Periodic Solutions of Hyperbolic Partial Differential Equations.

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To Prof. Beniamino SEGRE on the occasion of his 70-th birthday.

Summary. – *Starting with the problem of finding a mass-distribution on a sphere, admitting in each point P a density, function of the point P and of the mass contained in a certain domain depending on P, the author proves the existence and the uniqueness of the solution of* (5) *under assumption* (1), (2), (3). *As a generalization, he also studies hyperbolic partial differential equations having solutions periodic in one of the variables with period depending on the others.*

Introduction.

We start with the following problem: To find a mass-distribution on a sphere S of radius R, admitting a density in each point of S, and such that this density in a point P, be a function of the point P and of the mass contained in a certain domain of S depending on P. More precisely, if ϱ, θ, φ are the spherical coordinates of P with regard to the center of S, δ is the domain

$$(\delta) \qquad \varrho \leqslant \gamma \leqslant R , \quad 0 \leqslant t \leqslant \theta , \quad 0 \leqslant s \leqslant \varphi ,$$

and u the mass of δ, then the density at P will be

$$\frac{1}{\varrho^2 \sin\theta} \frac{\partial^3 u}{\partial\varrho\,\partial\theta\,\partial\varphi}$$

and the stated problem will lead to an equation of the form:

$$\frac{1}{\varrho^2 \sin\theta} \frac{\partial^3 u}{\partial\varrho\,\partial\theta\,\partial\varphi} = f(\varrho, \theta, \varphi, u(\varrho, \theta, \varphi)) .$$

On the other hand, it is obvious that the function $u(\varrho, \theta, \varphi)$ representing the mass of δ, satisfies the relation

$$u(\varrho, \theta, \varphi + 2\pi) = u(\varrho, \theta, \varphi) + 2\pi\lambda(\varrho, \theta) ,$$

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$2\pi\lambda(\varrho, \theta)$ being the mass of

$$(\delta_0) \qquad \varrho \leqslant \gamma \leqslant R, \quad 0 \leqslant t \leqslant \theta, \quad 0 \leqslant s < 2\pi,$$

whereas the density is periodic in φ. It follows that f must satisfy the condition

$$f(\varrho, \theta, \varphi + 2\pi, u + 2\pi\lambda(\varrho, \theta)) = f(\varrho, \theta, \varphi, u).$$

This problem will be considered in **A**, **B**, **C**.

As a generalization, in **D** we will study the case of hyperbolic partial differential equations admitting periodic solutions in one of the variables, with period depending on the others.

The method used is inspired by this of CAESARI [1], [2], [3]. DAN PETROVANU [4], [5] and the authors quoted in these papers.

A. – Statement of the problem.

1) Suggested by the considerations above, we consider the following problem:

A) Let $f(\varrho, \theta, \varphi, u)$ *be an n-dimensional vector function defined on*

$$(\mathrm{D}) \qquad 0 \leqslant \varrho \leqslant R, \quad 0 \leqslant \theta \leqslant \pi, \quad -\infty < \varphi < +\infty, \quad u \in R^n,$$

and satisfying the condition:

A_1) *It exists an n-dimensional vector function* $\lambda(\varrho, \theta)$ *defined on*

$$(\mathrm{D}_1) \qquad 0 \leqslant \varrho \leqslant R, \quad 0 \leqslant \theta \leqslant \pi,$$

satisfying the conditions:

$$\lambda(\varrho, 0) = 0,$$

so that, for every $(\varrho, \theta, \varphi, u) \in D$, *we have:*

$$(1) \qquad f(\varrho, \theta, \varphi + 2\pi, u + 2\pi\lambda(\varrho, \theta)) = f(\varrho, \theta, \varphi, u).$$

