On the Periodic Motions Near a given Periodic Motion of a Dynamical System.

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§ 1. **Introduction.** — In a recent note (Comptes Rendus, 1921), BIRKHOFF proved a $2n$-dimensional generalization of a simple special case of POINCARÉ'S two-dimensional geometric theorem. It was there suggested how this theorem might be useful in establishing the existence of infinitely many periodic motions (of a dynamical system with fixed energy constant) in the neighborhood of a given periodic motion of general stable type. This application is carried out for the first time in the present paper. A summary of the necessary preliminaries is also given.

Suppose we have a dynamical system with $n+1$ degrees of freedom and a given periodic motion of general stable type. By a change of variables and a reduction of the order of the system with the help of the energy integral and the elimination of the time, the system can be written in the Hamiltonian form,

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = \frac{\partial H}{\partial x_i}, \qquad i = 1, 2, \dots n, \tag{1.1}$$

where H is an analytic function of $x_1, y_1, x_2, y_2, \dots x_n, y_n$ and t, and admits the period 2π in t. The periodic motion appears as a « generalized equilibrium » point, $x_1 = y_1 = x_2 = y_2 = \dots = x_n = y_n = 0$, and any further periodic solutions of (1.1), near this equilibrium point and having a period which is an integral multiple of 2π, correspond to periodic motions in the original $(2n+2)th$ order system near the given periodic motion.

Let

$$\begin{aligned} x_i &= f_i(x_{10}, y_{10}, x_{20}, y_{20}, \dots x_{n0}, y_{n0}, t) \\ y_i &= g_i(x_{10}, y_{10}, x_{20}, y_{20}, \dots x_{n0}, y_{n0}, t) \end{aligned} \qquad i = 1, 2, \dots n,$$

be the solution of (1.1) which takes on the initial values, $x_{10}, y_{10}, \dots x_{n0}, y_{n0}$, for $t = 0$, and let

$$\begin{aligned} x_{i1} &= f_i(x_{10}, y_{10}, \dots x_{n0}, y_{n0}, 2\pi) \\ y_{i1} &= g_i(x_{10}, y_{10}, \dots x_{n0}, y_{n0}, 2\pi). \end{aligned}$$

These equations define a transformation T of the neighborhood of the origin into itself, and evidently there is a one-to-one correspondence between the periodic solutions of period $2m\pi$ and the points that are invariant under T^m, the *mth* iterate of T. The following method for detecting these invariant points was given by BIRKHOFF as a generalization of POINCARÉ'S geometric theorem:

Let $x_{1m}, y_{1m}, \dots x_{nm}, y_{nm}$ represent the point into which the point $x_{10}, y_{10}, \dots x_{n0}, y_{n0}$ is carried by T^m. On account of the well known relative integral invariants of (1.1), it is seen that

$$dJ = \sum_{i=1}^{n} (x_{im}dy_{im} - y_{im}dx_{im} - x_{i0}dy_{i0} + y_{i0}dx_{i0})$$

is an exact differential. Changing the variables to the modified polar coördinates, $u_i = x_i^2 + y_i^2$ and $\theta_i = \tan^{-1}(y_i/x_i)$, we find that

$$dJ = \sum_{i=1}^{n} (u_{im}d\theta_{im} - u_{i0}d\theta_{i0}).$$

Now suppose that we are able to find a manifold defined by the equations,

$$u_{i0} = B_i(\theta_{10}, \theta_{20}, \dots \theta_{n0}), \quad i = 1, 2, \dots n \qquad (B_i \text{ analytic, periodic}) \tag{1.2}$$

such that along this manifold θ_{im} always differs from θ_{i0} by some integral multiple of 2π, i. e. $\theta_{im} - \theta_{i0} = 2k_i\pi$. Then we have $d\theta_{im} = d\theta_{i0}$ and hence

$$dJ = \sum_{i=1}^{n} (u_{im} - u_{i0})d\theta_{i0}$$

along the manifold. Integrating, we get J as a single valued function of $(x_{10}, y_{10}, \dots x_{n0}, y_{n0})$, unique save for an additive constant, defined over the manifold. Considered as a function of the θ_0's, it must therefore be periodic and must have at least 2^n critical points (1). But any critical point of J on the manifold is obviously invariant under T^m, since $dJ = 0$ implies that $u_{im} = u_{i0}$, while we already know that for the point in question $\theta_{im} = \theta_{i0} + 2k_i\pi$.

The existence of periodic motions therefore depends upon the existence

(1) A critical point is a point for which $dJ = 0$; these are to be counted with their proper multiplicity. The existence of two critical points — maximum and minimum — is obvious. An easy method of establishing the existence of $2^n - 2$ other critical points is to apply M. MORSE'S critical point relations (see, for instance, his paper, *Relations between the Critical Points of a Real Function of* n *Real Variables*, « Trans. Am. Math. Soc. », vol. 27 (1925), pp. 345-356) to the n dimensional torus for which the connectivity numbers (mod 2) are the binomial coefficients.

of manifolds of the type (1.2). To prove that these manifolds really do exist, we make use of a preliminary normalization of equations (1.1) ([1]). In terms of conjugate imaginary variables $p_1, q_1, \dots p_n, q_n$, the transformation T can be written in the form

$$p_{i_1} = p_{i_0} e^{\sqrt{-1} M_i(p_0, q_0)} + \Phi_i(p_0, q_0)$$
$$q_{i_1} = q_{i_0} e^{-\sqrt{-1} M_i(p_0, q_0)} + \Psi_i(p_0, q_0).$$

The $\Phi_i(p_0, q_0)$ and $\Psi_i(p_0, q_0)$ are convergent power series in $p_{10}, \dots q_{n0}$ beginning with terms of degree $2\mu + 1$, where μ is arbitrarily large. The $M_i(pq)$ are polynomials with real coefficients of degree μ at most in the n products $p_1 q_1$, $p_2 q_2, \dots p_n q_n$. Setting $p_i q_i = u_i$, we accordingly write $M_i(u) = \psi_i + \sum_{j=1}^{n} c_{ij} u_j + \dots$. The significance of the fact that we are dealing with a given periodic motion of *general* stable type is that there are no homogeneous linear relations with integral coefficients (not all zero) connecting the ψ_i and 2π, and that the determinant $|c_{ij}|$ is not zero. We shall regularly denote by c^{ij} the cofactor of c_{ij} divided by the determinant itself, so that

$$\sum_{k=1}^{n} c_{jk} c^{lk} = \sum_{k=1}^{n} c_{kj} c^{kl} = \delta_{jl}.$$

We now change back to real coördinates, $x_i = \frac{p_i + q_i}{2}$, $y_i = \frac{p_i - q_i}{2\sqrt{-1}}$. It is to be remembered that these changes in coördinates do not destroy the Hamiltonian form of equations (1.1). The transformation T now appears in the form,

$$x_{i_1} = x_{i_0} \cos \varphi_i - y_{i_0} \sin \varphi_i + X_i(x_0, y_0)$$
$$y_{i_0} = x_{i_0} \sin \varphi_i + y_{i_0} \cos \varphi_i + Y_i(x_0, y_0),$$

where, for abbreviation, we have set $M_i(x_0^2 + y_0^2) = \varphi_i$. The $\bar{X}_i(x, y)$ and $\bar{Y}_i(x, y)$ are real convergent power series in $x_1, y_1, x_2, y_2, \dots x_n, y_n$ beginning with terms of degree $2\mu + 1$. Finally on introducing modified polar coördinates, $u_i = x_i^2 + y_i^2$, $\theta_i = \tan^{-1}(y_i / x_i)$, the transformation T takes the form

$$u_{i_1} = u_{i_0} + U_i(u_0, \theta_0)$$
$$\theta_{i_1} = \theta_{i_0} + M_i(u_0) + \Theta_i(u_0, \theta_0).$$

