Geometries involving affine connections and general linear connections.

An extension of the recent Einstein-Mayer geometry.

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Summary. - *Part I of this paper is concerned with the theory of differential invariants of a symmetric affine connection and a general linear connection of the KÖNIG type. Part II deals with a geometry in which the components of the affine connection are CHRISTOFFEL symbols, and the general linear connection is of a special sort. This section can be considered in part as an extension of the EINSTEIN-MAYER ⁽⁵⁾ geometry,*

INTRODUCTION

The study of the general linear connection was initiated by ROBERT KÖNIG ⁽¹⁾ and has been continued by L. SCHLESINGER ⁽²⁾, J. A. SCHOUTEN ⁽³⁾, J. H. C. WHITEHEAD ⁽⁴⁾, and others. A particular linear connection (with $m=5$, $n=4$) of the KÖNIG type has been used recently by A. EINSTEIN and W. MAYER ⁽⁵⁾ in the development of a geometry suitable for their unified field theory. Special cases, in which the associated spaces are tangent spaces, have been considered by H. WEYL, É. CARTAN, and others. For further references see O. VEULEN ⁽⁶⁾, L. P. EISENHART ⁽⁷⁾, and A. D. MICHAL ⁽⁸⁾.

In Part I of this paper we consider a manifold endowed with a symmetric affine connection and a general linear connection, and the composite tensors arising from the simultaneous consideration of these connections. With the aid of normal representations, normal tensors based on the linear connection and extensions of composite tensors are constructed. This leads to a general replacement theorem for simultaneous tensor differential invariants of a symmetric affine connection and a general linear connection. This includes as a special case the MICHAL-THOMAS ⁽⁹⁾ replacement theorem for differential invariants of an affine connection. E. BORTOLOTTI'S ⁽¹⁰⁾ two interesting memoirs on asymmetric connections and non-linear displacements suggest generalizations of our work. Furthermore, it is possible to consider invariants under projective changes of the two connections. Development of these theories will not be expounded here.

In Part II we consider a particular instance of this geometry, which is an extension of the EINSTEIN-MAYER geometry ⁽⁵⁾. In this case the manifold is a RIEMANNIAN space, the components of the affine connection are the CHRISTOFFEL symbols, and the linear connection is of a particular sort. For the general case ($m - n \neq 1$) we find that the calculation of the $m - n$ « exceptional directions » is not unique; and furthermore an additional postulate on the linear connection is necessary. Several of the theorems give new results for $m = 5$, $n = 4$, the EINSTEIN-MAYER case. It is of interest to note that the dependence of their field equations on V_4 tensors only is not a property peculiar to the tensors involved in those equations, but is a property common to all equations involving tensors of the form (11.4). See Theorem 28, below.

Without further statement, all functions considered in the paper will be assumed to be analytic in the region under consideration.

Numbers in parenthesis, as ⁽⁸⁾, refer to the papers listed at the end of the paper.

PART I.

Geometries involving affine connections and general linear connections.

1. **Definitions and Notation.** — Consider an n -dimensional manifold, with coordinates $(x^1, x^2, \dots, x^n)$, in which is defined a symmetric affine connection $\Gamma_{jk}^i(x)$ and a general linear connection $L_{\beta\alpha}^\alpha(x)$. By a symmetric affine connection we mean an invariant whose law of transformation under a change of coordinates

$$(1.1) \quad \bar{x}^i = \bar{x}^i(x^1, \dots, x^n), \quad \left| \frac{\partial \bar{x}^i}{\partial x^j} \right| \neq 0$$

is

$$(1.2) \quad \bar{\Gamma}_{jk}^i(\bar{x}) = \left(\Gamma_{bc}^a(x) \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} + \frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k} \right) \frac{\partial \bar{x}^i}{\partial x^a}.$$

A general linear connection is an invariant whose law of transformation under a change of representation is

$$(1.3) \quad M_\sigma^\sigma \bar{L}_{\beta\alpha}^\sigma(\bar{x}) = \left(L_{\lambda r}^\alpha(x) M_\beta^\lambda + \frac{\partial M_\beta^\alpha}{\partial x^r} \right) \frac{\partial x^r}{\partial \bar{x}^\sigma}.$$

A change of representation ^(*) denotes the composite of (i) a change of coor-

^(*) WHITEHEAD, loc. cit.; WHITEHEAD uses the terminology « frame of reference » instead of « vector coordinate system ».

dinates (1.1), (ii) a change of vector coordinates

$$V^\alpha = M^\alpha_\beta \bar{V}^\beta.$$

In the above formulas, as throughout the paper, we shall understand that small Latin indices range from 1 to n , and small Greek indices range 1 to m . Indices repeated in the same term are to be summed over their respective ranges.