B) Let $\alpha(\theta, \varphi)$, $\beta(\varrho)$, $\gamma(\varrho, \theta)$ *be some three n-dimensional vector-functions, such that:*

a) $\alpha(\theta, \varphi)$ *is defined for* $0 \leqslant \theta \leqslant \pi$, $-\infty < \varphi < +\infty$,

b) $\beta(\varrho)$ *is defined for* $0 \leqslant \varrho \leqslant R$,

c) $\gamma(\varrho, \theta)$ *is defined for* $0 \leqslant \varrho \leqslant R$, $0 \leqslant \theta \leqslant \pi$, *which satisfy the compatibility conditions:*

$$\begin{aligned} \alpha(0, \varphi) &= \beta(R), \\ \beta(\varrho) &= \gamma(\varrho, 0), \\ \gamma(R, \theta) &= \alpha(\theta, 0), \end{aligned} \tag{2}$$

and the generalized periodicity condition:

$$\alpha(\theta, \varphi + 2\pi) = \alpha(\theta, \varphi) + 2\pi\lambda(R, \theta). \tag{3}$$

We denote by K_3 the class of continuous vector-functions defined on

$$0 \leqslant \varrho \leqslant R, \quad 0 \leqslant \theta \leqslant \pi, \quad -\infty < \varphi < +\infty, \tag{d}$$

and having on (d) *continuous first and second order derivatives, and also the continuous third order derivative* $\partial^3 u / \partial\varrho\, \partial\theta\, \partial\varphi$.

We denote by K_{3p} the class of functions u satisfying the condition above and also the generalized periodicity condition,

$$u(\varrho, \theta, \varphi + 2\pi) = u(\varrho, \theta, \varphi) + 2\pi\lambda(\varrho, \theta). \tag{4}$$

We are looking for an n-dimensional vector-function $u(\varrho, \theta, \varphi) \in K_{3p}$ satisfying the equation

$$\frac{1}{\varrho^2 \sin\theta} \frac{\partial^3 u}{\partial\varrho\, \partial\theta\, \partial\varphi} = f(\varrho, \theta, \varphi, u), \tag{5}$$

and the conditions on characteristics:

$$\begin{aligned} u(R, \theta, \varphi) &= \alpha(\theta, \varphi), \\ u(\varrho, 0, \varphi) &= \beta(\varrho), \\ u(\varrho, \theta, 0) &= \gamma(\varrho, \theta). \end{aligned} \tag{6}$$

C) Suppose now $\lambda(\varrho, \theta)$ is a continuous function having first derivatives and also the derivative $\partial^2\lambda/\partial\varrho\, \partial\theta$, so that

$$\frac{1}{\varrho^2 \sin\theta} \frac{\partial^2 \lambda}{\partial\varrho\, \partial\theta} \quad \textit{is bounded}.$$

Then, taking

$$v(\varrho, \theta, \varphi) = u(\varrho, \theta, \varphi) - \varphi\lambda(\varrho, \theta),$$

we see that v is a vector-function periodic with respect to φ, if u satisfies (4). The eq. (5), becomes

$$\frac{\partial^3 v}{\partial\varrho\,\partial\theta\,\partial\varphi} = g(\varrho, \theta, \varphi, v)\,, \tag{7}$$

where

$$g(\varrho, \theta, \varphi, v) = f(\varrho, \theta, \varphi, v + \varphi\lambda)\varrho^2 \sin\theta - \frac{\partial^2\lambda}{\partial\varrho\,\partial\theta}\,,$$

and taking into account the fact that $\lambda(r, 0) = 0$, relations (6) become:

$$\begin{aligned} v(R, \theta, \varphi) &= \alpha(\theta, \varphi) - \varphi\lambda(R, \theta) = \bar{\alpha}(\theta, \varphi)\,, \\ v(\varrho, 0, \varphi) &= \beta(\varrho)\,, \\ v(\varrho, \theta, 0) &= \gamma(\varrho, 0)\,; \end{aligned} \tag{8}$$

$\bar{\alpha}$ is now a periodic vector function with regard to φ.