([1]) Cf. G. D. Birkhoff, *Dynamical Systems*, Chapter III, particularly § 9. Also Chapter VI, § 1.

The formal expressions for U_i and Θ_i are readily written down:

$$U_i(u_0,\theta_0)=2X_iu_{i0}^{\frac{1}{2}}\cos(\theta_{i0}+\varphi_i)+2Y_iu_{i0}^{\frac{1}{2}}\sin(\theta_{i0}+\varphi_i)+X_i^2+Y_i^2$$

$$\Theta_i(u_0,\theta_0)=\arctan\frac{-X_i\sin(\theta_{i0}+\varphi_i)+Y_i\cos(\theta_{i0}+\varphi_i)}{u_{i0}^{\frac{1}{2}}+X_i\cos(\theta_{i0}+\varphi_i)+Y_i\sin(\theta_{i0}+\varphi_i)}.$$

$U_i(u,\theta)$ may be represented as a convergent power series in $u_1^{\frac{1}{2}}, u_2^{\frac{1}{2}},\dots u_n^{\frac{1}{2}}$, with coefficients which are analytic periodic functions of $\theta_1,\theta_2,\dots\theta_n$ of period 2π. It begins with terms of degree $2\mu+2$ in the $\sqrt{u}$'s. The expression for $\Theta_i(u,\theta)$ is not so simple and will be discussed later.

§ 2. **Some Fundamental Inequalities.** — Let it be understood once and for all that the capital letter A, followed perhaps by a subscript, is used throughout this paper to denote a suitably chosen positive number, independent of $u_1,\theta_1,\dots u_n,\theta_n$. Thus, for example, we know from the power series development of $U_i(u,\theta)$ in powers of $u_1^{\frac{1}{2}}, u_2^{\frac{1}{2}},\dots u_n^{\frac{1}{2}}$ that

$$|U_i(u,\theta)|\leqq A_1\cdot\left[\sum_{j=1}^{n}u_j^{\frac{1}{2}}\right]^{2\mu+2}\leqq A_1n^{\mu+1}\cdot\left[\sum_{j=1}^{n}u_j\right]^{\mu+1},$$

provided that the u's are sufficiently small. Thus, we may write:

$$(2.1)\qquad |U_i(u,\theta)|\leqq A_2\cdot\left[\sum_{j=1}^{n}u_j\right]^{\mu+1}.$$

The point in $2n$ dimensional space whose modified polar coördinates are represented by $u_1,\theta_1,u_2,\theta_2,\dots u_n,\theta_n$ will be denoted by the symbol (u,θ). Sometimes, when the θ's are not being emphasized and no confusion is likely to result, this same point will be denoted by the more abbreviated symbol (u). The « distance » between two such points, (u,θ) and (u',θ') is defined as $\sqrt{\sum_{j=1}^{n}(u_j'-u_j)^2}$. The « distance » is thus independent of the θ's and is equal to the ordinary distance between two corresponding points (u) and (u') in n dimensional space. The distance of (u,θ) from the origin will be denoted by ζ.

Let α denote a fixed positive number less than 1/2. We shall show that the following inequalities hold as long as ζ is sufficiently small and $u_i/u_j\geqq\alpha$

for all pairs of indices, $i, j = 1, 2, \ldots n$:

$$(2.2)\qquad \begin{cases} |U_i(u, \theta)| \leqq A \cdot \zeta^{\mu+1} \\ |\Theta_i(u, \theta)| \leqq A \cdot \zeta^{\mu} \\ |\partial U_i/\partial u_j| \leqq A \cdot \zeta^{\mu} \\ |\partial U_i/\partial \theta_j| \leqq A \cdot \zeta^{\mu+1} \\ |\partial \Theta_i/\partial u_j| \leqq A \cdot \zeta^{\mu-1} \\ |\partial \Theta_i/\partial \theta_j| \leqq A \cdot \zeta^{\mu}. \end{cases}$$

The first of these inequalities follows immediately from (2.1) from the fact that $(\Sigma\, u_j)^2 \leqq n \sum_{j=1}^{n} u_j^2 = n\zeta^2$. In order to prove the second inequality, we consider briefly the function $\Theta_i(u, \theta)$. It is of the form, $\arctan \dfrac{f(u, \theta)}{u_i^{\frac{1}{2}} - g(u, \theta)}$, where f and g are convergent power series in $u_1^{\frac{1}{2}}, u_2^{\frac{1}{2}}, \ldots u_n^{\frac{1}{2}}$ with coefficients which are analytic periodic functions of $\theta_1, \theta_2, \ldots \theta_n$. They begin with terms of degree $2\mu + 1$. Consequently we have $|f(u, \theta)|$, $|g(u, \theta)| \leqq A_3 \cdot \left[\sum_{j=1}^{n} u_j^{\frac{1}{2}}\right]^{2\mu+1}$ for $u_1, \ldots u_n$ sufficiently small. Let us temporarily make the definition:

$$\Theta_i(\xi;\, u, \theta) = \arctan \frac{\xi f(u, \theta)}{1 - \xi g(u, \theta)}, \quad \text{so that} \quad \Theta_i(u, \theta) = \Theta_i\left(\frac{1}{u_i^{\frac{1}{2}}};\, u.\ \theta\right).$$

We have $|\Theta_i(\xi;\, u, \theta)| \leqq A_4 |\xi f(u, \theta)| \leqq A_4 A_3 \cdot |\xi| \cdot \left[\sum_{j=1}^{n} u_j^{\frac{1}{2}}\right]^{2\mu+1}$ as long as $|\xi f(u, \theta)|$, $|\xi g(u, \theta)|$, and the u's are sufficiently small. Therefore

$$|\Theta_i(u, \theta)| \leqq A_3 A_4 \cdot \left|\frac{1}{u_i^{\frac{1}{2}}}\right| \cdot \left[\sum_{j=1}^{n} u_j^{\frac{1}{2}}\right]\left[\sum_{j=1}^{n} u_j^{\frac{1}{2}}\right]^{2\mu} = A_5 \cdot \left[\sum_{j=1}^{n} \left(\frac{u_j}{u_i}\right)^{\frac{1}{2}}\right] \cdot \left[\sum_{j=1}^{n} u_j^{\frac{1}{2}}\right]^{2\mu}.$$

