Induction in Orthodox Involution Categories (Orthodox Categories, 3) (*) (**).

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Sunto. – Si definiscono i morfismi indotti in una categoria involutiva regolare, fattorizzante (dotata di fattorizzazioni epi-mono «uniche») e ortodossa (composizione di endomorfismi idempotenti è idempotente); si prova che in tale ambito l'ortodossia è condizione necessaria e sufficiente per la componibilità degli isomorfismi canonici (indotti dai morfismi identità). Questo lavoro e gli altri della serie saranno utilizzati per studiare le relazioni indotte tra subquozienti in una categoria esatta.

Introduction.

This paper is No. 3 in a series of four ⁽¹⁾; in No. 1 the *canonical preorder* ($\mathbf{\Phi}$) and *canonical congruence* (Φ) for an *orthodox* category (1.24) are defined (1.26-27); in No. 2 every category $\mathcal{K}$ provided with a regular involution is supplied with an orthodox involution category $\mathcal{K}^\square$ (2.20) which we call the *orthodox expansion* of $\mathcal{K}$.

Here we define *induction* (3.8) and *canonical isomorphisms* between subobjects in a factorizing (i.e. having «unique» epic-monic factorizations: see 3.1) orthodox involution category: canonical isomorphisms are composable, and orthodoxy is a necessary and sufficient condition (3.17) for such a fact.

These considerations will be applied in a following paper out of this series, to the canonical symmetrization $\mathcal{K}$ of an exact category $\mathcal{E}$, to supply a «good» definition of induced relations and canonical isomorphisms between subquotients of $\mathcal{E}$ (note that, $\mathcal{K}$ being a factorizing regular involution category, $\mathcal{K}^\square$ is still factorizing by 3.7).

The hypothesis « $\mathcal{K}$ factorizing» can be erased in 3.5-6 and in the whole section II (but for the last remark in 3.13).

Conventions and terminology of Nos. 1, 2 are always effective.

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(**) Entrata in Redazione il 15 Novembre 1976.

⁽¹⁾ The previous papers being: *Canonical preorder and congruence in orthodox semi-groups and categories* (Orthodox Categories, 1), Boll. Un. Mat. Ital., 13-B (1976), pp. 634-650; *Regular and orthodox involution categories* (Orthodox Categories, 2), Boll. Un. Mat. Ital., 14-A (1977), pp. 38-48. Item 1·p (resp. 2·p, 3·p) is to be found in No. 1 (resp. No. 2, here).

I. – Monomorphisms and projections.

3.1. From now on, the regular involution category $\mathcal{K}$ is supposed to be *factorizing*, that is any morphism α has an epic-monic factorization

$$(1) \quad \alpha = \mu \circ \pi \quad (\pi \text{ epic; } \mu \text{ monic})$$

unique up to isomorphism, the *canonical factorization* of α . Actually, it is easy to see that the uniqueness of these factorizations is here a consequence of the presence of a regular involution, by which any monic (resp. epic) α has a left (resp. right) inverse: $\tilde{\alpha}$. Moreover:

$$(2) \quad c(\alpha) = \tilde{\alpha}\alpha = \tilde{\pi}\pi = c(\pi), \quad i(\alpha) = \alpha\tilde{\alpha} = \mu\tilde{\mu} = i(\mu).$$

3.2. The usual preorder and equivalence relation among monics are written $<_M$ and $\sim_M$ ⁽²⁾; in each class of equivalent monomorphisms, forcibly having all the same codomain A , one is chosen to be called a *subobject* of A ; in the class of isomorphisms having codomain A we choose always 1_A . Dually are treated the epimorphisms, their preorder and equivalence being written $<_P$ and $\sim_P$; the choice of quotients is here determined by the precedent one, as we want that:

$$(1) \quad \mu \text{ is a subobject of } A \text{ iff } \tilde{\mu} \text{ is a quotient of } A.$$

By the previous assumption, $\mathcal{K}$ is provided with coimages and images: if 3.1.1 is a canonical factorization of $\alpha \in \mathcal{K}(A, B)$, *coim* α and *im* α are respectively the quotient of A and the subobject of B such that:

$$(2) \quad \text{coim } \alpha \sim_P \pi, \quad \text{im } \alpha \sim_M \mu.$$

By (1), it is easy to see that:

$$(3) \quad \text{coim } \tilde{\alpha} = (\text{im } \alpha)^\sim.$$

3.3. LEMMA. – Let $\sigma \in \mathcal{K}(A) = \mathcal{K}(A, A)$:

a) σ is idempotent iff there exist μ, ν monomorphisms of $\mathcal{K}$ such that:

$$(1) \quad \sigma = \nu\tilde{\mu}, \quad \tilde{\mu}\nu = 1$$

⁽²⁾ $\mu <_M \nu$ iff μ and ν are monics in $\mathcal{K}$, and there exists ν' in $\mathcal{K}$ such that $\mu = \nu\nu'$; $\mu \sim_M \nu$ iff $\mu <_M \nu$ and $\nu <_M \mu$.

and then

$$(2) \quad \tilde{\mu} \sim_{\mathbf{P}} \text{coim } \sigma, \quad \nu \sim_{\mathbf{M}} \text{im } \sigma$$

b) σ is a projection iff, moreover, $\mu = \nu$.