Consequently, our problem is equivalent with the problem of finding a solution of the system (7)-(8), periodic in φ with period 2π, which, on the other hand, is equivalent to that of solving the Volterra non-linear equation

$$\begin{aligned} v(r, \theta, \varphi) = \bar{\alpha}(\theta, \varphi) - \bar{\alpha}(0, \varphi) - \bar{\alpha}(\theta, 0) + \bar{\alpha}(0, 0) + \gamma(r, \theta) - \\ - \int_\varrho^R \int_0^\theta \int_0^\varphi g(r, t, s, v(r, t, s))\, dr\, dt\, ds\,, \end{aligned} \tag{9}$$

with the periodicity condition

$$\int_\varrho^R \int_0^\theta \int_0^{2\pi} g(r, t, s, v(r, t, s))\, dr\, dt\, ds = 0\,.$$

Taking into account that this condition must be satisfied independently of ϱ and θ, it follows that the relation above can be replaced by

$$\int_0^{2\pi} g(r, t, s, v(r, t, s))\, ds = 0\,. \tag{10}$$

B. – Existence and uniqueness theorems.

It is easy to prove

THEOREM 1. – *If*

A_1') *g is a continuous and bounded function on* (D), *and periodic, with period* 2π *with regard to* φ,

A_2') $\bar{\alpha}(\theta, \varphi)$ *is a continuous and bounded function on*

$$0 \leqslant \theta \leqslant \pi, \qquad -\infty < \varphi < +\infty,$$

and periodic, with period 2π *in* φ.

A_3') $\gamma(\varrho, \theta)$ *is a continuous function on*

$$0 \leqslant \varrho \leqslant R, \qquad 0 \leqslant \theta \leqslant \pi,$$

then the equation (9) *has a unique solution* $v(\varrho, \theta, \varphi)$ *of class* K_3, *defined on every compact set of*

$$(\mathrm{D}_4) \qquad (\varrho, \theta, \varphi); \quad 0 \leqslant \varrho \leqslant R, \quad 0 \leqslant \theta \leqslant \pi, \quad -\infty \subset \varphi < \infty.$$

If, in addition g satisfies a Lipschitz condition with respect to v, then the above solution is unique.

By using the fixed points theorems of Schauder and Banach, the proof is standard.

As a consequence we have

THEOREM 2. – *If*

B_1) $f(\varrho, \theta, \varphi, u)$ *is a continuous and bounded vector-function defined on* (D),

B_2) $\alpha(\theta, \varphi)$ *is a continuous and bounded vector function defined for* $0 \leqslant \theta < \pi$, $|\varphi| < \infty$,

B_3) $\beta(\varrho)$ *is a continuous vector function defined on*

$$0 \leqslant \varrho \leqslant R,$$

B_4) $\gamma(\varrho, \theta)$ *is a continuous vector-function defined on*

$$0 \leqslant \varrho \leqslant R, \qquad 0 \leqslant \theta \leqslant \pi,$$

B_5) $\lambda(\varrho, \theta)$ *is a continuous vector function defined on the same set as* γ, *which satisfies* $\lambda(r, 0) = 0$ *and has first order continuous derivatives and also the continuous mixed derivative* $\partial^2\lambda/\partial\varrho\,\partial\theta$.

B_6) *The functions* α, β, γ *satisfy the conditions* (2);

then the system (5), (6) *has a solution in every compact sef of* (D_4).

If in addition f satisfies a Lipschitz condition, the solution is unique.

C. – The periodicity condition.

We come now to condition that v be periodic. *We shall prove that, given α, β, λ, there exists a function γ so that v be periodic.*

To this end, denote first:

$$A(\theta,\varphi)=\bar{\alpha}(\theta,\varphi)-\bar{\alpha}(0,\varphi)-\bar{\alpha}(\theta,0)+\bar{\alpha}(0,0)=\alpha(\theta,\varphi)-\alpha(0,\varphi)-\alpha(\theta,0)+\alpha(0,0)-\varphi\lambda(R,0),$$

$A(\theta,\varphi)$ satisfies the relation

$$A(\theta,\varphi+2\pi)=A(\theta,\varphi).$$

With this, (9) becomes:

$$v(\varrho,\theta,\varphi)=\gamma(\varrho,\theta)+A(\theta,\varphi)-\int_\varrho^R\int_0^\theta\int_0^\varphi g(r,t,s,v(r,t,s))\,dr\,dt\,ds. \tag{11}$$

Suppose now g has continuous first derivative in v, and let $\gamma_i(\varrho,\theta)$ $(i=1,2)$ be two given functions satisfying the same conditions as $\gamma(\varrho,\theta)$ and $v_i(\varrho,\theta,\varphi)$ the two solutions of (11), corresponding to $\gamma=\gamma_i$.