But, we are assuming that $u_i/u_j \geqq \alpha$, and therefore we get

$$|\Theta_i(u, \theta)| \leqq A_5 n \sqrt{\frac{1}{\alpha}} \cdot \left[\sum_{j=1}^{n} u_j^{\frac{1}{2}}\right]^{2\mu} \leqq A_6 \cdot \zeta^{\mu},$$

as long as ζ is sufficiently small; here $A_6 = A_5 n^{\frac{3}{2}\mu+1}/\sqrt{\alpha}$. Similar considerations applied to the partial derivatives of $\Theta_i(\xi;\, u, \theta)$, with respect to ξ, the $u_j^{\frac{1}{2}}$, and the θ's, enable us to obtain the appraisals for $\partial\Theta_i/\partial u_j$ and $\partial\Theta_i/\partial\theta_j$. The appraisals for $\partial U_i/\partial u_j$ and $\partial U_i/\partial \theta_j$ are even easier.

§ 3. **A Simplification of the Coördinate System** (u, θ). — If we make the change of variables $\overline{u}_j = \sum_{k=1}^{n} c^{kj}[M_k(u) - \psi_k]$, the transformation T is readily seen $\left(\text{since} \sum_{k=1}^{n} c_{kj}c^{kj} = \delta_{ij}\right)$ to take the simpler form:

$$(3.1) \qquad \begin{aligned} u_{i_1} &= u_{i0} + U_i(u_0, \theta_0) \\ \theta_{i_1} &= \theta_{i0} + \psi_i + \sum_{j=1}^{n} c_{ij}u_{j0} + \Theta_i(u_0, \theta_0), \end{aligned}$$

where in accordance with our later notation the dashes over the new variables have been omitted. This change of variables is such that

$$(3.2) \qquad \overline{u}_i = u_i + p_i(\overline{u}); \quad u_i = \overline{u}_i + P_i(\overline{u}),$$

where $p_i(u)$ is a polynomial in $u_1, u_2, \ldots u_n$ which lacks constant and linear terms, and $P_i(u)$ is a convergent power series beginning with quadratic terms.

We must show that the inequalities of the previous paragraph still hold for these new variables as long as $\overline{\zeta}$ is sufficiently small and $\overline{u}_i/\overline{u}_j \geqq \overline{\alpha}$ for all pairs of indices $i, j = 1, 2, \ldots n$. Here $\overline{\alpha}$ is a fixed positive number less than 1/2, and $\overline{\zeta} = \sqrt{\sum_{j=1}^{n} \overline{u}_j^2}$.

Let α be any number such that $0 < \alpha < \overline{\alpha} < \frac{1}{2}$. Then starting with (3.2) it is easily shown that the fact that $\overline{u}_j/\overline{u}_k \leqq 1/\overline{\alpha}$ (for all pairs of indices j and k) implies that $u_j/u_k \leqq 1/\alpha$, provided that the $\overline{u}$'s are taken sufficiently small. It also follows from (3.2) that $\zeta \leqq A_1 \cdot \overline{\zeta}$, and that the derivatives $\partial u_j/\partial \overline{u}_i$ are bounded fos small values of the $\overline{u}$'s. This is all that is needed to verify the validity of inequalities (2.2) and (2.1) in the new variables.

Hereafter these new variables will be used exclusively with the dashes omitted.

§ 4. **The Behavior of the Image of a Point Under the Iterates of T.** — Let the m^{th} iterate of T take the point (u_0, θ_0) into (u_m, θ_m). In this paragraph we prove two fundamental theorems about the behavior of (u_m) for large values of m and small values of the u's.

THEOREM I. — *If* (u_0, θ_0) *is at a sufficiently small distance from the origin, then the distance* ζ_m, *of* (u_m, θ_m) *from the origin does not exceed* $n\zeta_0$ *as long as* $m \leqq A_{10}\zeta_0^{-\mu}$ $(n \geqq 2)$.

Proof: From (3.1) and (2.1) we have

$$|\Delta u_{im}| = |u_{im+1} - u_{im}| \leqq A_2 \cdot \left[\sum_{j=1}^{n} u_{jm}\right]^{\mu+1}.$$

Hence u_{im} increases less rapidly with m than as if $\dfrac{du_{im}}{dm} = \dfrac{A_8}{n}\left[\sum_{j=1}^{n} u_{jm}\right]^{\mu+1}$. It follows that $\sum_{j=1}^{n} u_{jm}$ can not increase to $\sqrt{n}$ times its initial value for $m=0$ until

$$m \geqq \frac{1}{\mu A_8}\left(1 - n^{-\frac{\mu}{2}}\right)\left[\sum_{j=1}^{n} u_{j0}\right]^{-\mu} \text{i. e. } m \geqq A_9\left[\left(\sum_{j=2}^{n} u_{j0}\right)^2\right]^{-\frac{\mu}{2}} \geqq A_9\left[n\sum_{j=1}^{n} u_{j0}^2\right]^{-\frac{\mu}{2}} = A_{10}\zeta_0^{-\mu}.$$

Hence, as long as $m \leqq A_{10}\cdot\zeta_0^{-\mu}$, we have

$$\sum_{j=1}^{n} u_{jm}^2 \leqq \left[\sum_{j=1}^{n} u_{jm}\right]^2 \leqq n\left[\sum_{j=1}^{n} u_{j0}\right]^2 \leqq u^2 \sum_{j=1}^{n} u_{j0}^2$$

since $u_j \geqq 0$, i. e. $\zeta_m \leqq n\zeta_0$ as long as $m \leqq A_{10}\cdot\zeta_0^{-\mu}$, q. e. d.

THEOREM II. — *If the point* (u_0) *is such that* $u_{j0}/u_{k0} \geqq 2\alpha$ *for all ordered pairs of indices* j *and* k, *then* $u_{jm}/u_{km} \geqq \alpha$ *as long as* $m \leqq A_{15}\cdot\zeta_0^{-\mu}$, *provided that* ζ_0 *is sufficiently small.*

Proof: As in the proof of theorem I, we have

$$\Delta u_{im} \leqq A_2 \cdot \left[\sum_{j=1}^{n} u_{jm}\right]^{\mu+1} \leqq A_{11}\zeta_m^{\mu+1}.$$

As long as $m \leqq A_{10}\zeta_0^{-\mu}$, we have from theorem I, $\Delta u_{im} \leqq A_{12}\zeta_0^{\mu+1}$. Hence $\sqrt{\sum_{i=1}^{u} \Delta u_{im}^2} \leqq A_{13}\zeta_0^{\mu+1} = d$, which is not less than the greatest distance the point (u_m) can move at each application of the transformation T.