PROOF. – a) From condition (1): $\sigma\sigma = \nu\tilde{\mu}\nu\tilde{\mu} = \nu(\tilde{\mu}\nu)\tilde{\mu} = \nu\tilde{\mu} = \sigma$. Vice versa let $\sigma = \nu\tilde{\mu}$ be a canonical factorization of σ , an idempotent; then

$$\tilde{\mu}\nu = (\tilde{\nu}\nu)\tilde{\mu}\nu(\tilde{\mu}\mu) = \tilde{\nu}(\nu\tilde{\mu})(\nu\tilde{\mu})\mu = \tilde{\nu}(\nu\tilde{\mu})\mu = 1.$$

b) Sufficiency of condition $\mu = \nu$ being obvious, let us prove its necessity for $\sigma = \tilde{\sigma}$

$$\mu = \mu\tilde{\nu}\nu = \tilde{\sigma}\nu = \sigma\nu = \nu\tilde{\mu}\nu = \nu 1 = \nu.$$

3.4. For any object A in $\mathcal{K}$, let $\mathbf{M}_{\mathcal{K}}(A)$ and $\mathbf{M}'_{\mathcal{K}}(A)$ denote the «set» of $\mathcal{K}$ -subobjects of A and, respectively, of $\mathcal{K}$ -monomorphisms of codomain A , provided with their canonical preorder $<_{\mathbf{M}}$. We write also $\mathbf{M}(A)$ and $\mathbf{M}'(A)$ if no confusion may arise.

As in [1], 3.18, we consider two mappings

$$(1) \quad \mathbf{M}_{\mathcal{K}}(A) \rightarrow \mathcal{K}_0(A), \quad \mu \mapsto i(\mu) = \mu\tilde{\mu}$$

$$(2) \quad \mathcal{K}_0(A) \rightarrow \mathbf{M}_{\mathcal{K}}(A), \quad \varepsilon \mapsto \text{im } \varepsilon$$

It is easy to prove these mappings to be reciprocal (use 3.3 to prove that $i(\text{im } \varepsilon) = \varepsilon$); moreover, by 3.1.2, (2) takes $c(\alpha)$ in $(\text{coim } \alpha)^{\sim}$ and $i(\alpha)$ in $\text{im } \alpha$. The following conditions are easily seen to be equivalent, for $\mu, \nu \in \mathbf{M}'(A)$

$$(3) \quad \mu <_{\mathbf{M}} \nu$$

$$(4) \quad \mu = \nu\tilde{\nu}\mu$$

$$(5) \quad \mu\tilde{\mu} = \nu\tilde{\nu}\mu\tilde{\mu}$$

$$(6) \quad \mu\tilde{\mu} = \mu\tilde{\mu}\nu\tilde{\nu}$$

$$(7) \quad i(\mu) < i(\nu) \quad \text{in } \mathcal{K}_0(A) \quad (1.5)$$

Consequently:

$$(8) \quad \mu \sim_{\mathbf{M}} \nu \quad \text{iff} \quad i(\mu) = i(\nu).$$

3.5. For $\mu, \nu \in \mathbf{M}'(A)$, we consider the morphism

$$(1) \quad \tilde{\nu}\mu: \text{Dom } \mu \rightarrow \text{Dom } \nu.$$

The conditions (2), (2') (resp. (3), (3')) are easily seen to be equivalent:

- (2) $\tilde{\nu}\mu$ is monic
- (2') $i(\mu) = i(\mu) i(\nu) i(\mu)$
- (3) $\tilde{\nu}\mu$ is an isomorphism
- (3') $i(\mu) = i(\mu) i(\nu) i(\mu)$ and $i(\nu) = i(\nu) i(\mu) i(\nu)$.