We obtain from these:

$$v_2(\varrho,\theta,\varphi)-v_1(\varrho,\theta,\varphi)=\gamma_2(\varrho,\theta)-\gamma_1(\varrho,\theta)-\int_\varrho^R\int_0^\theta\int_0^\varphi\frac{\partial g}{\partial v}(r,t,s,w(r,t,s))(v_2-v_1)\,dr\,dt\,ds, \tag{12}$$

where $w(r,t,s)$ is a vector function of the form:

$$w(r,t,s)=v_1(r,t,s)+\eta[v_2(r,t,s)-v_1(r,t,s)],\qquad |\eta|\leqslant 1, \tag{13}$$

$\partial g/\partial v$ is the functional matrix $(\partial g_i/\partial v_j)$, operating here on the vector v_2-v_1.

Denoting by $\mathfrak{R}(\varrho,\theta,\varphi,r,t,s)$ the resolvent kernel of the kernel $\partial g/\partial v$, we obtain:

$$v_2-v_1=\gamma_2-\gamma_1-\int_\varrho^R\int_0^\theta\int_0^\varphi\mathfrak{R}(\varrho,\theta,\varphi,r,t,s)(\gamma_2-\gamma_1)\,dr\,dt\,ds. \tag{14}$$

Consider now the operator T defined by:

(15) $$T\gamma = \int_0^{2\pi} \left\{ \frac{1}{2\pi} \gamma(\varrho, \theta) + \mu g(\varrho, \theta, s, v(\varrho, \theta, s)) \right\} ds\,, \qquad \mu = \text{const.}$$

where v is the solution of (11) corresponding to γ.

Concerning this operators we shall prove

THEOREM 3. – *If g is a continuous bounded function on* (D), *has a continuous and bounded derivative $\partial g/\partial v$, is 2π-periodic in φ,*

(16) $$\|g\| \leqslant M\,, \qquad \left\| \frac{\partial g}{\partial v} \right\| \leqslant M_2\,,$$

and satisfies the condition that there exists a constant μ so that, for every ϱ, θ, φ, w, the relation

(17) $$\left\| I + 2\pi\mu \frac{\partial g}{\partial v} \right\| \leqslant 2\pi\sigma \qquad \sigma = const.\,,$$

is satisfied, then (15) *has a unique fixed point. If γ_0 is this fixed point, then* (15) *reduces to the periodicity condition* (10).

PROOF. – In the Banach space B of continuous functions on $[0, R] \times [0, 2\pi]$, with sup-norm, it is easy to see that T transforms B into itself, and that

$$\|T\gamma_2 - T\gamma_1\| = \left\| \int_0^{2\pi} \left\{ \frac{1}{2\pi} (\gamma_2 - \gamma_1) + \mu \frac{\partial g}{\partial v} (\varrho, \theta, s, w)(v_2 - v_1) \right\} ds \right\|,$$

Replacing $v_2 - v_1$ by his value from (14), we have:

(18) $$\|T\gamma_2 - T\gamma_1\| = \left\| \int_0^{2\pi} \left\{ \left(\frac{I}{2\pi} + \mu \frac{\partial g}{\partial v} (\varrho, \theta, s, w(\varrho, \theta, s)) \right) (\gamma_2 - \gamma_1) - \right.\right.$$
$$\left.\left. - \mu \frac{\partial g}{\partial v} \Big((\varrho, \theta, s, w(\varrho, \theta, s)) \Big) \int_\varrho^R \int_0^\theta \int_0^s R(\varrho, \theta, s; r_1, t_1, s_1)(\gamma_2 - \gamma_1) dr_1 dt_1 ds_1 \right\} ds \right\|, \quad 0 \leqslant s \leqslant 2\pi\,.$$