Let $u_{j0}/u_{k0} = 2\beta_{jk} \geqq 2\alpha$. Also let $\lambda_i = u_{i0}/\zeta_0$, so that the λ_i are the « direction cosines » of the ray from the origin through (u_0). We have $\lambda_j/\lambda_k \geqq 2\alpha$. Hence $n\lambda_j^2 \geqq 4\alpha^2 \sum_{k=1}^{n} \lambda_k^2 = 4\alpha^2$. Therefore $\lambda_j \geqq \dfrac{2\alpha}{\sqrt{n}}$.

We consider some other point (u), which for a certain pair of indices, j and k, is such that $u_j/u_k = \alpha$. The distance between the point (u_0) [regarded as fixed] and the point (u) [regarded as variable subject to the condition $u_j - \alpha u_k = 0$] is given by $\sqrt{\sum_{i=1}^{n}(u_i - u_{i0})^2}$, the minimum value, D, of which is found by elementary methods to be

$$D = \frac{u_{j0} - \alpha u_{k0}}{\sqrt{1+\alpha^2}} = \frac{(2\beta_{jk} - \alpha)u_{k0}}{\sqrt{1+\alpha^2}} \geqq \frac{\alpha u_{k0}}{\sqrt{1+\alpha^2}} = \frac{\alpha\lambda_k\zeta_0}{\sqrt{1+\alpha^2}} \geqq \frac{2\alpha^2}{\sqrt{n(1+\alpha^2)}}\zeta_0 = A_{14}\cdot\zeta_0.$$

This distance cannot be traversed by the point (u_m) upon successive iterations of T until $m \geqq D/d \geqq (A_{14}/A_{13})\zeta_0^{-\mu}$, unless perhaps m first becomes greater than $A_{10}\zeta_0^{-\mu}$, at which point the necessary information from Theorem I would no longer be forthcoming.

Hence the theorem is true as stated, if we denote by A_{15} the lesser of the two numbers A_{10} and (A_{14}/A_{13}).

Let the region $R(\eta, \alpha)$ denote the collection of points for which $\zeta \leqq \eta$ and $u_j/u_k \geqq \alpha$ for all pairs of indices j and k. Theorems I and II show us that if (u_0, θ_0) is a point of $R(\eta, 2\alpha)$, then the image point (u_m, θ_m) under T^m must lie within $R(n\eta, \alpha)$ as long as $m \leqq A_{15}\zeta_0^{-\mu}$, provided that ζ_0 is sufficiently small.

§ 5. **The Non-Vanishing Property of the Jacobian.** — We now proceed to prove

THEOREM III. — *If* K *is any positive number and if* η *is a sufficiently small positive number, then for* (u_0, θ_0) *in* $R(\eta, 2\alpha)$ *the derivative* $\frac{\partial \theta_{im}}{\partial u_{j0}}$ *differs from* mc_{ij} *by a quantity which tends to zero with* ζ_0, *as long as* m *does not exceed* $K\zeta_0^{-\frac{\mu}{4}+1}$. *This tendency to zero is uniform with respect to* m.

Proof: We introduce the notation $\frac{\partial \theta_{im}}{\partial u_{k0}} = v_{ik}(m)$, $\frac{\partial u_{im}}{\partial u_{k0}} = w_{ik}(m)$. $[m, k]$ will be used as a symbol to denote any linear homogeneous function of $v_{1k}(m), v_{2k}(m), \ldots v_{nk}(m), w_{1k}(m), w_{2k}(m), \ldots w_{nk}(m)$, whose coefficients, depending upon m and (u_0, θ_0), are infinitesimals of at least the $(\mu-1)^{th}$ order in ζ_0 for $m \leqq K\zeta_0^{-\mu+1}$ uniformly for (u_0, θ_0) in $R(\eta, 2\alpha)$. The sum of any definite number N of the symbols $[m, k]$ is another symbol $[m, k]$, N being assumed independent of m or ζ_0. Let

$$(5.1) \qquad \begin{cases} a_{ik} = \dfrac{\partial \theta_{i1}}{\partial u_{k0}} = c_{ik} + \dfrac{\partial \Theta_i}{\partial u_{k0}}(u_0, \theta_0) \\ v_{ik}(1) = a_{ik}, \quad v_{ik}(0) = 0, \quad w_{ik}(0) = \delta_{ik}. \end{cases}$$

Now by the elementary rules for partial differentiation we find

$$(5.2) \qquad \begin{cases} v_{ik}(m+1) = \sum\limits_{j=1}^{n} \dfrac{\partial \theta_{im+1}}{\partial \theta_{jm}} v_{jk}(m) + \sum\limits_{j=1}^{n} \dfrac{\partial \theta_{im+1}}{\partial u_{jm}} w_{jk}(m) \\ w_{ik}(m+1) = \sum\limits_{j=1}^{n} \dfrac{\partial u_{im+1}}{\partial \theta_{jm}} v_{jk}(m) + \sum\limits_{j=1}^{n} \dfrac{\partial u_{im+1}}{\partial u_{jm}} w_{jk}(m). \end{cases}$$

But

$$\frac{\partial\theta_{im+1}}{\partial\theta_{jm}} = \delta_{ij} + \frac{\partial\Theta_i(u_m, \theta_m)}{\partial\theta_{jm}}, \quad \frac{\partial\theta_{im+1}}{\partial u_{jm}} = c_{ij} + \frac{\partial\Theta_i(u_m, \theta_m)}{\partial u_{jm}},$$

$$\frac{\partial u_{im+1}}{\partial\theta_{jm}} = \frac{\partial U_i(u_m, \theta_m)}{\partial\theta_{jm}}, \quad \frac{\partial u_{im+1}}{\partial u_{jm}} = \delta_{ij} + \frac{\partial U_i(u_m, \theta_m)}{\partial u_{jm}}.$$

We now use the inequalities (2.2) with reference to the point (u_m, θ_m). These inequalities are here applicable, because from Theorem II $u_{jm}/u_{km} \geqq \alpha$ for all pairs of indices j and k, m being restricted in such a way that $m \leqq A_{15}\zeta_0^{-\mu}$. Remembering from Theorem I that $\zeta_m \leqq n\zeta_0$, and introducing the symbols $[m, k]$, we therefore get from (5.2)

$$(5.3) \qquad \begin{cases} \text{I.} \quad v_{ik}(m+1) = v_{ik}(m) + \sum_{j=1}^{n} c_{ij} w_{jk}(m) + [m, k] \\ \text{II.} \quad w_{ik}(m+1) = w_{ik}(m) + [m, k]. \end{cases}$$