In fact (2) means that:

$$(4) \quad 1 = (\tilde{\nu}\mu)^\sim (\tilde{\nu}\mu) = \tilde{\mu}\nu\tilde{\mu}$$

which is equivalent to

$$(5) \quad \mu\tilde{\mu} = \mu\tilde{\mu}\nu\tilde{\mu}\tilde{\mu}.$$

3.6. DEFINITION. – For $\mu, \nu \in \mathbf{M}'(A)$ we write

$$(1) \quad \mu \mathbf{\Phi}_M \nu$$

iff the equivalent conditions 3.5.2-2' are satisfied; and

$$(2) \quad \mu \mathbf{\Phi}_M \nu$$

iff $\mu \mathbf{\Phi}_M \nu$ and $\nu \mathbf{\Phi}_M \mu$, that is iff 3.5.3-3' are true.

Dually, for epics π, π' having the same domain, we write

$$(3) \quad \pi \mathbf{\Phi}_P \pi' \quad \text{iff} \quad \tilde{\pi} \mathbf{\Phi}_M \tilde{\pi}'$$

$$(4) \quad \pi \mathbf{\Phi}_P \pi' \quad \text{iff} \quad \tilde{\pi} \mathbf{\Phi}_M \tilde{\pi}'.$$

Relations (1), (2) may be non-transitive in $\mathbf{M}'(A)$ ⁽³⁾; nevertheless, when $\mathcal{K}$ is orthodox, relations 3.5.2'-3' on projections $i(\mu), i(\nu)$ become the domination $\mathbf{\Phi}$ and the canonical congruence $\mathbf{\Phi}$ of $\mathcal{K}$ (1.26 and 2.13), which proves that $\mathbf{\Phi}_M$ is a preorder and $\mathbf{\Phi}_M$ an equivalence relation: it becomes possible to speak of canonically isomorphic subobjects of $\mathcal{K}$ (see the next section). Before, let us ensure of the following fact.

3.7. THEOREM. – If $\mathcal{K}$ is a factorizing regular involution category, its orthodox expansion $\mathcal{K}^\square$ is factorizing.

⁽³⁾ If $\mathcal{K}$ is the category of relations among left R -modules (R a ring), relations (2) and (3) are proved ([2], p. 53) to be non-transitive in $\mathbf{M}'_{\mathcal{K}}(A \oplus A)$, where A is a non-zero left R -module. This is the reason to « replace » $\mathcal{K}$ by $\mathcal{K}^\square$, an orthodox category.

PROOF. – Let

$$(1) \quad \alpha' = (\alpha, S, T): (A, S) \rightarrow (B, T)$$

be a morphism in $\mathcal{K}^\square$, and

$$(2) \quad \alpha = \mu\pi, \quad \text{Cod}\pi = \text{Dom}\mu = C$$

a canonical factorization of α in $\mathcal{K}$; $\tilde{\pi}\pi = \tilde{\alpha}\alpha \in S$, hence (2.19) $\pi_\square(S)$ is an idempotent involution subsemigroup of C and

$$(3) \quad \pi' = (\pi, S, \pi_\square^\pi(S)): (A, S) \rightarrow (C, \pi_\square(S))$$

is an epimorphism (2.23) in $\mathcal{K}^\square$; likewise

$$(4) \quad \mu' = (\mu, \mu^\square(T), T): (C, \mu^\square(T)) \rightarrow (B, T)$$

is a monomorphism. Moreover:

$$(5) \quad \pi_\square(S) = \pi S \tilde{\pi} = \tilde{\mu}(\mu\pi S \tilde{\pi} \tilde{\mu})\mu = \mu^\square(\alpha_\square(S)) \subset \mu^\square(T)$$

$$(6) \quad \mu^\square(T) = \tilde{\mu}T\mu = \pi(\tilde{\pi}\tilde{\mu}T\mu\pi)\tilde{\pi} = \pi_\square(\alpha^\square(T)) \subset \pi_\square(S)$$

which proves that $\pi_\square(S) = \mu^\square(T)$; thus π' and μ' are composable, and

$$(7) \quad \alpha' = \mu'\pi'$$

the uniqueness of this factorization being a consequence of the regular involution on $\mathcal{K}^\square$ (3.1)

II. – Induction in factorizing orthodox involution categories.

In this section, $\mathcal{K}$ is always a factorizing (3.1) orthodox involution category (2.13), provided with its canonical preorder (or domination) $\mathbf{I}$ and its canonical congruence Φ (2.13). A is an object in $\mathcal{K}$.

3.8. DEFINITION. – If in the following (possibly non-commutative) diagram

$$(1) \quad \begin{array}{ccc} & \xrightarrow{\alpha} & \\ \uparrow \mu & & \uparrow \nu \\ & \xrightarrow{\alpha'} & \end{array}$$

μ and ν are monics, we shall say that α' is induced by α on μ and ν iff:

$$(2) \quad \alpha' \mathbf{I} \tilde{\nu} \alpha \mu .$$

This definition will be justified in section IV (3.19-21).