Choose now μ such that formula (17) be satisfied, and denoting

(19) $$\nu = \left\| \frac{\partial g}{\partial v} (\varrho, \theta, s, w)\ \mathcal{R}(\varrho, \theta, s; r_1, t_1, s_1) \right\|,$$

relation (18) gives:

$$\|T\gamma_2 - T\gamma_1\| \leqslant (\sigma + 2\pi|\mu|\nu\pi^2 R)\|\gamma_2 - \gamma_1\| \,.$$

It is obvious that the constants ν and σ depend on g and $\partial g/\partial v$; we have namely

$$\mathfrak{K}(\varrho, \theta, \varphi; r, t, s) = \sum_{i=1}^{\infty} k_i(\varrho, \theta, \varphi; r, t, s)\,,$$

$$k_1(\varrho, \theta, \varphi, r, t, s) = \frac{\partial g}{\partial v}(r, t, s; w(r, t, s))\,,$$

$$k_n(\varrho, \theta, \varphi; r, t, s) = \int_{\varrho}^{R}\int_{0}^{\theta}\int_{0}^{\varphi} k_{n-1}(\varrho, \theta, \varphi; r, t, s) k_1(r, t, s)\, dr\, dt\, ds\,,$$

which gives

$$\|\mathfrak{K}\| \leqslant \sum_k \frac{(2\pi^2 n M_2 R)^k}{(k!)^3} M_2\,.$$

From (19), it follows

$$\nu = M_2^2 \sum_k \frac{(2\pi^2 n M_2 R)^k}{(k!)^3}\,.$$

If we can choose now σ, μ, ν and R so that

$$\sigma + 2\pi|\mu|\cdot\nu\pi^2 R < 1\,,$$

the operator T is a contraction, and, as a consequence, he has a fixed point, which is the function we are looking for.

As a consequence we get:

THEOREM 4. – *If f is a continuous bounded function on D, has a continuous and bounded derivative $\partial g/\partial v$, satisfies* (1), *with $\lambda(\varrho, \theta)$ satisfying the condition $\lambda(\varrho, 0) = 0$ and B_5 of theorem* 2, *then* (7) *has a unique periodic solution.*

D. – Generalization of the problem.

The problem studied in the above chapters suggests a new one, namely the problem of finding solutions of a hyperbolic equation periodic with regard to one of the variables, with period depending on the others.

A partial answer to this problem for a particular case analogous to the preceeding one, will be given in the

THEOREM 5. – *If*

1_1) *$h(x, y, z, u)$ is an n-dimensional vector function defined on*

$$(\mathfrak{D}) \qquad 0 \leqslant x \leqslant a\,, \quad 0 \leqslant y \leqslant b\,, \quad -\infty < z < +\infty\,, \quad u \in R^n\,,$$

1_2) *$h(x, y, z, u)$ is continuous and bounded in (D), has a bounded derivative $(\partial h/\partial v)(x, y, z, w)$, and, as a consequence satisfies a Lipschitz condition*

$$(20_1) \qquad \|h\| \leqslant M_0\,, \qquad \left\|\frac{\partial h}{\partial u}\right\| < M_1\,,$$

2_1) *$\alpha(y, z)$, $\beta(x, z)$, $\gamma(x, y)$ are three n-dimensional continuous vector functions, defined on:*

$$(20_2) \qquad [0, b] \times (-\infty, \infty)\,, \quad [0, a] \times (-\infty, +\infty)\,, \quad [0, a] \times [0, b]\,,$$

respectively, satisfying the compatibility conditions

$$(20_3) \qquad \begin{aligned} \alpha(0, z) &= \beta(0, z)\,, \\ \beta(x, 0) &= \gamma(x, 0)\,, \\ \gamma(0, y) &= \alpha(y, 0)\,, \end{aligned}$$

3) *The function $p(x, y)$ is strictly positive, bounded, has bounded first derivatives and bounded mixed derivative*:

$$(20_4) \qquad p(x, y) \geqslant q > 0\,, \qquad |p(x, y)|\,, \ \left|\frac{\partial p}{\partial x}\right|, \ \left|\frac{\partial p}{\partial y}\right|, \ \left|\frac{\partial^2 p}{\partial x\,\partial y}\right| \leqslant p_1\,.$$