We proceed to show how the w's can be eliminated from equations (5.3). Replacing m by $m+1$ in equations (5.3) I., we have

$$v_{ik}(m+2) = v_{ik}(m+1) + \sum_{j=1}^{n} c_{ij} w_{jk}(m+1) + [m+1, k].$$

Subtracting I from this, we get after transposing,

$$\Delta^2 v_{ik}(m) = v_{ik}(m+2) - 2v_{ik}(m+1) + v_{ik}(m) =$$
$$= \sum_{j=1}^{n} c_{ij}[w_{jk}(m+1) - w_{jk}(m)] + [m+1, k] + [m, k].$$

We eliminate the $w_{jk}(m+1)$ from these equations with the help of (5.3) II and thus obtain

$$(5.4) \qquad \Delta^2 v_{ik}(m) = [m+1, k] + [m, k],$$

where now the $w_{ik}(m+1)$ have already been eliminated from the symbol $[m+1, k]$. We now solve equations (5.3) I for the $w_{ik}(m)$ in terms of the $v_{ik}(m)$ and $v_{ik}(m+1)$. We can clearly do this, since the determinant of the coefficients of the unknowns is precisely the non-zero determinant $|c_{ij}|$ plus an infinitesimal in ζ_0 of order $\mu - 1$ at least. Substituting the resulting expressions for the $w_{ik}(m)$ into the right members (5.4), we see that the required elimination has been completely effected.

For convenience, let us now introduce the functions $y_{ik}(m)$ and the numbers b_{ik} as follows:

$$(5.5) \qquad \begin{cases} b_{ik} = a_{ik} \quad \text{and} \quad y_{ik}(m) = v_{ik}(m), \quad \text{if} \quad a_{ik} \geqq 0, \\ b_{ik} = -a_{ik} \quad \text{and} \quad y_{ik}(m) = -v_{ik}(m) \quad \text{if} \quad a_{ik} < 0. \end{cases}$$

We first wish to find out how large the $y_{ik}(m)$ can become while m is restricted by the inequality $m \leqq 2K\zeta_0^{-\frac{\mu}{4}+1}$. Evidently, on account of (5.4), the $|y_{ik}|$ can not increase as rapidly as they would if

$$\Delta^2 y_{ik}(m) = +\frac{2\rho}{n}\left[\sum_{j=1}^{n} y_{jk}(m+1)\right],$$

where for abbreviation $\rho = A_{16}\cdot\zeta_0^{\mu-1}$. A solution (unique for integral values of m) of this system of difference equations, under the initial conditions,

$$(5.6) \qquad y_{ik}(0) = 0, \quad y_{ik}(1) = b_{ik},$$

is readily found to be

$$(5.7) \quad y_{ik}(m) = b_{ik}m + \frac{1}{n}\left(\sum_{j=1}^{n} b_{jk}\right)\left[\frac{(1+\rho+\sqrt{2\rho+\rho^2})^m - (1+\rho-\sqrt{2\rho+\rho^2})}{2\sqrt{2\rho+\rho^2}} - m\right]$$

$$= b_{ik}m + \frac{1}{n}\left(\sum_{j=1}^{n} b_{ik}\right)\rho^{-\frac{1}{2}}\Omega\left(m\rho^{\frac{1}{2}},\ \rho^{\frac{1}{2}}\right),$$

where Ω is a convergent power series in powers of $m\rho^{\frac{1}{2}}$ and $\rho^{\frac{1}{2}}$, which lacks constant and linear terms.

Let us see how (5.7) behaves as we let ζ_0 (and consequently ρ) approach zero and allow m to take on values in the range,

$$0 < m \leqq 2K\zeta_0^{-\frac{\mu}{4}+1},$$

$$(5.8) \qquad \text{i. e. } 0 < m \leqq A_{17}\rho^{-\frac{\mu}{4}+\omega} = 2K\zeta_0^{-\frac{\mu}{4}+1}, \quad \text{where} \quad \omega = \frac{3}{4(\mu-1)} > 0.$$

We have from (5.5), (5.1), and (2.2):

$$(5.9) \qquad |b_{ik} - |c_{ik}|| \leqq \rho \quad (|c_{ik}| = \text{absolute value of } c_{ik} \text{ and not the determinant})$$

provided that ζ_0 is sufficiently small. Here we increase the number A_{16} defining ρ, if necessary. From the power series development of Ω, we have

$$\left|\Omega\left(m\rho^{\frac{1}{2}},\ \rho^{\frac{1}{2}}\right)\right| \leqq A_{18}\left(m\rho^{\frac{1}{2}} + \rho^{\frac{1}{2}}\right)^2 = A_{18}(m^2\rho + 2m\rho + \rho),$$

as long as both $m\rho^{\frac{1}{2}}$ and $\rho^{\frac{1}{2}}$ are taken sufficiently small. On account of (5.8) this requirement is surely fulfilled, if ζ_0 is sufficiently small; therefore

$$\left|\rho^{-\frac{1}{2}}\Omega\left(m\rho^{\frac{1}{2}},\ \rho^{\frac{1}{2}}\right)\right| \leqq A_{18}\left(m^2\rho^{\frac{1}{2}} + 2m\rho^{\frac{1}{2}} + \rho^{\frac{1}{2}}\right) \leqq A_{18}\left(A_{17}{}^2\rho^{2\omega} + 2A_{17}\rho^{\frac{1}{4}+\omega} + \rho^{\frac{1}{2}}\right)$$

by (5.8). Hence, since $\sum_{j=1}^{n} b_{jk}$ is bounded, we see that the second term in the right hand member of (5.7) tends to zero with ρ and ζ_0; (5.7) and (5.9) thus yield the result that

$$y_{ik}(m) - |c_{ik}|\, m = \bar{\iota}_{ik}(\zeta_0, m)$$

where $\lim_{\zeta_0=0} \bar{\iota}(\zeta_0, m) = 0$ uniformly in m, for the range (5.8).

Now let us see how rapidly the slopes of the $y_{ik}(m)$ can decrease, while we limit m to lie in the range,

$$(5.10) \qquad 0 < m \leqq A_{17}\rho^{-\frac{1}{4}+\omega} - 1 = 2K\zeta_0^{-\frac{\mu}{4}+1} - 1.$$