3.9. Induction respects identities (1 induces 1 on μ and μ), involution (α' is induced by α on μ and ν iff $\tilde{\alpha}'$ is induced by $\tilde{\alpha}$ on ν and μ) and composition:

$$(1) \quad \begin{array}{ccccc} & \xrightarrow{\alpha} & & \xrightarrow{\beta} & \\ \uparrow \mu & & \uparrow \nu & & \uparrow \xi \\ & \xrightarrow{\alpha'} & & \xrightarrow{\beta'} & \end{array}$$

α' and β' being induced by α and β on μ and ν , ν and ξ , we have

$$(2) \quad \beta' \alpha' \mathbf{I} (\xi \beta \nu)(\tilde{\nu} \alpha \mu) = \xi \beta (\nu \tilde{\nu}) \alpha \mu \mathbf{I} \xi \beta \alpha \mu$$

which proves that $\beta' \alpha'$ is induced by $\beta \alpha$ on μ and ξ .

3.10. If α induces an isomorphism α' on μ and ν , $\alpha' \mathbf{I} \tilde{\nu} \alpha \mu$ and, by 1.27g,

$$(1) \quad \alpha' = \tilde{\nu} \alpha \mu .$$

The induced isomorphism is then unique (when existing), and maximum among induced morphisms.

Composition of induced isomorphisms is still an induced morphism by 3.9, and obviously an isomorphism, hence the only possible.

3.11. Let us consider, particularly, a fixed object A and its subobjects: μ and ν being such, any morphism α' induced by 1_A on μ and ν :

$$(1) \quad \alpha' \mathbf{I} \tilde{\nu} \mu$$

will be said *induced on μ and ν* .

3.12. The following conditions on μ , ν , subobjects of A , are equivalent:

- (1) there is an isomorphism induced on μ and ν (by 1_A),
- (2) $\tilde{\nu} \mu$ is an isomorphism,
- (3) $\mu \Phi_M \nu$.

In fact (1) and (2) are equivalent by 3.10 (take $\alpha = 1_A$), (2) and (3) by 3.6.2. These conditions being satisfied, the isomorphism induced on μ and ν is forcibly $\tilde{\nu}\mu$ (3.10.1): it will be called the *canonical isomorphism* from μ to ν while μ and ν will be said to be *canonically isomorphic* (as subobjects of A). Canonical isomorphisms are composable by 3.10.

Having to use such isomorphisms in the case of a factorizing regular involution category (possibly non-orthodox), they will be called *pseudo-canonical isomorphisms*, in order to recall the possibility of non-componibility (see 3.19).

3.13. We consider now the inverse category $\mathcal{K} = \overline{\mathcal{K}}/\Phi$: monomorphisms in $\overline{\mathcal{K}}$ are just the images of monomorphisms in $\mathcal{K}$ (1.27h), thus $\overline{\mathcal{K}}$ is factorizing; moreover α induces α' on μ and ν in $\mathcal{K}$ iff $\tilde{\alpha}$ induces $\tilde{\alpha}'$ on $\tilde{\mu}$ and $\tilde{\nu}$ in $\overline{\mathcal{K}}$ (1.27e). In $\overline{\mathcal{K}}$, as in every factorizing inverse category, one has the advantage that, $\mathbf{I}$ being an order, there exists a unique maximum induced morphism (by $\tilde{\alpha}$ on $\tilde{\mu}$ and $\tilde{\nu}$: precisely $\tilde{\nu}\tilde{\alpha}\tilde{\mu}$); composition of maximum induced morphisms may, of course, not be maximum, but in the case of induced isomorphisms.

Remark also that, for $\tilde{\mu}, \tilde{\nu} \in M'_{\overline{\mathcal{K}}}(A)$:

$$\begin{aligned} (1) \quad & \tilde{\mu} \mathbf{I}_M \tilde{\nu} \quad \text{iff} \quad \tilde{\mu} <_M \tilde{\nu} \\ (2) \quad & \tilde{\mu} \Phi_M \tilde{\nu} \quad \text{iff} \quad \tilde{\mu} \sim_M \tilde{\nu} \end{aligned}$$

namely $\tilde{\mu} \mathbf{I}_M \tilde{\nu}$ iff $i(\tilde{\mu}) \mathbf{I} i(\tilde{\nu})$ by 3.6.1; $\tilde{\mu} <_M \tilde{\nu}$ iff $i(\tilde{\mu}) < i(\tilde{\nu})$ by 3.4; now $\mathbf{I}$ and $<$ coincide in $\overline{\mathcal{K}}_0(A)$, commutative idempotent semigroup.