We shall deal with composite tensors, i. e. tensors which may involve both Greek and Latin indices. The transformation law of the composite tensor $X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}(x)$ is

$$(1.4) \quad X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}(x) \cdot M^\mu_\alpha \cdot \frac{\partial x^\alpha}{\partial \bar{x}^\mu} = \bar{X}^{\lambda\tau\eta\theta}_{\mu\nu\rho\sigma}(\bar{x}) \cdot M^\alpha_\lambda \cdot \frac{\partial x^\alpha}{\partial \bar{x}^\mu}.$$

As special cases we have V_n tensors, i. e. tensors which involve only Latin indices (usual tensors), and V_m tensors, i. e. tensors which involve only Greek indices.

2. The absolute calculus of an affine connection and a general linear connection. — It follows from (1.2), (1.3), and (1.4) by the usual differentiation and elimination process that the functions $X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma;c}$ defined in terms of the composite tensor $X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}$ by

$$(2.1) \quad X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma;c} \equiv \frac{\partial X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}}{\partial x^c} + \dots + L^\alpha_{\lambda c} X^{\lambda\beta\gamma\delta}_{\mu\nu\rho\sigma} + \dots + \Gamma^\alpha_{hc} X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} + \dots \\ \dots - L^\lambda_{\beta c} X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} - \dots - \Gamma^h_{bc} X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} - \dots$$

are the components of a composite tensor, to be known as the covariant derivative of the composite tensor $X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}$. The covariant differential of the composite tensor $X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}$ is defined to be

$$\delta X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} = X^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma;c} dx^c.$$

The invariant theory of the linear connection leads to the consideration of the composite tensor $P^\alpha_{\beta ab}$ defined by

$$(2.2) \quad P^\alpha_{\beta ab} \equiv \frac{\partial L^\alpha_{\beta a}}{\partial x^b} - \frac{\partial L^\alpha_{\beta b}}{\partial x^a} + L^\sigma_{\beta a} L^\alpha_{\sigma b} - L^\sigma_{\beta b} L^\alpha_{\sigma a}.$$

The vanishing of this tensor is a necessary and sufficient condition that there exist a representation in which the components of the connection vanish. In this case $L^\alpha_{\beta a}$ is said to be an integrable connection.

where $P(\dots)$ denotes the sum of terms derived from $(\dots)$ by cyclic permutation of the Latin indices. Thus the functions $L_{\lambda a_1 \dots a_p}^\alpha$ are symmetric in the indices $a_1 \dots a_p$. The derivatives $\frac{d^k X^\alpha}{ds^k}$ are then given by

$$(3.4) \quad \frac{d^k X^\alpha}{ds^k} = -L_{\lambda a_1 \dots a_k}^\alpha X^\lambda \frac{dx^{a_1}}{ds} \dots \frac{dx^{a_k}}{ds}, \quad (k=1, 2, \dots)$$

and the analytic solution of (3.2) is

$$(3.5) \quad X^\alpha = V_\lambda^\alpha X_0^\lambda$$

where

$$(3.6) \quad V_\lambda^\alpha = \delta_\lambda^\alpha - \sum_{k=1}^{\infty} (L_{\lambda a_1 \dots a_k}^\alpha)_g \xi^{a_1} \dots \xi^{a_k} \frac{s^k}{k!}.$$

Along the path we have

$$y^i = \xi^i s$$

where y^i are affine normal coordinates ⁽⁶⁾. Hence $N_\lambda^\alpha(y)$, defined by

$$(3.7) \quad N_\lambda^\alpha(y) = \delta_\lambda^\alpha - \sum_{k=1}^{\infty} (L_{\lambda a_1 \dots a_k}^\alpha(x))_g \frac{y^{a_1} \dots y^{a_k}}{k!},$$

are analytic functions of the coordinates y^i . Obviously the determinant $|N_\beta^\alpha|$ does not vanish in the neighborhood of $y^i = 0$.

DEFINITION. — The representation defined implicitly from a given representation with coordinates x^i and vector coordinates $V^\alpha(x)$, by means of the transformations

$$(3.8) \quad x^i = g^i + y^i - (\Gamma_{jk}^i(x))_g \frac{y^j y^k}{2!} - (\Gamma_{jkl}^i(x))_g \frac{y^j y^k y^l}{3!} - \dots$$

and

$$(3.9) \quad V^\alpha(x) = N_\lambda^{\alpha \dagger} V^\lambda(y),$$

where $N_\lambda^\alpha(y)$ are defined by (3.7), will be called a *normal representation* with base point $x^i = g^i$.