From the above results, it follows that the right hand members of (5.4) can not exceed, in absolute value, an expression of the form $A_{19}\rho^{\frac{3}{4}+\omega}$. Hence the y_{ik} can not decrease as rapidly as they would if they satisfied the difference relation $\Delta^2 y_{ik}(m) = -A_{19}\rho^{\frac{3}{4}+\omega}$. But, using the initial conditions (5.6), this yields $y_{ik}(m) = b_{ik}m - A_{19}\rho^{\frac{3}{4}+\omega} \cdot \frac{1}{2} m(m-1)$. Also

$$\left| A_{19}\rho^{\frac{3}{4}+\omega} \frac{1}{2} m(m-1) \right| = \frac{A_{19}}{2} \left| m^2\rho^{\frac{3}{4}+\omega} - m\rho^{\frac{3}{4}+\omega} \right| \leqq \frac{A_{19}}{2} \left| A_{17}{}^2\rho^{\frac{1}{4}+3\omega} + A_{17}\rho^{\frac{1}{2}+2\omega} \right|,$$

which tends to zero with ρ. Hence, we again obtain the result that $y_{ik}(m)$ differs from $|c_{ik}|\, m$ by an infinitesimal in ζ_0. Thus the true value of $y_{ik}(m)$ must also have this property. That is, from (5.5),

$$v_{ik}(m) \equiv \frac{\partial \theta_{im}}{\partial u_{k0}} \equiv \pm y_{ik}(m) = c_{ik}m + \iota_{ik}(\zeta_0, m),$$

where $\iota_{ik}(\zeta_0, m)$ represents an infinitesimal in ζ_0, uniformly with respect to m, as long as m lies on the range (5.10). It remains only to note that m will surely lie on the range (5.10), if it satisfies the inequalities

$$0 < m \leqq K\zeta_0^{-\frac{\mu}{4}+1}$$

provided, as always, that ζ_0 is sufficiently small.

Incidentally the theorem just proved shows us that $\frac{1}{m^n} \frac{\partial(\theta_{1m}, \theta_{2m}, \theta_{3m}, \ldots \theta_{nm})}{\partial(u_{10}, u_{20}, u_{30}, \ldots u_{n0})}$ differs from the determinant $|c_{ij}|$ by an infinitesimal in ζ_0 and hence can not vanish for ζ_0 sufficiently small.

§ 6. **A Simple Special Case.** — We consider here the degenerate case where, in the transformation (3.1), defining T, the U_i and the Θ_i are identically zero. In this case the m^{th} iterate of T becomes simply:

$$(6.1) \qquad \begin{cases} u_{im} = u_{i0} = u_i \\ \theta_{im} = \theta_{i0} + m\psi_i + m \sum_{j=1}^{n} c_{ij} u_j . \end{cases}$$

We shall use a theorem proved by BOREL [1] in the consideration of the approximation of irrational numbers by continued fractions. An immediate corollary of BOREL'S theorem is the following

LEMMA. — Corresponding to any positive number γ not less than 11 and to any real number β (rational or irrational), two integers p and q can always be found such that $\left|\beta - \frac{p}{q}\right| < \frac{1}{\sqrt{5}} \frac{1}{q^2}$ and $\gamma \leqq q \leqq 15\gamma^2 \cdot \left[p \gtreqless 0\right]$.

We consider a positive number, $\frac{1}{2}\eta$, which we regard as fixed but as having been chosen sufficiently small in advance. Let $\bar{u}_1 = \bar{u}_2 = \ldots = \bar{u}_n = \frac{\eta}{2\sqrt{n}}$.

Let $v_i = \sum_{j=1}^{n} c_{ij} \bar{u}_j$. Then

$$(6.2) \qquad \bar{u}_k = \sum_{j=1}^{n} c^{ik} v_i .$$

Let ν be a fixed positive number $\geqq 1$. Then, if η is sufficiently small, we have $\eta^{-\nu} \geqq 11$. Using the lemma, we choose integers m_i and k_i' such that $\left|\frac{k_i'}{m_i} - \frac{\psi_i + v_i}{2\pi}\right| < \frac{1}{\sqrt{5}} \frac{1}{m_i^2}$ and $\eta^{-\nu} \leqq m_i \leqq 15\eta^{-2\nu}$.

Write $m = m_1 \cdot m_2 \cdot m_3 \ldots m_n$ so that $\eta^{-n\nu} \leqq m \leqq 15^n \eta^{-2n\nu}$. And let $k_i = m_1 \cdot m_2 \ldots m_{i-1} \cdot k_i' \cdot m_{i+1} \cdot m_{i+2} \ldots m_n$. Then the above inequalities yield $\left|\frac{2k_i\pi}{m} - \psi_i - v_i\right| < \frac{2\pi}{\sqrt{5}} \frac{1}{m_i^2} \leqq \frac{2\pi}{\sqrt{5}} \eta^{2\nu}$. In other words $\frac{2k_i\pi}{m} - \psi_i = v_i + \varepsilon_i$, where $|\varepsilon_i| \leqq A_{20}\eta^{2\nu}$.

Now, using the above definitions for the integers $k_1, k_2, \ldots k_n$ and m, we define $u_1, u_2, \ldots u_n$ by means of the linear equations, $2k_i\pi = m\psi_i + m \sum_{j=1}^{n} c_{ij} u_j$, and hence $\sum_{j=1}^{n} c_{ij} u_j = v_i + \varepsilon_i$. Solving these equations for the u_i, we get with the help of (6.2)

$$u_k = \sum_{i=1}^{n} c^{ik}(v_i + \varepsilon_i) = \bar{u}_k + \bar{\varepsilon}_k , \quad \text{where} \quad |\bar{\varepsilon}_k| = \left|\sum_{i=1}^{n} c^{ik} \varepsilon_i\right| \leqq A_{21}\eta^{2\nu} .$$

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(1) ÉMILE BOREL, *Leçons sur la théorie de la croissance*, p. 149.

The distance of the point (u) from the point $(\bar{u})$ is

$$\sqrt{\sum_{k=1}^{n}(u_k - \bar{u}_k)^2} = \sqrt{\sum_{k=1}^{n}\bar{\varepsilon}_k^{\,2}} \leqq A_{22}\eta^{2\nu}.$$

On the other hand the distance of the point $(\bar{u})$ from any of the manifolds, $u_i - 2\alpha u_j = 0$ $\left[\alpha < \frac{1}{2}\right]$, is found to be $\dfrac{\bar{u}_i - 2\alpha\bar{u}_j}{\sqrt{1+4\alpha^2}} = \dfrac{1-2\alpha}{\sqrt{n(1+4\alpha^2)}}\cdot\dfrac{\eta}{2}$. Hence, if η is sufficiently small, the point (u) will lie well within the region $R(\eta, 2\alpha)$, its distance from the boundary exceeding $A_{23}\cdot\eta$.

In order to conform with the notation of the rest of the paper we replace ν by $\dfrac{\mu-4}{8n}$ $[\mu \geqq 8n+4]$ and u_i by u_{i_0}'. We collect the results of this paragraph in the following

THEOREM IV. — *Let the positive numbers* $\alpha\left(<\frac{1}{2}\right)$ *and* $\mu(\geqq 8n+4)$ *be chosen in advance and then held fast. Then it is possible to choose the positive number* η *so small that one can always find integers,* $k_1, k_2, \ldots k_n$, *and* m, *dependent upon* η *and having the following two properties*:

1. $\eta^{-\frac{\mu}{8}+\frac{1}{2}} \leqq m \leqq K\eta^{-\frac{\mu}{4}+1}$ $[K = 15^n]$.