Last, $M_{\overline{\mathcal{K}}}(A)$ has intersections, as $\overline{\mathcal{K}}_0(A)$:

$$(3) \quad \tilde{\mu} \cap_M \tilde{\nu} = im(i(\tilde{\mu})i(\tilde{\nu})).$$

III. – Necessity of orthodoxy.

Let $\mathcal{K}$ be a factorizing regular involution category, A an object to $\mathcal{K}$.

3.14. We associate to any set M of subobjects of A the involution subsemigroup ⁽⁴⁾ $S(M)$ of $\mathcal{K}(A)$ generated by all the projections $\mu\tilde{\mu}$, for $\mu \in M$; we associate to any involution subsemigroup S of $\mathcal{K}(A)$ the set $M(S)$ of subobjects μ of A such that $\mu\tilde{\mu} \in S$ ⁽⁵⁾.

⁽⁴⁾ We recall (see 2.19, footnote ⁽⁴⁾) that an *involution subsemigroup* is here a subsemigroup stable for involution and *having the same identity of the semigroup* in object.

⁽⁵⁾ Equivalent condition: $\mu = im(\alpha)$, for some $\alpha \in S$; in fact, μ being a subobject we have $\mu = im(\mu\tilde{\mu})$; viceversa, $\mu = im(\alpha)$, with $\alpha \in S$, implies $\mu\tilde{\mu} = \alpha\tilde{\alpha} \in S$.

3.15. These mappings constitute a *Galois connection* [3, 4]:

$$\begin{aligned} (1) \quad & M \subset M' \Rightarrow \mathbf{S}(M) \subset \mathbf{S}(M'), \quad S \subset S' \Rightarrow \mathbf{M}(S) \subset \mathbf{M}(S') \\ (2) \quad & \mathbf{MS}(M) \supset M \quad \quad \quad \mathbf{SM}(S) \subset S. \end{aligned}$$

An involution subsemigroup S of $\mathcal{K}(A)$ is closed in this connection (i.e. $S = \mathbf{SM}(S)$) iff it is generated by its projections; a characterization of closed sets of subobjects of A will be given in No. 4.

3.16. MAIN THEOREM. – Let S be an involution subsemigroup of $\mathcal{K}(A)$, generated by its projections; if $M = \mathbf{M}(S)$, the following conditions are equivalent

- (1) S is idempotent
- (2) pseudo-canonical isomorphisms (3.12) among elements of M (as subobjects of A) are composable; i.e., if $\mu_i \in M$ ($i = 1, 2, 3$) and $\tilde{\mu}_2\mu_1, \tilde{\mu}_3\mu_2$ are isomorphisms then $(\tilde{\mu}_3\mu_2)(\tilde{\mu}_2\mu_1) = \tilde{\mu}_3\mu_1$.

PROOF. – (1) $\Rightarrow$ (2). Let S be idempotent, $\mu_i: L_i \rightarrow A$ elements of M ($i = 1, 2, 3$) and $\tilde{\mu}_2\mu_1, \tilde{\mu}_3\mu_2$ isomorphisms.

In $\mathcal{K}^\square(2.20)$ we consider the following objects and monomorphisms ($i = 1, 2, 3$):

$$\begin{aligned} (3) \quad & A' = (A, S), \quad L'_i = (L_i, \mu_i^\square(S)) \\ (4) \quad & \mu'_i = (\mu_i, \mu_i^\square(S), S) \end{aligned}$$

(which is justified by 2.23, as $M = \mathbf{M}(S)$ and $\mu_i \in M$ gives $\mu_i\tilde{\mu}_i \in S$).

In $\mathcal{K}^\square$, a factorizing orthodox involution category, canonical isomorphisms are composable (3.10) so that

$$(\tilde{\mu}'_3\mu'_2)(\tilde{\mu}'_2\mu'_1) = \tilde{\mu}'_3\mu'_1$$

and in $\mathcal{K}$

$$(\tilde{\mu}_3\mu_2)(\tilde{\mu}_2\mu_1) = \tilde{\mu}_3\mu_1.$$

A direct proof, not using $\mathcal{K}^\square$, will be given in 3.18.

(2) $\Rightarrow$ (1). Let ε and σ be respectively a projection and an idempotent of S ; by 2.6 it is sufficient to prove that $\sigma\varepsilon$ is idempotent.

First, σ being idempotent in S , we have (3.3a) two monomorphisms μ, ν of $\mathcal{K}$ such that

$$(5) \quad \sigma = \nu\tilde{\mu}, \quad \tilde{\mu}\nu = 1$$

moreover $\mu\tilde{\mu} = \tilde{\sigma}\sigma$ and $\nu\tilde{\nu} = \sigma\tilde{\sigma}$ are in S (that is, μ and ν —which we cannot pretend to be *both* subobjects—are equivalent to elements of M , which we shall write μ , $\nu \in M'$).

