On the Concept of Strongly Transitive Systems in Topology.

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Dedicated to Professor Beniamino SEGRE
on the occasion of his 70-th birthday.

Summary. – *The notion of strongly transitive systems of sets was considered by G. D. Birkhoff in connection with ergodic theory (see [1]) using the concept of measure. An analogue of this notion in Topology is obtained by replacing measurable sets by sets having Baire property and sets of measure zero by meager sets.*

In this paper, some examples of strongly transitive systems in the topological sense are discussed and the Baire property is shown essential in the definition of this notion.

1. – Introduction and terminology.

A subset P of a topological space X is said to be *meager* or of the *first category* if it is a countable union of nowhere dense sets:

$$P = N_1 \cup N_2 \cup \ldots, \qquad \mathrm{Int}(\bar{N}_n) = \emptyset .$$

A set $A \subset X$ is said meager *at a point* p (belonging to A or not) if there is an open set $\mathcal{G}$ containing p such that $A \cap \mathcal{G}$ is meager.

Let us note that, if A is meager at each of its points, then A is a meager set.

A set B is said to have the *Baire property* (or to be open modulo the ideal of meager sets), if $B = (\mathcal{G} - P_1) \cup P_2$ where $\mathcal{G}$ is open and P_1 and P_2 are meager. This turns out to be equivalent to say that each non-empty open set contains a point where either B or $X - B$ is meager.

A family Q of subsets of X is said to be a *partition* of X if these sets are non-empty, disjoint and $X = \cup Q$.

DEFINITION. – A partition Q is said to be *strongly transitive* (in the topological sense), if given a union Z of some members of Q either Z or $X - Z$ is meager, provided Z has the Baire property.

REMARKS. – *On the duality between measure and Baire property.*

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As mentioned in the Summary, the notion of strong transitivy in the sense of measure theory, is dual to the notion just defined. That means that its definition is obtained from the preceding one replacing the terms Baire property by measurability and meager sets by set of measure zero. This is one more example of the duality between measure and Baire property. This duality has been extensively studied by J. C. OXTOBY [8] and various authors (comp. [7-11]).

Let us quote some examples.

A set is measurable iff it can be represented as the union of an F_σ-set and a set of measure 0; a set has the property of Baire iff it can be represented as the union of a $\mathcal{G}_\delta$-set and a meager set.

Let $E \subset X \times Y$ (where we assume for the sake of simplicity that $X = Y =$ the space of reals). If E is of plane measure zero, then the vertical sections E^x of E are of linear measure zero for all $x \in X$ except a set of linear measure zero in X (Theorem of Fubini [2]).

If $E \subset X \times Y$ is meager, then the set E^x is meager in Y for all $x \in X$ except a meager set in X (see [6], p. 249 or [3], vol. I, p. 247).

Duality principle (see [8], p. 75, in connection with duality theorems of Erdös and Sierpinski). Let α be any proposition involving solely the notions of measure zero, meager set, and notions of pure set theory. Let α^* be the proposition obtained from α by interchanging the terms « set of measure zero » and « meager set » wherever they appear. Then each of the propositions α and α^* implies the other, assuming the continuum hypothesis.

2. – Examples of strongly transitive partitions.

Obviously, the property of a partition to be strongly transitive becomes significant only if the space X is not meager on itself; for instance if X is complete.

A simple example of a strongly transitive partition of the space of reals E is the well known partition (of Vitali) into sets such that two reals belong to the same set if their difference is rational. In other words, for the following equivalence relation

$$xRy \equiv x - y \text{ is rational} \tag{1}$$

the quotient-space E/R is the required partition.

Its strong transitivity follows at once from the Theorem which we are going now to prove.

THEOREM 1. – Let X be a complete space and G a group of autohomeomorphisms $h: X \to X$. Consider the equivalence relation R defined by the formula

$$xRy \equiv \exists h \in G: y = h(x) , \tag{2}$$

and consider the quotient space $Q = X/R$.

Assume that each $E \in Q$ is dense in X. Then Q is strongly transitive.

PROOF. – Let Z be a union of some members of Q, i.e. $Z = h(Z)$. First, we shall show that if Z is meager at some point, then Z is a meager set. In other words, we have to prove that Z is meager at each point $x \in Z$.

Our assumption about Z means that there exists a non-empty open set U such that the set $U \cap Z$ is meager. Let $x \in E \in Q$. Since E is dense in X, we have $E \cap U \neq \emptyset$, and hence there is $h \in G$ such that $h(x) \in U$. Since $U \cap h(Z)$ is meager (because $h(Z) = Z$), it follows that the set $h(Z)$ is meager at the point $h(x)$. Hence Z is meager at x, because h is a homeomorphism of X onto X.