2. *The solution of the linear equations* $2k_i\pi = m\psi_i + m\sum_{j=1}^{n} c_{ij}u_{j_0}'$, *yields a point* (u_0') *lying within the region* $R(\eta, 2\alpha)$, *its distance from the boundary exceeding* $A_{23}\eta$.

Here A_{23} is a suitably chosen positive number, dependent upon n, c_{ij}, μ, and α, but independent of η.

§ 7. **The General Case.** — We need the following elementary lemma:

LEMMA. — Let $f_i(u_1, u_2, \ldots u_n, t)$ $[i = 1, 2, \ldots n]$ be defined for $0 \leqq t \leqq 1$ and for (u) in some closed n dimensional region S. Let all the $f_i(u_1, u_2, \ldots u_n, 0)$ vanish together for one and only one set of values for the u's; viz. $u_i = u_i'$. Let K denote the shortest distance from (u') to the boundary of S. Suppose that f_i is of class C'' and that the Jacobian,

$$J = \frac{\partial(f_1, f_2, \cdots f_n)}{\partial(u_1, u_2, \ldots u_n)},$$

is nowhere zero for (u) in S and $0 \leqq t \leqq 1$. Let $g_{ij}(u_1, u_2, \ldots u_n, t)$ represent the cofactor of $\partial f_i/\partial u_j$ in J divided by J itself, so that

$$\sum_{j=1}^{n} g_{ij}\frac{\partial u_j}{\partial f_k} = \sum_{j=1}^{n} g_{ji}\frac{\partial f_j}{\partial u_k} = \delta_{ik}.$$

Finally denote by M an upper bound for the functions,

$$F_k(u_1, u_2, \ldots u_n, t) \equiv -\sum_{i=1}^{n} g_{ik} \frac{\partial f_i}{\partial t}.$$

That is, $|F_i| \leq M$ for (u) in S and $0 \leq t \leq 1$.

Then there exists a unique set of functions $u_1(t), u_2(t), \ldots u_n(t)$, of class C'', defined on some interval, $0 \leq t \leq t_1$; such that $f_i[u_1(t), u_2(t), \ldots u_n(t), t] \equiv 0$. Furthermore $u_i(0) = u_i'$ and t_1 is the lesser of the two numbers 1 and $\dfrac{\varkappa}{\sqrt{n}M}$.

This lemma is for our purposes more advantageous than the usual « implicit function theorem », because it gives us a definite appraisal for the interval on which the functions $u_1(t), \ldots u_n(t)$ are defined. We give a brief indication of the proof: Necessary and sufficient conditions on a set of functions, $u_1(t), \ldots u_n(t)$, that all the $f_i[u_1(t), \ldots u_n(t), t] \equiv 0$, are that

$$\frac{df_i}{dt} \equiv \sum_{j=1}^{n} \frac{\partial f_i}{\partial u_j}\frac{du_j}{dt} + \frac{\partial f_i}{\partial t} \equiv 0 \quad \text{and} \quad f_i[u_1(0), \ldots u_n(0), 0] = 0.$$

Solving for the derivatives of the u_i we get a system of differential equations, $\dfrac{du_i}{dt} = F_i(u_1, u_2, \ldots u_n, t)$, in the standard form. These equations are to be solved under the initial conditions $u_i(0) = u_i'$. The lemma now follows from the known existence theorems for systems of ordinary differential equations.

We consider values for η, not only sufficiently small for the validity of theorem IV, but also so small that all the results of the preceding paragraphs hold if $\zeta_0 \leq \eta$. We forthwith choose a set of values for $k_1, k_2, \ldots k_n$, and m, depending on η and satisfying the conditions of theorem IV,

$$\eta^{-\frac{\mu}{8}+\frac{1}{2}} \leq m \leq K\eta^{-\frac{\mu}{4}+1} \leq K\zeta_0^{-\frac{\mu}{4}+1}. \tag{7.1}$$

Now it is easily proved by induction that T^m may be written in the form

$$\left\{\begin{aligned} u_{im} &= u_{i0} + U_{im}(u_0, \theta_0) \\ \theta_{im} &= \theta_{i0} + m\psi_i + m\sum_{j=1}^{n} c_{ij}u_{j0} + \Theta_{im}(u_0, \theta_0) \end{aligned}\right. \tag{7.2}$$

where

$$\Theta_{im}(u_0, \theta_0) = \sum_{\nu=0}^{m-2}(m-1-\nu)\left[\sum_{j=1}^{n} c_{ij}U_j(u_\nu, \theta_\nu)\right] + \sum_{\nu=0}^{m-1}\Theta_i(u_\nu, \theta_\nu). \tag{7.3}$$

We wish to show that the equations, $\theta_{im} - \theta_{i0} = 2k_i\pi$ can be solved for the $u_{10}, u_{20}, \ldots u_{n0}$ in terms of the $\theta_{10}, \theta_{20}, \ldots \theta_{n0}$. For this purpose we regard the θ's as fixed and try to solve the equations,

$$(7.4) \qquad \Phi_i(u_{10}, \dots u_{n0}) \equiv -\frac{2k_i\pi}{m} + \psi_i + \sum_{j=1}^{n} c_{ij}u_{j0} + \frac{1}{m}\Theta_{im}(u_0, \theta_0) = 0,$$

for the u_0's. Since $m\Phi_i$ differs from θ_{im} by a constant, $2k_i\pi + \theta_{i_0}$, it appears that $\frac{\partial \Phi_i}{\partial u_{j0}} \equiv \frac{1}{m}\frac{\partial \theta_{im}}{\partial u_{j0}}$, which, according to theorem III, differs from c_{ij} by an infinitesimal in ζ_0, uniformly with respect to m, m being required not to exceed $K\zeta_0^{-\frac{\mu}{4}+1}\left(\geq K\eta^{-\frac{\mu}{4}+1}\right)$. This means that $\frac{\partial}{\partial u_{j0}}\left[\frac{1}{m}\Theta_{im}(u_0, \theta_0)\right]$ tends to zero with η, uniformly with respect to m, and, hence, in particular, if m is determined as in theorem IV. We assume always that (u_0) lies in $R(\eta, 2\alpha)$.

It will be convenient at this stage to introduce a parameter t and consider the equations,

$$(7.5) \qquad f_i(u_1, \dots u_{n0}, t) \equiv -\frac{2k_i\pi}{m} + \psi_i + \sum_{j=1}^{n} c_{ij}u_{j0} + \frac{t}{m}\Theta_{im}(u_0, \theta_0) = 0.$$

We allow t to assume values on the interval $0 \leq t \leq 1$, and we shall try to solve (7.5) for the u_0's as functions of t. Evidently, if we can do this, all we need to do is to set $t = 1$ to get the required solution of (7.4).