In the same way, $(\mu\tilde{\mu})\varepsilon$, idempotent by 2.3, has a factorization:

$$(6) \quad \mu\tilde{\mu}\varepsilon = v_1\tilde{\mu}_1, \quad \tilde{\mu}_1 v_1 = 1 \quad (\mu_1, v_1 \in M')$$

and we have the following (possibly non-commutative) diagram:

$$(7) \quad \begin{array}{ccccc} A & \xrightarrow{\varepsilon} & A & \xrightarrow{\sigma} & A \\ \uparrow \mu_1 & & \uparrow v_1 & \nwarrow \mu & \nearrow \nu \\ & \bullet \xrightarrow{\quad} \bullet & & & \\ & v_0 & & & \end{array}$$

From

$$(8) \quad v_1 \sim_M im(\mu\tilde{\mu}\varepsilon) <_M \mu$$

it follows the existence of a monomorphism v_0 in $\mathcal{C}$ such that $v_1 = \mu v_0$. Hence:

$$(9) \quad \sigma\varepsilon = v\tilde{\mu}\varepsilon = v\tilde{\mu}(\mu\tilde{\mu}\varepsilon) = v\tilde{\mu}v_1\tilde{\mu}_1 = v\tilde{\mu}\mu v_0\tilde{\mu}_1 = (vv_0)\tilde{\mu}_1.$$

we prove that $vv_0 \in M'$; by (5):

$$(10) \quad vv_0\tilde{v}_0\tilde{v} = v(\tilde{v}\mu)v_0\tilde{v}_0(\tilde{\mu}\nu)\tilde{v} = (v\tilde{v})(v_1\tilde{v}_1)(v\tilde{v})$$

and the last product is in S , as $v, v_1 \in M'$.

We use now the componibility of pseudo-canonical isomorphisms among elements of M' (immediately following from the same property for M), for $\mu_1, v_1, (vv_0)$

$$(11) \quad \tilde{v}_1\mu_1 = 1 \quad \text{by (6)}$$

$$(12) \quad (vv_0) \sim v_1 = \tilde{v}_0\tilde{v}\mu v_0 = \tilde{v}_0 v_0 = 1$$

hence $(vv_0) \sim \mu_1 = 1$. Last, $\sigma\varepsilon$ is idempotent by (9) and 3.2a.

3.17. COROLLARY. — If $\mathcal{C}$ is a factorizing regular involution category, its pseudo-canonical isomorphisms among subobjects are composable iff $\mathcal{C}$ is orthodox.

3.18. Let us give a direct proof of (1) $\Rightarrow$ (2) in 3.16.

We have $\mu_1, \mu_2, \mu_3 \in M$, $\iota = \tilde{\mu}_2\mu_1$ and $\varkappa = \tilde{\mu}_3\mu_2$ being isomorphisms; we want to

prove that $\kappa\iota = \tilde{\mu}_3\mu_1$. Now the morphism (2)

$$(1) \quad \begin{array}{ccccc} & A & & A & & A \\ & \uparrow & & \uparrow & & \uparrow \\ \mu & \cdot & \xrightarrow{\iota} & \cdot & \xrightarrow{\kappa} & \cdot & v \\ & \uparrow & & \uparrow & & \uparrow \\ & B & & B & & B \end{array}$$

$$(2) \quad \mu_3\kappa\tilde{\mu}_1 = (\mu_3\tilde{\mu}_3)(\mu_2\tilde{\mu}_2)(\mu_1\tilde{\mu}_1)$$

is in $\mathcal{S}$, then an idempotent; by 3.3a there exist monics μ, v such that

$$(3) \quad \mu_3\kappa\tilde{\mu}_1 = v\tilde{\mu}, \quad \tilde{\mu}v = 1$$

$$(4) \quad v \sim im(\mu_3\kappa\tilde{\mu}_1) < \mu_3$$

$$(5) \quad \mu \sim im(\mu_1\tilde{\mu}\tilde{\mu}_3) < \mu_1$$

and we have monomorphisms v_1, v_3 such that

$$(6) \quad \mu = \mu_1v_1, \quad v = \mu_3v_3$$

Hence:

$$(7) \quad \kappa\iota = \tilde{\mu}_3(\mu_3\kappa\tilde{\mu}_1)\mu_1 = \tilde{\mu}_3v\tilde{\mu}\mu_1 = \tilde{\mu}_3\mu_3v_3\tilde{v}_1\tilde{\mu}_1\mu_1 = v_3\tilde{v}_1$$

which proves that v_1, v_3 are isomorphisms so that:

$$(8) \quad \tilde{\mu}_3\mu_1 = v(\tilde{v}_3\tilde{\mu}_3\mu_1v_1)\tilde{v}_1 = v_3(\tilde{v}\mu)\tilde{v}_1 = v_3\tilde{v}_1 = \kappa\iota.$$