The transformation (3.8) is the well known one that defines an affine normal coordinate system y^i determined by a coordinate system x^i and a point $x^i = g^i$.

The symbol $(\dagger)$ preceding a component of a tensor, connection, etc., will denote that that component is referred to and evaluated in a normal representation.

THEOREM 1. — *Under a change of representation*

$$(3.10) \quad \begin{aligned} x^i &= f^i(\bar{x}) \\ V^\alpha(x) &= M_\beta^\alpha(x) \bar{V}^\beta(\bar{x}) \end{aligned}$$

the normal representation with base point $x^i = g^i$

$$y^i, {}^{\dagger}V^{\alpha}(y)$$

undergoes the transformations

$$(3.11) \quad y^i = \left(\frac{\partial x^i}{\partial \bar{x}^j} \right)_g \bar{y}^j$$

$$(3.12) \quad {}^{\dagger}V^{\alpha}(y) = (M^{\alpha}_{\beta}(x))_g {}^{\dagger}\bar{V}^{\beta}(\bar{y}).$$

Proof: The relation (3.11) between two affine normal coordinate systems with the same origin is well known ⁽⁶⁾.

If X^{α} is a vector field parallel to an arbitrarily chosen initial value $X^{\alpha}(g) \equiv X^{\alpha}_0$ along a path in V_n , then ${}^{\dagger}X^{\alpha}$ given by

$$X^{\alpha} = N^{\alpha}_{\lambda} {}^{\dagger}X^{\lambda}$$

is a constant along the path and is equal to X^{α}_0 (cf. (3.5) and (3.6)). If $\bar{X}^{\alpha}$ is given by

$$X^{\alpha} = M^{\alpha}_{\beta} \bar{X}^{\beta}$$

then $\bar{X}^{\alpha}$ satisfies the system

$$\frac{d\bar{X}^{\alpha}}{ds} + \bar{L}^{\alpha}_{\lambda a} \bar{X}^{\lambda} \frac{d\bar{x}^a}{ds} = 0$$

and the initial values $\bar{X}^{\alpha}_0$ are given by $X^{\alpha}_0 = (M^{\alpha}_{\beta})_g \bar{X}^{\beta}_0$. Also ${}^{\dagger}\bar{X}^{\alpha}$ is constant along the curve and is equal to $\bar{X}^{\alpha}_0$. Hence

$$(3.13) \quad {}^{\dagger}X^{\alpha} = (M^{\alpha}_{\beta})_g {}^{\dagger}\bar{X}^{\beta}$$

at all points of the path, and consequently at all points in the region of convergence of (3.5). The transformation of vector coordinates between the normal representations is given by

$$(3.14) \quad {}^{\dagger}V^{\alpha} = \tilde{N}^{\alpha}_{\lambda} M^{\lambda}_{\mu} \bar{N}^{\mu}_{\beta} {}^{\dagger}\bar{V}^{\beta}$$

where $\|\tilde{N}^{\alpha}_{\lambda}\|$ is the inverse matrix of $\|N^{\alpha}_{\lambda}\|$ and $\bar{N}^{\mu}_{\beta}$ is defined similarly to N^{μ}_{β} (cf. (3.7)). Since (3.14) must also hold for the vector field ${}^{\dagger}X^{\alpha}(y)$ or ${}^{\dagger}\bar{X}^{\alpha}(\bar{y})$ we see that

$$\tilde{N}^{\alpha}_{\lambda} M^{\lambda}_{\mu} \bar{N}^{\mu}_{\beta} = (M^{\alpha}_{\beta})_g.$$

This completes the proof of the theorem.

THEOREM 2. — *The transformation law of the invariants $L^{\alpha}_{\lambda a_1 \dots a_p}(x)$ under a change of representation*

$$\begin{aligned} x^i &= f^i(\bar{x}) \\ V^{\alpha} &= M^{\alpha}_{\beta}(x) \bar{V}^{\beta} \end{aligned}$$

is

$$(3.15) \quad L_{\lambda a_1 \dots a_p}^\alpha(x) M_\beta^\lambda \frac{\partial x^{\alpha_1}}{\partial \bar{x} r_1} \dots \frac{\partial x^{\alpha_p}}{\partial \bar{x} r_p} = M_\sigma^\alpha \bar{L}_{\beta r_1 \dots r_p}^\sigma + [\dots] - \frac{\partial^p M_\beta^\alpha}{\partial \bar{x} r_1 \dots \partial \bar{x} r_p}$$

where [...] indicates terms in $\bar{\Gamma}_{r_1 \dots r_{k+1}}^\alpha$ and $\bar{L}_{\beta r_1 \dots r_k}^\sigma$, ($k < p$).