4) *Denoting by $\mathfrak{R}(x, y, z; \xi, \eta, \zeta)$ the resolvent kernel of the kernel $\partial h/\partial v$, let M_2 be the bound of $\mathfrak{R}$, $\partial\mathfrak{R}/\partial x$, $\partial\mathfrak{R}/\partial y$, $\partial^2\mathfrak{R}/\partial x\,\partial y$, for $0 \leqslant z \leqslant m > 0$; i.e.*

$$\|\mathfrak{R}\|\,, \ \left\|\frac{\partial\mathfrak{R}}{\partial x}\right\|, \ \left\|\frac{\partial\mathfrak{R}}{\partial y}\right\|, \ \left\|\frac{\partial^2\mathfrak{R}}{\partial x\,\partial y}\right\| \leqslant M_2 \quad \text{(depending, obviously on } M_1)\,.$$

5) *Suppose that the constants a, b, p_1, q, M_1, M_2 satisfy the relation*

$$\frac{p_1}{q}\left[1 + \left(\frac{p_1}{q} + 1\right)\left(a + b + ab\left(\frac{p_1}{q} + \frac{1}{2}\right)\right)\right] + \\ + \mu p_1[M_1(1 + a)(1 + b) + M_2(1 + 2a)(1 + 2b)] < 1\,,$$

then

1) *the equation*

$$(21) \qquad \frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = h(x, y, z, u)\,,$$

has a unique solution belonging to K_3 *on* $[0, a]\times[0, b]\times[0, m]$, *and satisfying the conditions*

$$u(x, y, 0) = \gamma(x, y)\,,$$
$$(22)\qquad u(x, 0, z) = \beta(x, z)\,,$$
$$u(0, y, z) = \alpha(y, z)\,.$$

2) *It is possible to choose* $\gamma(x, y)$ *such that the equation* (21) *has a solution* u *periodic in* z *which satisfies*

$$(23)\qquad u(x, y, z, p(x, y)) = u(x, y, 0)\,.$$

First, we see that the given problem is equivalent to the one of finding a solution of the Volterra non-linear integral equation

$$(24)\qquad u(x, y, z) = \alpha(y, z) + \beta(x, z) + \gamma(x, y) - \gamma(x, 0) - \gamma(0, y) - \\ - \beta(0, z) + \beta(0, 0) + \int_0^x\int_0^y\int_0^z h(\xi, \eta, \zeta, u(\xi, \eta, \zeta))\, d\xi\, d\eta\, d\zeta\,.$$

satisfying the generalized periodicity condition

$$(25)\qquad B(x, y, 0) + \int_0^x\int_0^y\int_0^z h(\xi, \eta, \zeta, u(\xi, \eta, \zeta))\, d\xi\, d\eta\, d\zeta = 0\,,$$

where

$$B(x, y, z) = \alpha\big(y, z + p(x, y)\big) - \alpha(y, z) + \beta\big(x, z + p(x, y)\big) - \beta(x, z) - \\ - \beta\big(0, z + p(x, y)\big) + \beta(0, 0)\,.$$

The proof of the existence and uniqueness of a solution of (24) is standard, by using Banach's fixed point theorem. Therefore we shall not insist on it.

Concerning the existence of a solution of (24) which, in addition satisfy (25), we consider the operator

$$(26)\qquad T\gamma = \frac{\partial^2 B(x, y, 0)}{\partial xy\partial y} + \frac{\partial^2}{\partial x\,\partial y}\left\{\int_0^x d\xi\int_0^y d\eta\int_0^{p(x,y)}\left[\frac{1}{p(x, y)}\,\gamma(\xi, \eta) + \mu h(\xi, \eta, \zeta), u(\xi, \eta, \zeta)\right]d\zeta\right\}$$

where $u(\xi, \eta, \zeta)$ is the solution of (24) corresponding to $\gamma(x, y)$ for $z = 0$, and μ a constant.