IV. – Why to define induction by domination.

3.19. Let $\mathcal{K}$ be a (factorizing) orthodox involution category: the necessity of orthodoxy has been proved in the preceding section, just in order to preserve the componibility of pseudo-canonical isomorphisms, whose definition does *not* require domination. We want now to justify our definition of induction (3.8).

3.20. We assume that a notion of *non-univocal induction* ⁽⁶⁾ has been defined (which we call **-induction* to avoid confusion with 3.8) so that:

⁽⁶⁾ This means that, for any triple (α, μ, v) where α is a morphism, μ and v are monics and $Cod \mu = Dom \alpha$, $Cod v = Cod \alpha$, a set $I(\alpha, \mu, v) \subset \mathcal{K}(Dom \mu, Dom v)$ is assigned, whose elements we say *induced by α on μ and v* .

- a) in any diagram 3.8.1 $\tilde{v}\alpha\mu$ is $*$ -induced by α on μ and v ;
- b) composition is respected (i.e. in any diagram 3.9.1, if α' and β' are respectively $*$ -induced by α on μ and v , and by β on v and ξ , then $\beta'\alpha'$ is $*$ -induced by $\beta\alpha$ on μ and ξ).

We want to prove that any $\alpha' \mathbf{I} \tilde{v}\alpha\mu$ is $*$ -induced by α on μ and v .

In other words, our definition of induction (3.8) will result the « more restricted » one among those which satisfy a) and b).

3.21. Let $\alpha' \mathbf{I} \tilde{v}\alpha\mu$ (3.8.1); there exist idempotents σ, τ such that

$$(1) \quad \alpha' = \tau \tilde{v} \alpha \mu \sigma = \tilde{v}(\nu \tau \tilde{v}) \alpha(\mu \sigma \tilde{\mu}) \mu.$$

Now $\mu \sigma \tilde{\mu} = \mu_{\square}(\sigma)$ and $\nu \tau \tilde{\nu} = \nu_{\square}(\tau)$ are idempotent (2.18) hence (2.4) composition of two projections, so that (3.3b) we can write:

$$(2) \quad \mu \sigma \tilde{\mu} = (\xi_2 \xi_2)(\xi_1 \xi_1), \quad \nu \tau \tilde{\nu} = (\xi_4 \xi_4)(\xi_3 \xi_3)$$

where all ξ_i are monics; in the diagram

$$(3) \quad \begin{array}{ccccccc} \bullet & \xrightarrow{1} & \bullet & \xrightarrow{1} & \bullet & \xrightarrow{\alpha} & \bullet & \xrightarrow{1} & \bullet & \xrightarrow{1} & \bullet \\ \uparrow \mu & & \uparrow \xi_1 & & \uparrow \xi_2 & & \uparrow \xi_3 & & \uparrow \xi_4 & & \uparrow \nu \\ \bullet & \xrightarrow{\tilde{\xi}_1 \mu} & \bullet & \xrightarrow{\tilde{\xi}_2 \xi_1} & \bullet & \xrightarrow{\tilde{\xi}_3 \alpha \xi_3} & \bullet & \xrightarrow{\tilde{\xi}_4 \xi_3} & \bullet & \xrightarrow{\tilde{\nu} \xi_4} & \bullet \end{array}$$

$\xi_1 \mu$ is $*$ -induced by 1 on μ and ξ_1 (3.20a), and so on; by 3.20b, the composition of the low row in (3) is $*$ -induced by α on μ and v ; by (1) and (2) this composition is just α' .

V. – Characterizations of domination in orthodox involution categories.

$\mathcal{K}$ is an orthodox involution category; A and B are objects in $\mathcal{K}$.

3.22. LEMMA. – If $\mu, \alpha \in \mathcal{K}(A, B)$ and μ is monic, the following conditions are equivalent:

- a) $\mu \mathbf{I} \alpha$;
- b) $\mu \mathbf{\Phi} \alpha$;
- c) $\tilde{\alpha} \mu = 1_A$ ($\tilde{\mu} \alpha = 1_A$).

If they are satisfied α too is monic, and $\mu \mathbf{\Phi}_M \alpha$; if $\mathcal{K}$ is inverse, the preceding conditions are equivalent to: $\mu = \alpha$.