Proof: From the covariant sets of equations (6)

$$\frac{d^k \bar{x}^a}{ds^k} + \bar{\Gamma}_{c_1 \dots c_k}^a \frac{d \bar{x}^{c_1}}{ds} \dots \frac{d \bar{x}^{c_k}}{ds} = 0$$

we have, along a path,

$$(3.16) \quad \frac{d^k M_\beta^\alpha}{ds^k} = \left[\frac{\partial^k M_\beta^\alpha}{\partial \bar{x} c_1 \dots \partial \bar{x} c_k} + (\dots) \right] \frac{d \bar{x}^{c_1}}{ds} \dots \frac{d \bar{x}^{c_k}}{ds}$$

where (...) indicates terms in $\bar{\Gamma}_{c_1 \dots c_i}^a$, ($i \leq k$). We also have the covariant sets of equations

$$(3.17) \quad \frac{d^k \bar{X}^\beta}{ds^k} + \bar{L}_{\bar{\mu} r_1 \dots r_k}^\beta \bar{X}^{\bar{\mu}} \frac{d \bar{x}^{r_1}}{ds} \dots \frac{d \bar{x}^{r_k}}{ds} = 0.$$

From (3.16) and (3.17) we have, along a path,

$$(3.18) \quad \frac{d^p X^\alpha}{ds^p} = \left[\frac{\partial^p M_\beta^\alpha}{\partial \bar{x} r_1 \dots \partial \bar{x} r_p} + (\dots) - M_\lambda^\alpha \bar{L}_{\beta r_1 \dots r_p}^\lambda \bar{X}^\beta \frac{d \bar{x}^{r_1}}{ds} \dots \frac{d \bar{x}^{r_p}}{ds} \right]$$

where (...) indicates terms in $\bar{\Gamma}_{r_1 \dots r_{k+1}}^\alpha$ and $\bar{L}_{\beta r_1 \dots r_k}^\alpha$, ($k < p$). The theorem follows by substituting from (3.10) and (3.18) in the equation

$$\frac{d^p X^\alpha}{ds^p} + L_{\lambda a_1 \dots a_p}^\alpha X^\lambda \frac{dx^{a_1}}{ds} \dots \frac{dx^{a_p}}{ds} = 0.$$

COROLLARY. — *The invariants $L_{\lambda a_1 \dots a_p}^\alpha$, ($p = 1, 2, 3, \dots$), all vanish when referred to a normal representation and evaluated at the base point of that representation. That is*

$$(3.19) \quad ({}^\dagger L_{\lambda a_1 \dots a_p}^\alpha(y))_0 = 0. \quad p = 1, 2, 3, \dots$$

4. Tensor extensions of composite tensors and König normal tensors. —

With the aid of the above results we have the following two theorems.

THEOREM 3. — *The functions $X_{\beta \dots b \dots, c_1 \dots c_p}^{\alpha \dots a \dots}(x)$ defined in terms of the composite tensor $X_{\beta \dots b \dots}^{\alpha \dots a \dots}$ by*

$$(4.1) \quad X_{\beta \dots b \dots, c_1 \dots c_p}^{\alpha \dots a \dots}(g) = \left(\frac{\partial^p X_{\beta \dots b \dots}^{\alpha \dots a \dots}}{\partial y^{c_1} \dots \partial y^{c_p}} \right)_0,$$

where $x^i = g^i$ is the base point of the normal representation, are the components of a composite tensor of the indicated rank.

THEOREM 4. — The functions $K_{\beta a c_1 \dots c_p}^\alpha(x)$ defined by

$$(4.2) \quad K_{\beta a c_1 \dots c_p}^\alpha(g) = \left(\frac{\partial^p \dagger L_{\beta a}^\alpha}{\partial y^{c_1} \dots \partial y^{c_p}} \right)_0,$$

where $x^i = g^i$ is the base point of the normal representation, are the components of a composite tensor of the indicated rank.

The proofs of these two theorems parallel those of corresponding well known theorems for affine extensions of affine tensors, and affine normal tensors.

DEFINITION. — The tensors $X_{\dots, c_1 \dots c_p}(x)$ and $K_{\beta a c_1 \dots c_p}^\alpha(x)$ defined by (4.1) and (4.2) will be referred to as the p^{th} extension of the tensor $X_{\dots}$ and the p^{th} KÖNIG normal tensor, respectively.