We try to apply the lemma. In the first place

$$\frac{\partial f_i}{\partial u_{j0}} = c_{ij} + t\frac{\partial}{\partial u_{j0}}\left[\frac{1}{m}\Theta_{im}(u_0, \theta_0)\right]$$

which differs from c_{ij} by an infinitesimal in η. Hence, if we denote by g_{ij} the cofactor of $\frac{\partial f_i}{\partial u_{j0}}$ in $\frac{\partial(f_1, f_2, \dots f_n)}{\partial(u_{10}, u_{20}, \dots u_{n0})}$, divided by the Jacobian itself, we see that g_{ij} differs from c^{ij} by an infinitesimal in η. Hence $|g_{ij}| \leq A_{24}$. Also it follows from (7.3), (2.2), (7.1), and from the elementary fact that $\sum_{\nu=0}^{m-2}(m-1-\nu) = \frac{1}{2}m(m-1)$ that

$$\left|\frac{\partial f_i}{\partial t}\right| = \left|\frac{1}{m}\Theta_{im}(u_0, \theta_0)\right| \leq A_{25}\eta^{\frac{3}{4}\mu+2}.$$

Hence, we find that

$$|F_k| \equiv \left|-\sum_{j=1}^{n} g_{ik}\frac{\partial f_i}{\partial t}\right| \leq nA_{24}A_{25}\eta^{\frac{3}{4}\mu+2} = A_{26}\eta^{\frac{3}{4}\mu+2}$$

which is the M of the lemma.

In the second place, we know from theorem IV that for $t = 0$ equations (7.5) have one and only one solution, $u'_{10}, u'_{20}, \dots u'_{n0}$, which lies well within the region $R(\eta, 2\alpha)$. For the region S of the lemma, we shall take a certain $n-1$ dimensional sphere together with its interior. The center of S is to be the point (u^1) and the radius is to be $A_{23}\eta^2$. By theorem IV the whole of S lies within $R(\eta, 2\alpha)$. The « K » of the lemma is therefore equal to $A_{23}\eta^2$.

Hence a unique set of solving functions $u_{10}(t), u_{20}(t), \ldots u_{n0}(t)$ exists, the interval of definition being $0 \leq t \leq t_1$, where t_1 is the lesser of the two numbers, $A_{27}^{-\frac{3}{4}\mu}$ and 1. Hence, if η is taken sufficiently small, $t_1 = 1$, and $u_{10}(1), u_{20}(1), \ldots u_{n0}(1)$ satisfy (7.4), as required. The solution (u_0), thus obtained, also has the property that its distance from (u_0') does not exceed $A_{23}\eta^2$. Hence ζ_0 exceeds an infinitesimal of the first order in η.

For a fixed sufficiently small value of η and with a corresponding fixed choice for the k_i and m, the solution (u_0) is unique, at least so far as the region $R(\eta, 2\alpha)$ is concerned. For suppose there were a second solution $(\bar{u}_0)$ corresponding to each element of an infinite sequence of η's tending to zero. We can join the two points (u_0) and $(\bar{u}_0)$ with a straight line segment, which lies wholly within $R(\eta, 2\alpha)$ and whose direction cosines we denote by $\lambda_1, \lambda_2, \ldots \lambda_n$. Since θ_{im} has the same value (viz. $\theta_{i0} + 2k_i\pi$) at both ends of the segment, its directional derivative, $\sum_{j=1}^{n} \frac{\partial \theta_{im}}{\partial u_{j0}} \lambda_j$, must vanish at some intermediate point P_i. We know, by theorem III, that

$$\frac{1}{m} \sum_{j=1}^{u} \frac{\partial \theta_{im}}{\partial u_{j0}}\Big|_P \lambda_j = \sum_{j=1}^{n} c_{ij}\lambda_j + \tau_i(P, \eta)$$

where $\tau_i(P, \eta)$ tends to zero with η, uniformly as to P or the λ's. Hence

$$\sum_{i=1}^{n} \left[\sum_{j=1}^{n} c_{ij}\lambda_j + \tau_i(P_i, \eta) \right]^2 = 0 = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij}\lambda_j \right)^2$$

$+$ an infinitesimal in η, where, of course, the λ_i depend upon η. Let $\Lambda_1, \Lambda_2, \ldots \Lambda_n$ be the set of values for the λ_i which, under the condition $\sum_{j=1}^{n} \lambda_k^2 = 1$ makes $\sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij}\lambda_j \right)^2$ a minimum. Then we infer that $\sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij}\Lambda_j \right)^2 \leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij}\lambda_j \right)^2 =$ an infinitesimal in η. But, since the Λ_i are independent of η, this implies that $\sum_{i=1}^{n} c_{ij}\Lambda_j = 0$ for Λ_i not all zero; and thus we obtain a contradiction of the hypothesis that the determinant of the c_{ij} is different from zero.

The obtained unique values for $u_{10}, u_{20}, \ldots u_{n0}$, which satisfy (7.4) may be regarded as single valued functions, $B_i(\theta_{10}, \theta_{20}, \ldots \theta_{n0})$ of the θ_0's. They are obviously periodic of period 2π, since the Φ_i are also. The implicit function theorem, furthermore, shows that they are analytic in the neighborhood of any point $(\theta_{10}, \theta_{20}, \ldots \theta_{n0})$. We have thus proved the following theorem, which is the main result of this paper:

THEOREM V. — *It is possible to find a manifold in the space of the* 2n *variables,* $u_{10}, \theta_{10}, \ldots u_{n0}, \theta_{n0}$ (*or the original variables* $x_{i0} = u_{i0}^{\frac{1}{2}} \cos \theta_{i0}$, $y_{i0} = u_{i0}^{\frac{1}{2}} \sin \theta_{i0}$) *defined by equations of the type,*

$$u_{i_0} = B_i(\theta_{10}, \theta_{20}, \ldots \theta_{n0}), \qquad i = 1, 2, \ldots n,$$

along which, for a suitable choice of the integer m, *the* θ_{im} *differ from the* θ_{i0} *by integral multiples of* 2π. *The* B_i *are analytic single-valued non-vanishing periodic functions of period* 2π *in the* θ_0's. *It is assumed that* μ (*which appears in the equations defining* T) *is not less than than* $8n + 4$.

Furthermore this manifold may be taken in such a way that, given a positive number $\alpha < \frac{1}{2}$, *the* u_{i_0} *satisfy the following relations*:

1. $\frac{u_{i_0}}{u_{j_0}} \geqq 2\alpha$ *for all pairs of indices* i, j = 1, 2,... n.
2. $\sum_{j=1}^{n} u_{j_0}^2 [\equiv \zeta_0^2]$ *is arbitrarily small.*
3. $0 \leqq m \leqq K \zeta_0^{-\frac{\mu}{4}+1}$, *where* $K = 15^n$.

Item 2, of course, implies that there are an infinite number of manifolds of the type described in the theorem.