$f(a) = \cap f(K_x),$$

where all such $f(K_x) \neq f(K_a)$ are involved that there exist such $f(K_b) \neq f(K_a)$, $f(K_x)$ that $f(K_b) \subset f(K_a) \cup f(K_x)$. The sets $f(K_a)$ being cones K'_y etc., we conclude that

$$f(a) = \cap f(K_x) \supset \cap K'_y,$$

where all such $K'_y \neq K'_d$ are involved that there exist $K'_v \neq K'_b, K'_y$ that $K'_v \subset K'_b \cup K'_y$. But applying 7.5.2 to the cone K'_b , we see that

$$\cap K'_y = (d).$$

Thus, $d = f(a)$, q.e.d.

7.5.4. – $f(A) = A'$ and for every $y \in A'$ $f^{-1}(K'_y) = K_{f^{-1}(y)}$ i.e. the mapping f^{-1} has the same properties as f .

PROOF. – Evidently, $A = \cup K_x$, where all K_x with x belonging to a given cone K_a are involved. Hence, using 7.5.3 (and putting $f(x) = y$)

$$f(A) = \cup f(K_x) = \cup K'_{f(x)} = \cup K'_y.$$

Since $f(K_a) = K'_{f(a)}$, all K'_y with $y \in K'_{f(a)}$ are involved here, and, therefore, $f(A) = A'$.

Now $y \in A'$ being given there exists $x = f^{-1}(y)$ and, owing to 6.5.2, $f(K_x) = K'_{f(x)} = K'_y$, i.e. $f^{-1}(K'_y) = K_x = K_{f^{-1}(y)}$, q.e.d.

7.5.5. – For every a $f(C_a) = C'_{f(a)}$.

PROOF. – If $x \in C_a$, 7.4.2 implies that there exists such a cone $K_b \neq K_a, K_x$ that

$$K_a \cap K_x \subset K_b \subset K_a \cap K_x.$$

Applying the mapping f and using 7.5.3 (i.e. $f(K_a) = K'_{f(a)}$ etc.) we conclude that there exists such $K'_a \neq K'_{f(a)}, K'_{f(x)}$ that

$$K'_{f(a)} \cap K'_{f(x)} \subset K'_a \subset K'_{f(a)} \cup K'_{f(x)}.$$

It means, owing to 7.4.2 applied to the cones K' , that $f(x) \in C'_{f(a)}$. Therefore, $f(C_a) \subset C'_{f(a)}$.

But f^{-1} having the same properties as, we conclude that $C'_{f(a)} \subset f^{-1}(C'_{f(a)}) = C_a$. Therefore, $f(C_a) = C'_{f(a)}$, q.e.d.

Thus, the condition $f(K_x) = K'_y$ implies $f(C_a) = C'_y$, and the result of § 6 leads to the conclusion that f is an affine mapping of A onto A' .

7.6. – Now we turn to the cones Q .

7.6.1. – $a \in Q_x$ if and only if there exist such cones $Q_b \neq Q_a, Q_x$ that $Q_b \subset Q_a \cup Q_x$.

PROOF. – We may put $Q_x = Q$. Then we have to prove two statements.

(1) If $a \in Q$, then there exist $Q_b \subset Q_a \cup Q$, $Q_b \neq Q_a, Q$.

(2) If $a \notin Q$, no such Q_b exist.

The first follows from 7.2.1, if we observe that $Q_b = \mathring{K}_b \cup (b)$. If $a \in Q$, $\lambda \in (0, 1)$, then 7.2.1 implies that $Q_{\lambda a} \subset Q_a \cup Q$.

Let now $a \notin Q$. The proposition 7.2.2 implies that if $a \notin K$ then no $Q_b \neq Q_a$, Q is contained in $Q_a \cup Q$.

Suppose $a \in K$, i.e. $a \in K \setminus Q = C \setminus (0)$ and, therefore, $0 \in C_a \setminus (a)$. Suppose there is a cone $Q_b \subset Q_a \cup Q$, ($b \neq a, 0$) and let $b \in Q$ (if $b \in Q_a$ we take a for the origin). Then $0 \in Q_b$, $b \neq 0$, and, therefore, since $a \in C$, we have $\lambda a \in Q_b$ for sufficiently small $|\lambda|$. But, since $a \in C$, so $\lambda a \notin Q_a \cup Q$, and therefore $Q_b \not\subset Q_a \cup Q$.

Thus proposition 7.6.1 is proved.

7.6.2. – Proposition 7.6.1 being quite analogous to 7.4.1, we may apply to the cones Q the same considerations that were applied above to the cones K . Thus, by full analogy with 7.5.3, 7.5.4, we shall prove that if for every x $f(Q_x) = Q'_x$, then $f(Q_x) = Q'_{f(x)}$ and the converse mapping f^{-1} has the same properties as f .