COROLLARY. — In case the tensor $X_{\dots}$ is a pure V_n tensor, the tensor $X_{\dots, c_1 \dots c_p}$ is the p^{th} affine extension of the tensor $X_{\dots}$.

THEOREM 5. — The König normal tensor $K_{\beta a c_1 \dots c_p}^\alpha$ is symmetric in the indices $c_1 \dots c_p$,

$$(4.3) \quad K_{\beta a c_1 \dots c_p}^\alpha = K_{\beta a c_r \dots c_s}^\alpha$$

where $c_r \dots c_s$ is any permutation of $c_1 \dots c_p$. Furthermore $K_{\beta a c_1 \dots c_p}^\alpha$ satisfies the identity

$$(4.4) \quad S(K_{\beta a c_1 \dots c_p}^\alpha) = 0$$

where $S(\dots)$ indicates the sum of the terms derived from $(\dots)$ by cyclic permutation of the Latin indices.

Proof: (4.3) is immediate from (4.2).

From the definition (3.3) of $L_{\beta a c_1 \dots c_p}^\alpha$, we see that

$$L_{\beta a c_1 \dots c_p}^\alpha = \frac{1}{p!} P \left[\frac{\partial^p \dagger L_{\beta a}^\alpha}{\partial x^{c_1} \dots \partial x^{c_p}} + (\dots) \right]$$

where $P[\dots]$ indicates the sum of the terms derived from $(\dots)$ by permuting the Latin indices, and where $(\dots)$ indicates terms in $L_{\beta a}^\alpha$ and $\frac{\partial^k L_{\beta a}^\alpha}{\partial x^{c_1} \dots \partial x^{c_k}}$, ($k < p$). Hence at the base point of a normal representation we have

$$0 = P \left[\frac{\partial^p \dagger L_{\beta a}^\alpha}{\partial y^{c_1} \dots \partial y^{c_p}} + (\dots) \right]_0$$

or

$$(4.5) \quad P[K_{\beta a c_1 \dots c_p}^\alpha(g) + (\dots)] = 0$$

where (...) indicates terms which vanish with $P[K_{\beta a c_1 \dots c_k}^\alpha(g)]$, ($k < p$). For $p = 1$ (4.5) is equivalent to (4.4) for $p = 1$. Assume that (4.4) holds for the k^{th} normal tensor, ($k = 1, \dots, p - 1$). Then (4.5) establishes (4.4) for the p^{th} normal tensor. This proves the theorem as stated.

It is possible to calculate the explicit form of the extensions of tensors and the KÖNIG normal tensors.

THEOREM 6. — *The first König normal tensor is given by*

$$(4.6) \quad K_{\beta ab}^\alpha = \frac{1}{2} P_{\beta ab}^\alpha$$

where $P_{\beta ab}^\alpha$ is the curvature tensor defined in (2.2).

Proof: From (2.2) we see that

$$(4.7) \quad P_{\beta ab}^\alpha = K_{\beta ab}^\alpha - K_{\beta ba}^\alpha.$$

From (4.4) we have

$$(4.8) \quad 0 = K_{\beta ab}^\alpha + K_{\beta ba}^\alpha.$$

Addition of (4.7) and (4.8) establishes the theorem.

THEOREM 7. — *The first extension of a composite tensor is the covariant derivative, as given by (2.1).*

This follows without difficulty from the definitions of the covariant derivative and the first extension, and the properties of a normal representation.

THEOREM 8. — *The second extension of the composite tensor $X_{\beta b}^{\alpha a}$ is*

$$X_{\beta b, cd}^{\alpha a} = X_{\beta b; c; d}^{\alpha a} - K_{\lambda cd}^\alpha X_{\beta b}^{\lambda a} - A_{hcd}^a X_{\beta b}^{\alpha h} + K_{\beta cd}^\lambda X_{\lambda b}^{\alpha a} + A_{bcd}^h X_{\beta b}^{\alpha h}$$

where A_{bcd}^a is the first affine normal tensor ⁽⁶⁾.

Proof: By Theorem 7

$$X_{\beta b; c; d}^{\alpha a}(g) = \left(\frac{\partial \dagger X_{\beta b; c}^{\alpha a}}{\partial y^d} \right)_0.$$

Expanding $\dagger X_{\beta b; c}^{\alpha a}$ according to the definition (2.1), differentiating the result, and evaluating at the base point of the normal representation proves the theorem. The generalization to any composite tensor is immediate.