Let now B be the Banach space of continuous functions on $[0, a]\times[0, b]$ with the sup-norm; taking into account (20), one can easily see that T transforms B into itself.

If now γ_1 and γ_2 are two functions satisfying the same conditions as γ, and u_1 and u_2 are the corresponding solutions of (24), we have:

$$u_2(x, y, z) - u_1(x, y, z) = \gamma_2(x, y) - \gamma_1(x, y) + \\ + \int_0^x \int_0^y \int_0^z \frac{\partial h}{\partial u}(\xi, \eta, \zeta, \omega, (\xi, \eta, \zeta))(u_2 - u_1)\, d\xi\, d\eta\, d\zeta ,$$

where

$$w(\xi, \eta, \zeta) = u_1(\xi, \eta, \zeta) + \eta[u_2(\xi, \eta, \zeta) - u_1(\xi, \eta, \zeta)] , \qquad |\eta| < 1 ,$$

and $\partial h/\partial u$ is the Fréchet derivative of h.

Denoting by $\mathfrak{R}(x, y, z, \xi, \eta, \zeta)$ the resolvent kernel of the kernel $\partial h/\partial u$, we can write:

$$(27) \qquad u_2(x, y, z) - u_1(x, y, z) = \gamma_2(x, y) - \gamma_1(x, y) + \\ + \int_0^x \int_0^y \int_0^z \mathfrak{R}(x, y, z; \xi, \eta, \zeta)(\gamma_2(\xi, \eta) - \gamma_1(\xi, \eta))\, d\xi\, d\eta\, d\zeta .$$

On the other hand, we have

$$(28) \qquad T\gamma_2 - T\gamma_1 = \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y \int_0^{p(x,y)} \Big\{ \Big[\frac{1}{p(x, y)} + \mu \frac{\partial h}{\partial u}(\xi, \eta, \zeta, w(\xi, \eta, \zeta)) \Big] (\gamma_2 - \gamma_1) + \\ + \mu \frac{\partial h}{\partial u} \int_0^\xi \int_0^\eta \int_0^\zeta \mathfrak{R}(\xi, \eta, \zeta; \xi_1, \eta_1, \zeta_1)(\gamma_2 - \gamma_1)\, d\xi_1\, d\eta_1\, d\zeta_1 \Big\}\, d\xi\, d\eta\, d\zeta$$

and if we put:

$$(29) \qquad \mathfrak{R}_1(x, y, z, \xi, \eta, \zeta) = \int_{\xi_1}^x \int_{\eta_1}^y \int_{\zeta_1}^z \frac{\partial h}{\partial u}(\xi, \eta, \zeta, w)\, \mathfrak{R}(\xi, \eta, \zeta, \xi_1, \eta_1, \zeta_1)\, d\xi_1\, d\eta_1\, d\zeta_1 ,$$

(28) becomes:

$$(30) \qquad T\gamma_2 - T\gamma_1 = \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y \int_0^{p(x,y)} \Big[\frac{1}{p(x, y)} + \mu \frac{\partial h}{\partial u}(\xi, \eta, \zeta, w(\xi, \eta, \zeta)) + \\ + \mu \mathfrak{R}_1(x, y, z; \xi, \eta, \zeta) \Big] [\gamma_2(\xi, \eta) - \gamma_1(\xi, \eta)]\, d\xi\, d\eta\, d\zeta .$$

If we denote also

$$(31) \qquad \frac{1}{p(x, y)} I + \mu \frac{\partial h}{\partial u}(\xi, \eta, \zeta, w) + \mu \mathfrak{R}_1(x, y, z, \xi, \eta, \zeta) = A(x, y, z, \xi, \eta, \zeta) ,$$