PROOF. – $a) \Rightarrow c)$: $1 = \tilde{\mu}\mu \mathbf{I} \tilde{\alpha}\mu$, hence $1 = \tilde{\alpha}\mu$. $c) \Rightarrow b)$: $\tilde{\alpha}\mu = 1$ gives $\mu = \mu\tilde{\alpha}\mu$ and $\alpha = \alpha\tilde{\mu}\alpha$. $b) \Rightarrow a)$ and the last remark are obvious (1.27g).

3.23. LEMMA. – Let $\mu, \nu \in \mathbf{M}'_{\mathcal{K}}(A)$; the following are equivalent:

- a) $\mu \mathbf{I}_M \nu$;
- b) there exists a morphism α such that $\mu \mathbf{I} \nu\alpha$;
- c) there exists a morphism β such that $\mu \mathbf{I} \nu\beta$.

In such a case $\alpha \mathbf{I} \beta \mathbf{I} \tilde{\nu}\mu$ are monics. Moreover, $\mu \mathbf{I}_M \nu$ iff there exists an isomorphism ι such that $\mu \mathbf{I} \nu\iota$, and in such a case $\iota = \tilde{\nu}\mu$.

PROOF. – $a) \Rightarrow b)$: take $\alpha = \tilde{\nu}\mu$: $\mu = \mu\tilde{\mu}\mu \mathbf{I} \nu\tilde{\nu}\mu = \nu\alpha$. $b) \Rightarrow c)$: if $\mu \mathbf{I} \nu\alpha$ then $\mu \mathbf{I} \nu\alpha$ by 3.22; moreover $\alpha \mathbf{I} \tilde{\nu}\mu$, which is monic by 3.6. $c) \Rightarrow a)$: $\mu\tilde{\mu} \mathbf{I} \nu\beta\tilde{\beta}\tilde{\nu} \mathbf{I} \nu\tilde{\nu}$. The last assertion can be seen in a similar way; for the uniqueness of ι , recall that $\mathbf{I}$ -equivalent isomorphisms coincide (1.27g).

3.24. PROPOSITION. – Let $\alpha, \alpha' \in \mathcal{K}(A, B)$ have canonical factorizations:

$$(1) \quad \alpha = \mu\pi, \quad \alpha' = \mu'\pi'.$$

The following conditions are equivalent:

- a) $\alpha \mathbf{I} \alpha'$ (resp. $\alpha \mathbf{I} \alpha'$);
- b) $\pi \mathbf{I}_P \pi'$, $\mu \mathbf{I}_M \mu'$ and $\pi'\tilde{\pi} \mathbf{I} \tilde{\mu}'\mu$ (resp. $\pi \mathbf{I}_P \pi'$, $\mu \mathbf{I}_M \mu'$, $\pi'\tilde{\pi} = \tilde{\mu}'\mu$);
- c) $\pi'\tilde{\pi}$, $\tilde{\mu}'\mu$ are $\mathbf{I}$ -equivalent monics (resp. coincident isomorphisms);
- d) there is some monic (resp. some isomorphism) $\nu: \text{Cod } \pi \rightarrow \text{Cod } \pi'$ such that:
 $\nu\pi \mathbf{I} \pi'$ and $\mu'\nu \mathbf{I} \mu$.

If these conditions are satisfied:

$$(3) \quad \nu \mathbf{I} \pi' \tilde{\pi} \mathbf{I} \tilde{\mu}'\mu \quad (\text{resp. } \nu = \pi' \tilde{\pi} = \tilde{\mu}'\mu).$$

PROOF. – $a) \Rightarrow b)$: it is obvious that $\pi \mathbf{I}_P \pi'$ and $\mu \mathbf{I}_M \mu'$; moreover:

$$\begin{aligned} \pi' \tilde{\pi} &= \pi' \tilde{\pi} \tilde{\mu}\mu & \mathbf{I} & \pi' \tilde{\pi}' \tilde{\mu}'\mu = \tilde{\mu}'\mu \\ \pi' \tilde{\pi} &= \tilde{\mu}'\mu' \pi' \tilde{\pi} & \mathbf{I} & \tilde{\mu}'\mu \pi \tilde{\pi} = \tilde{\mu}'\mu. \end{aligned}$$

$b) \Leftrightarrow c)$: obvious by definition of $\mathbf{I}_M$, $\mathbf{I}_P$ (3.6). $b) \Rightarrow d)$ and the last remark (3) follow from 3.23. $d) \Rightarrow a)$: $\alpha = \mu\pi \mathbf{I} \mu'\nu\pi \mathbf{I} \mu'\pi' = \alpha'$. The conditions in parenthesis can be treated analogously; as in 3.23, apply 1.27g.

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