COROLLARY. — *The second extension of a tensor $X_{\gamma \dots \delta}^{\alpha \dots \beta}$ with no Latin indices is*

$$X_{\gamma \dots \delta, cd}^{\alpha \dots \beta} = \frac{1}{2} [X_{\gamma \dots \delta; c; d}^{\alpha \dots \beta} + X_{\gamma \dots \delta; d; c}^{\alpha \dots \beta}].$$

THEOREM 9. — *The $(p + 1)^{\text{st}}$ König normal tensor is given by*

$$(4.9) \quad K_{\beta a b c_1 \dots c_p}^\alpha = \frac{1}{p + 2} \left(P_{\beta a b, c_1 \dots c_p}^\alpha + \sum_{k=1}^p P_{\beta a c_k, c_1 \dots c_{k-1} b c_{k+1} \dots c_p}^\alpha \right) \quad (p \geq 1).$$

Proof: From (2.2) we have

$$(4.10) \quad P_{\beta ab, c_1 \dots c_p}^\alpha = K_{\beta abc_1 \dots c_p}^\alpha - K_{\beta bac_1 \dots c_p}^\alpha + \\ + \sum_{k=1}^{p-1} C[K_{\beta ac_1 \dots c_k}^\alpha K_{\beta bc_{k+1} \dots c_p}^\alpha - K_{\beta bc_1 \dots c_k}^\alpha K_{\beta ac_{k+1} \dots c_p}^\alpha]$$

where $C[\dots]$ denotes the sum of the different terms derived from $[\dots]$ by taking permutations of the indices $c_1 \dots c_p$.

From Theorem 5 we have

$$S(K_{\beta abc_1 \dots c_p}^\alpha) = 0.$$

If we add to (4.10) this equation and the p equations derived from (4.10) by interchanging b with c_i , ($i = 1, \dots, p$), we have (making use of the symmetry relation (4.3))

$$(4.11) \quad (p+2)K_{\beta abc_1 \dots c_p}^\alpha = P_{\beta ab, c_1 \dots c_p}^\alpha + \\ + \sum_{k=1}^p P_{\beta ac_k, c_1 \dots c_{k-1} bc_{k+1} \dots c_p}^\alpha + (\dots),$$

where $(\dots)$ denotes sums of terms in the KÖNIG normal tensors of lower order, which sums vanish by (4.4). The vanishing of those terms is easily checked by direct calculation for $p = 1, 2, 3$, and may be proved for the general case. The details will be omitted for the sake of brevity.

Another general method that will apply to the calculation of the explicit form of a KÖNIG normal tensor of any order will be indicated. Write the relations

$$(4.12) \quad N_\sigma^\alpha + L_{\beta\sigma}^\alpha(y) = +L_{\lambda\sigma}^\alpha(x)N_\beta^\lambda \frac{\partial x^\sigma}{\partial y^\alpha} + \frac{\partial N_\beta^\alpha}{\partial y^\sigma}.$$

Take the p^{th} partial derivative with respect to the normal coordinates y^i , and evaluate at the base point of the normal representation. The equations (3.7) and (3.8) yield

$$(4.13) \quad \left\{ \begin{array}{l} (N_\beta^\alpha)_0 = \delta_\beta^\alpha \\ \left(\frac{\partial^k N_\beta^\alpha}{\partial y^{c_1} \dots \partial y^{c_k}} \right)_0 = - (L_{\beta c_1 \dots c_k}^\alpha(x))_g \\ \left(\frac{\partial x^i}{\partial y^j} \right)_0 = \delta_j^i \\ \left(\frac{\partial^k x^i}{\partial y^{c_1} \dots \partial y^{c_k}} \right)_0 = - (\Gamma_{c_1 \dots c_k}^i(x))_g. \end{array} \right.$$

Making these substitutions in the equations derived from (4.12) gives an expression for $K_{\beta ac_1 \dots c_p}^\alpha(x)$ in terms of KÖNIG normal tensors of lower order, derivatives of $L_{\beta c}^\alpha$, $\Gamma_{c_1 \dots c_{k+1}}^i$, and $L_{\beta c_1 \dots c_k}^\alpha(x)$, ($k = 1, \dots, p$).

THEOREM 10. — *If $L_{\beta\alpha}^z(x)$ is an integrable connection all of the KÖNIG normal tensors vanish.*

This theorem follows from (4.6) and Theorem 9.