Assume now that Z has the property of Baire. Then—by definition of this property—there is p such that either Z or $X - Z$ is meager at p. It follows that either Z or $X - Z$ is a meager set.

This completes the proof.

REMARKS. – A number of examples of strongly transitive partitions can be obtained by virtue of Theorem 1 using the following statement (shown in [5]).

Let T be a Polish space, i.e. complete and separable, and let $f\colon T \to X$ be an open mapping onto. Let Q be a partition of X and denote by $f^{-1}(Q)$ the partition of T into sets $f^{-1}(E)$ where $E \in Q$. Then if Q is a strongly transitive partition, so is $f^{-1}(Q)$.

A particular, important case of this statement is obtained by letting $T = X \times Y$, where Y is Polish and f denotes the projection of $X \times Y$ on the X-axis.

Another way of obtaining interesting strongly transitive partitions is based on the following theorem (see [9], p. 162; as pointed out by Oxtoby, this theorem is the category analogue of the Kolmogorov zero-one law in the theory of probability).

THEOREM 2. – Let $X_1, X_2, \ldots$ be an infinite sequence of Polish spaces and let $X = X_1 \times X_2 \times \ldots$. Denote by R the relation

$$xRy \equiv (x_n = y_n \text{ except for a finite number of indices}) .$$

Then the quotient $Q = X/R$ is strongly transitive.

PROOF. – Let Z be the union of some members of Q. One sees easily that if xRy, then Z contains both x and y or neither. That means—in terminology of Oxtoby—that Z is a *tail* set. Now, as shown in [9], 3.1, if a tail set has the Baire property, then either this set or its complement is meager.

3. – Partitions of indecomposable continua into composants.

A continuum X, i.e. a compact connected space, is called *indecomposable* if it cannot be represented as the union of two proper subcontinua.

The *composant* of a point $x \in X$ is the union of all proper subcontinua of X containing x. One sees easily that the family of composants of X is a partition of X.

The relevant equivalence relation is the following: $(xRy) \equiv$ (there is a continuum $C \neq X$ containing x and y).

The simplest indecomposable continuum, called the B_0 continuum of Knaster, is defined as follows ([3], vol. II, p. 205).

Let us represent the Cantor discontinuum C as the union

$$C = C_1 \cup C_2 \cup \ldots \cup C_n \cup \ldots \cup \{0\}\,,$$

where $C_n = \{x \in C: 2/3^n \leqslant x \leqslant 1/3^{n-1}\}$ for $n = 1, 2, \ldots$.

Denote by H_0 the union of all half-circles having the point $\frac{1}{2}$ as centre, intersecting all points of C, and lying in the upper half-plane. For $n \geqslant 1$, let H_n be the union of all half-circles having the point $5/(2\cdot 3^n)$ as centre (i.e. the centre of C_n), intersecting all points of C_n and lying in the lower half-plane. We define

$$B_0 = H_0 \cup H_1 \cup \ldots \cup H_n \cup \ldots .$$

It has been shown (see [4], p. 254) that the *family of all composants of B_0 is strongly transitive*. The same result can be also deduced from Theorem 1 of § 2 (see Remarks).

A problem not solved thus far is the following.

PROBLEM. – Is the family of all composants of *any* indecomposable continuum strongly transitive?

4. – The role of the Baire property in the definition of strong transitivity.

The fact that the Baire property is essential in the definition of strong transitivity (i.e. it cannot be dropped) is an immediate consequence of the following

THEOREM 3. – If D is a partition of a Polish space X of cardinality $\mathfrak{c}$ into meager sets, then there exists a set Z, union of some members of D, such that neither Z nor $X - Z$ is of the first category.

We will deduce this Theorem, from the following statement belonging to the General Set Theory (compare an analogous statement in [8], p. 76).

AUXILIARY THEOREM. – Let X be an arbitrary set of cardinality $\aleph_1$ and I a proper (i.e. $X \notin I$) σ-ideal dominated by an $F \subset I$ of cardinality $\aleph_1$ (i.e. for each A in I there is B in F such that $A \subset B$).

Let $D \subset I$ be a partition of X. Then there exists a set Z, union of some members of D, such that neither Z nor $X - Z$ is a member of I.

To derive Theorem 3 from the Auxiliary Theorem, we will have to put $I =$ the ideal of all meager subsets of X, and $F =$ the family of all meager F_σ-sets. See also Remark.