(30) gives:

$$T\gamma_2 - T\gamma_1 = \int_0^{p(x,y)} A(x, y, z; x, y, \zeta)(\gamma_2(x, y) - \gamma_1(x, y))\, d\xi +$$

$$+ \frac{\partial p}{\partial x}\int_0^x A(x, y, z; \xi, y, p(x, y))(\gamma_2(\xi, y) - \gamma_1(\xi, y))\, d\xi +$$

$$+ \frac{\partial p}{\partial y}\int_0^y A(x, y, z; x, \eta, p(x, y))(\gamma_2(x, \eta) - \gamma_1(x, \eta))\, d\eta +$$

$$+ \frac{\partial^2 p}{\partial x \partial y}\int_0^x\int_0^y A(x, y, z; \xi, \eta, p(x, y))(\gamma_2(\xi, \eta) - \gamma_1(\xi, \eta))\, d\xi d\eta +$$

$$+ \int_0^y\int_0^{p(x,y)} \frac{\partial A}{\partial y}(x, y, z; x, \eta, \zeta))\gamma_2(x, \eta) - \gamma_1(x, \eta))\, d\eta d\zeta +$$

$$+ \int_0^x\int_0^{p(x,y)} \frac{\partial A}{\partial x}(x, y, z; \xi, y, \zeta)(\gamma_2(\xi, y) - \gamma_1(\xi, y))\, d\xi d\zeta +$$

$$+ \frac{\partial p}{\partial x}\int_0^x\int_0^y \frac{\partial A}{\partial y}(x, y, z; \xi, \eta, p(x, y))(\gamma_2(\xi, \eta) - \gamma_1(\xi, \eta))\, d\xi d\eta +$$

$$+ \frac{\partial p}{\partial y}\int_0^x\int_0^y \frac{\partial A}{\partial x}(x, y, z; \xi, \eta, p(x, y))(\gamma_2(\xi, \eta) - \gamma_1(\xi, \eta))\, d\xi d\eta +$$

$$+ \int_0^x\int_0^y\int_0^{p(x,y)} \frac{\partial^2 A}{\partial x \partial y}(x, y, z; \xi, \eta, \zeta)(\gamma_2(\xi, \eta) - \gamma_1(\xi, \eta))\, d\xi d\eta\,.$$

It follows

$$\|T\gamma_2 - T\gamma_1\| \leqslant \tau \|\gamma_2 - \gamma_1\|\,,$$

where

$$\tau = p_1(1 + a)(1 + b)\|A\| + p_1(a + b + 2ab)\left\|\frac{\partial A}{\partial x}\right\| + p_1 ab\left\|\frac{\partial^2 A}{\partial x \partial y}\right\|.$$

Taking into account (31) and the relation

$$\|\mathfrak{K}_1\| \leqslant \left\|\frac{\partial h}{\partial u}\right\| \|\mathfrak{K}\| abp_1 \leqslant M_1 M_2 abp_1 \quad \text{for} \quad 0 \leqslant \zeta \leqslant z \leqslant p\,,$$

it follows:

$$\|A\| \leqslant \frac{1}{2} + \mu(M_1 + M_2)\,,$$

$$\left\|\frac{\partial A}{\partial x}\right\|,\ \left\|\frac{\partial A}{\partial y}\right\| \leqslant \frac{p_1}{q^2} + \mu M_2\,,$$

$$\left\|\frac{\partial^2 A}{\partial x \partial y}\right\| \leqslant \frac{p_1^2}{q^4} + \frac{p_1}{q^2} + \mu M_2\,;$$

hence:

$$\tau \leqslant p_1\left\{\frac{1}{q}(1+a)(1+b) + \frac{p_1}{q^2}(a+b+2ab) + \left(2\frac{p_1^2}{q^3} + \frac{p_1}{q^2}\right)ab\right\} + \\ + \mu p_1\left\{(M_1+M_2)(1+a)(1+b)p_1 + M_2(a+b+2ab) + M_2 ab\right\},$$

or

$$\tau \leqslant \frac{p_1}{q}\left[1 + \left(\frac{p_1}{q} + 1\right)(a+b+ab)\left(\frac{p_1}{q} + \frac{1}{2}\right)\right] + \mu p_1 M(1+a)(1+b) + \\ + \mu p_1 M_2 (1+2a)(1+2b)\,,$$

that is, taking into account the hypotheses,

$$\tau < 1\,.$$

It then follows that T has a unique fixed point. But this fixed point satisfies obviously the generalized periodicity condition. Our theorem is then proved.

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