THEOREM 11. — *If the covariant derivatives of the tensors $P_{\beta ab}^z$ and R_{abc}^h vanish, then the KÖNIG normal tensors of even order all vanish.*

Proof: If $R_{abc;d}^h$ is zero the affine normal tensors of even order vanish (*). By hypothesis we have

$$(4.14) \quad \frac{\partial \dagger P_{\beta ab}^z}{\partial y^c} + \dagger L_{\lambda c}^z \dagger P_{\beta ab}^\lambda - \dagger L_{\beta c}^\lambda \dagger P_{\lambda ab}^z - \\ - \dagger \Gamma_{ac}^h \dagger P_{\beta hb}^\alpha - \dagger \Gamma_{bc}^h \dagger P_{\beta ah}^\alpha = 0.$$

These equations may also be written in a symbolic form convenient for our purposes,

$$(4.15) \quad \frac{\partial^2 \dagger L_{\beta\alpha}^z}{\partial y^b \partial y^c} - \frac{\partial^2 \dagger L_{\beta b}^z}{\partial y^a \partial y^c} + \left(\dagger L \frac{\partial \dagger L}{\partial y} + \dagger L \dagger P + \dagger \Gamma \dagger P \right) = 0.$$

From (4.9) we see that the second KÖNIG normal tensor vanishes. If we differentiate (4.14) twice with respect to y^i and evaluate at the base point of the normal representation we see that the third extension of $P_{\beta ab}^z$ is zero. If we differentiate (4.15) twice with respect to y^i and evaluate at the base point we find that the fourth KÖNIG normal tensor is symmetric in the first two Latin indices. Then, by Theorem 5, it is a zero tensor. We may then proceed by induction and show that if the $(2k-1)^{st}$ extensions of $P_{\beta ab}^z$ and the $(2k)^{th}$ KÖNIG normal tensors ($k=1, \dots, p$) all vanish, then the $(2p+1)^{st}$ extension of $P_{\beta ab}^z$ and the $(2(p+1))^{nd}$ KÖNIG normal tensor vanish.

5. Replacement theorem. — With the aid of theorems 1, 3, 4, and by methods similar those used in establishing the MICHAL-THOMAS theorem for differential invariants of an affine connection (*), the following fundamental theorem for simultaneous differential invariants of a general linear connection and a symmetric affine connection can be established. We leave the details of the proof to the reader.

THEOREM 12. — *Any tensor differential invariant of the form*

$$(5.1) \quad T_{\dots \beta \dots b \dots}^{z \dots a \dots} \left(L_{\beta\alpha}^z, \frac{\partial L_{\beta\alpha}^z}{\partial x^{c_1}}, \dots, \frac{\partial^r L_{\beta\alpha}^z}{\partial x^{c_1} \dots \partial x^{c_r}}; \Gamma_{ab}^i, \frac{\partial \Gamma_{ab}^i}{\partial x^{c_1}}, \dots \right. \\ \left. \dots, \frac{\partial^s \Gamma_{ab}^i}{\partial x^{c_1} \dots \partial x^{c_s}}; (E)F^{\dots}, \frac{\partial (E)F^{\dots}}{\partial x^{c_1}}, \dots, \frac{\partial^t (E)F^{\dots}}{\partial x^{c_1} \dots \partial x^{c_t}} \right),$$

(*) This was proved by A. D. MICHAL in his course on Modern Differential Geometry, 1931-1932. The proof is similar to that of Theorem 11 above.

where ${}_{(E)}F^{\dots}$ are the components of any set of composite tensors, may be expressed as the same functions of arguments that are affine and composite tensors

$$T^{\dots\alpha\dots\alpha}_{\dots\beta\dots\beta}(0, K^z_{\beta\alpha c_1}, \dots, K^z_{\beta\alpha c_1 \dots c_r}; 0, A^i_{abc_1}, \dots, A^i_{abc_1 \dots c_s}; \\ {}_{(E)}F^{\dots}, {}_{(E)}F^{\dots, c_1}, \dots, {}_{(E)}F^{\dots, c_1 \dots c_t})$$

by replacing $L^z_{\beta\alpha}$ by zero, derivatives of $L^z_{\beta\alpha}$ by the corresponding König normal tensors, Γ^i_{ab} by zero, derivatives of Γ^i_{ab} by the corresponding affine normal tensors, and derivatives of the components of any composite tensor by the corresponding extensions of that tensor.

COROLLARY. — Any tensor differential invariant of the form (5.1) may be expressed by functions of the König curvature tensor $P^z_{\beta ab}$, the affine curvature tensor B^h_{abc} , the tensors ${}_{(E)}F^{\dots}$, and successive covariant derivatives of $P^z_{\beta ab}$, B^h_{abc} , and ${}_{(E)}F^{\dots}$.