PROOF OF THE AUXILIARY THEOREM. – For each $E \subset X$, we denote by $\tilde{E}$ the union of all members of D which intersect E; so, for example

(0) $$Z = \tilde{Z}\,, \quad \widetilde{\bigcup} E_t = \bigcup \tilde{E}_t\,, \qquad (x \notin \tilde{E} \Rightarrow \tilde{x} \cap \tilde{E} = \emptyset)\,.$$

Arrange the family of sets whose complements belong to F in a transfinite sequence:

(1) $$A_0, A_1, \ldots, A_\alpha, \ldots, \qquad \alpha < \omega_1\,,$$

(2) $$(X - A_\alpha) \in F\,.$$

Since $F \subset I$, condition (2) implies that

(3) $$A_\alpha \notin I\,,$$

because otherwise $[(X - A_\alpha) \cup A_\alpha] \in I$, contrary to the assumption that $X \notin I$.

Now, we will define two sets

$$P = (p_0, p_1, \ldots, p_\alpha, \ldots)\,, \qquad \alpha < \omega_1\,,$$

$$Q = (q_0, q_1, \ldots, q_\alpha, \ldots)\,, \qquad \alpha < \omega_1\,,$$

so that, denoting

$$P_\alpha = (p_0, \ldots, p_\alpha) \qquad \text{and} \qquad Q_\alpha = (q_0, \ldots, q_\alpha)\,,$$

we have for each $\alpha < \omega_1$:

$(4)_\alpha$ $$\tilde{P}_\alpha \cap \tilde{Q}_\alpha = \emptyset\,.$$

Our aim is to show that the set $Z = \tilde{P}$ satisfies the conclusion of the Auxiliary Theorem.

We proceed by transfinite induction. Let $p_0 \in A_0$ be arbitrary (obviously $A_0 \neq \emptyset$) and $q_0 \in A_0 - \tilde{p}_0$; such q_0 exists, because otherwise we would have $A_0 \subset \tilde{p}_0$, and $A_0 \in I$ (since $\tilde{p}_0 \in D \subset I$) contrary to (3).

Since $q_0 \notin \tilde{p}_0$, we have $\tilde{q}_0 \neq \tilde{p}_0$ and condition $(4)_0$ follows (because the members of D are disjoint).

Now assume that $(4)_\xi$ is true for each $\xi < \alpha$ (where $\alpha > 0$). We define p_α and q_α as follows. Let

$$R_\alpha = (q_0, q_1, \ldots, q_\xi, \ldots)\,, \qquad \xi < \alpha\,.$$

Put

(5) $$p_\alpha \in A_\alpha - \tilde{R}_\alpha \qquad \text{and} \qquad q_\alpha \in A_\alpha - \tilde{P}_\alpha\,.$$

The existence of p_α and q_α follows from the fact that the sets R_α and P_α are countable and hence

$$\tilde{R}_\alpha \in I \quad \text{and} \quad \tilde{P}_\alpha \in I ,$$

while

$$A_\alpha - \tilde{R}_\alpha \neq \emptyset \neq A_\alpha - \tilde{P}_\alpha$$

because $A_\alpha \notin I$ by (3).

By (5) we have for $\xi < \alpha$

$$\tilde{p}_\alpha \cap \tilde{Q}_\xi = \emptyset = \tilde{q}_\alpha \cap \tilde{P}_\alpha ,$$

and since $\tilde{p}_\alpha \cap \tilde{q}_\alpha = \emptyset$ and

$$P_\alpha = \bigcup P_\xi \cup \{p_\alpha\} \quad \text{and} \quad Q_\alpha = \bigcup Q_\xi \cup \{q_\alpha\} ,$$

formula $(4)_\alpha$ follows from $(4)_\xi$ (by the additivity of the operation $\tilde{p}$, compare (0)).

Thus the sets P and Q are defined.

Put $Z = \tilde{P}$ and $W = \tilde{Q}$.

Formula (4) gives at once

$$Z \cap W \neq \emptyset \quad \text{and} \quad Z \cap A_\alpha \neq \emptyset \neq W \cap A_\alpha \tag{6}$$

for each $\alpha < \omega_1$ (by (5)).

The last statement implies that $Z \notin I$, because otherwise we would have $Z \subset B$ for some member B of F and hence $Z \cap A_\alpha = \emptyset$, where $A_\alpha = X - B$.

Similarly $W \notin I$, and hence $X - Z \notin I$, because $W \subset X - Z$ by (6).

REMARK. – Using the continuum hypothesis one can apply Theorem 3 to any Polish space.

COROLLARY. – In every indecomposable continuum X there is a set of composants such that neither the union nor the complement of this union is meager.

Because every composant of X is an F_σ-set meager in X (by a Theorem of Mazurkiewicz, see e.g. [3], vol. II, p. 212, Theorem 6).

* * *

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