Then we show that $f(C_x) = C'_{f(x)}$, the proof being the same as that of 7.5.5 with the only difference that 7.4.3 is applied instead of 7.4.2 (we use here that, because of $f(Q_x) = Q'_{f(x)}$, $f(\hat{K}_x) = \hat{K}'_{f(x)}$).

Thus, Case (III) of Theorem 4 is reduced to Case (I), and Theorem 4 is completely proved.

7.6.3. – It is more simple to characterize the vertex a of a cone Q_a by the following property. A point $z \in Q_a$ is the vertex, i.e. $z = a$, if and only if there exist such Q_b, Q_c that $z \notin Q_b \cup Q_c$ but $Q_a \setminus (z) \subset Q_b \cup Q_c$. (The proof is not difficult starting with the observation that for every b with $b^2 = 0$ and $c = -b$, $Q \setminus (0) \subset Q_b \cup Q_c$).

This property of the vertex ensures, evidently, that the vertex is mapped into vertex. After that we proceed as it is said above.

REFERENCES

- [1] A. D. ALEXANDROV, *On Lorentz transformations*, Uspehi Mat. Nauk, **5**, 3 (37) (1950), p. 187 (Russian).
- [2] A. D. ALEXANDROV - V. V. OVCHINNIKOVA, *Notes on the foundations of relativity theory*, Vestnik Leningrad. Univ., **14** (1953), p. 95 (Russian).
- [3] A. D. ALEXANDROV, *The space-time of the theory of relativity*, Jubilee of Relativity Theory, Proceedings, Basel, 1956.
- [4] A. D. ALEXANDROV, *Philosophical content and meaning of relativity theory*, Voprosy filosofii, **4** (1959), p. 67 (Russian).
- [5] A. D. ALEXANDROV, *Relativity theory as a theory of absolute space-time*, Philosophical problems of modern physics, Moscow, 1959, pp. 269-323 (Russian).
- [6] A. D. ALEXANDROV, *A contribution to chronogeometry*, Canad. J. Math., **49**, No. 6 (1967), pp. 1119-1128.
- [7] A. D. ALEXANDROV, *Cones with a transitive group*, Doklady Akad. Nauk SSSR, **189**, No. 4 (1969), pp. 691-698.
- [8] YU. F. BORISOV, *On transformations of a pseudo-euclidian space preserving isotopicity of vectors*, Izvestija Vysš. Učebn. Zaved. Matematika, No. 6 (19) (1960), p. 31 (Russian).
- [8a] Math. Rev., vol. **25**, No. 1480.
- [9] A. A. ROBB, *A theory of time and space*, Cambridge, 1914.

- [10] E. C. ZEEMAN, *Causality implies the Lorentz group*, J. Math. Phys., **5**, No. 4 (1964), p. 490.
- [11] O. S. ROTHUS, *Order isomorphisms of cones*, Proc. Am. Math. Soc., **17**, No. 6 (1966).
- [12] J. L. ALONSO - F. J. UNDURIAN, *On the Continuity of Causal Automorphisms of Space-time*, Commun. Math. Phys., **4**, No. 5 (1967), pp. 349-351.
- [13] H. BUSEMANN, *Time-like spaces*, Rospr. mat., No. 53 (1967).
- [14] G. TEPPATI, *An Algebraic Analogue of Zeeman's Theorem*, Nuovo Cimento, **54 A**, No. 3 (1968), pp. 800-804.
- [15] G. BARUCCHI - G. TEPPATI, *The Causality Group*, Nuovo Cimento, **52 A**, No. 1 (1967), pp. 50-61.
- [16] G. CEZAR - E. MIHUL, *Causal groups of space-time*, Commun. Math. Phys., **14**, No. 2 (1969), pp. 165-170.
- [17] J. RÄTZ, *Zur Definition der Lorentz-transformationen*, Math.-phys. Semesterber., **17**, No. 2 (1970), pp. 163-167.
- [18] H. J. BORCHERS - G. C. HEGERFELDT, *Über ein Problem der Relativitätstheorie: Wann sind Punktabbildungen des R^n linear?*, Nachrichten der Akademie der Wissenschaften in Göttingen, II, Math.-Phys. Klasse, Jahrgang, 1972, No. 10.
- [19] H. J. BORCHERS - G. C. HEGERFELDT, *The structure of space-time transformations*, Commun. Math. Phys., **28**, No. 3 (1972), pp. 259-266.
- [20] A. GAMBA - G. LUZZATTO, Nuovo Cimento, **33** (1964), p. 1732.
- [21] L. MICHEL, *Ecole d'ete de Physique Theorique de Cargese* (1965).
- [22] R. I. PIMENOV, *Mathematical theory of space-time*, Leningrad, 1969 (Russian).