This corollary follows from a theorem analogous to one for an affinely connected manifold $(^s, s)$, namely: (A) The p^{th} extension ($p \geq 2$) of any composite tensor T may be expressed as a rational integral function of T and the first p repeated covariant derivatives of T , the tensors $P^z_{\beta ab}$, B^h_{abc} , and the first $(p-2)$ repeated covariant derivatives of $P^z_{\beta ab}$ and B^h_{abc} ; (B) the p^{th} KÖNIG normal tensor is a rational integral function of $P^z_{\beta ab}$ and its first $(p-1)$ repeated covariant derivatives, and (when $p \geq 3$) B^h_{abc} and its first $(p-3)$ repeated covariant derivatives. Note that Theorem 8 gives (A) for $p=2$, and Theorem 9 (with $p=1$) gives (B) for $p=2$. The generalization follows by induction as in the affine case.

PART II.

An extension of the recent Einstein-Mayer geometry.

6. Preliminary remarks. — In this part of the paper we shall consider a special situation which is itself an extension of the geometry considered by EINSTEIN and MAYER in their recent treatment of the unified field theory ⁽⁵⁾.

Let V_n be an n -dimensional Riemannian space with each point of which is associated an m -dimensional linear vector space V_m ($m > n$) which is endowed with a symmetric metric tensor $G_{\alpha\beta}(x)$ of rank m , such that $G_{\alpha\beta}X^\alpha X^\beta$

is positive definite throughout V_n . Further assume the existence of a composite tensor γ_a^α for which $\|\gamma_a^\alpha\|$ is of rank n . Define $G^{\alpha\beta}$ to be the cofactor of $G_{\alpha\beta}$ in $|G_{\alpha\beta}|$, divided by $|G_{\alpha\beta}|$.

In addition to the previously mentioned summation conventions, capital Latin indices will have the range $n+1$ to m and such an index repeated in the same term is to be summed over its range.

THEOREM 13. — *There exist $m-n$ orthonormal vector fields which are solutions of the equations*

$$(6.1) \quad \gamma_a^\alpha X_\alpha = 0;$$

i. e. there exist vector fields A_α^E such that

$$(6.2) \quad \gamma_a^\alpha A_\alpha^E = 0$$

and

$$(6.3) \quad G^{\alpha\beta} A_\alpha^E A_\beta^F = \delta^{EF} \equiv \begin{cases} 1 & \text{if } E = F \\ 0 & \text{if } E \neq F. \end{cases}$$

Any two such sets of vector fields are related by

$$A_\alpha^E = L_F^E A_\alpha^F$$

where $\|L_F^E\|$ is orthogonal.

The orthonormal sets of solutions are build up by the SCHMIDT orthogonalization process familiar in the theory of integral equations.

THEOREM 14. — *The quadratic form*

$$(G_{\alpha\beta} \gamma_a^\alpha \gamma_b^\beta) dx^a dx^b$$

is positive definite and of rank n .

The first part of the theorem is obvious. The second follows by reductio ad absurdum.

We impose the condition that the Riemannian metric g_{ab} is given by

$$(6.4) \quad g_{ab} = G_{\alpha\beta} \gamma_a^\alpha \gamma_b^\beta.$$

As consequences of theorems 13 and 14 we have the corollary:

COROLLARY 1. — *The vector fields A_E^α , defined by*

$$(6.5) \quad A_E^\alpha = G^{\alpha\beta} A_\beta^E,$$

are orthonormal sets of solutions of

$$(6.6) \quad \gamma_\alpha^a X^a = 0;$$

that is

$$\gamma_x^\alpha A_E^\alpha = 0$$

and

$$(6.7) \quad G_{\alpha\beta} A_E^\alpha A_F^\beta = \delta_{EF} \equiv \begin{cases} 1 & \text{if } E = F \\ 0 & \text{if } E \neq F. \end{cases}$$

DEFINITION. — The vectors A_E^α will be referred to as exceptional vectors and any vector X^α which satisfies the equation

$$G_{\alpha\beta} X^\alpha A_E^\beta = 0$$

will be said to lie in the exceptional manifold of V_m .

7. Formulae relating the fundamental tensors.

THEOREM 15.

$$(7.1) \quad \gamma_\alpha^\alpha \gamma_\alpha^b = \delta_\alpha^b.$$

THEOREM 16.

$$(7.2) \quad \gamma_\alpha^\alpha \gamma_\beta^\alpha = \delta_\beta^\alpha - A_E^\alpha A_\beta^E.$$

The proof of these two theorems is similar to the EINSTEIN-MAYER case.

If we are given the components of a V_m vector we can construct the components of a corresponding V_n vector by the defining relations

$$(7.3) \quad b^a = \gamma_\alpha^a B^\alpha$$

$$(7.4) \quad b_a = \gamma_\alpha^a B_\alpha.$$

THEOREM 17. — *The inverses of (6.4), (7.3), and (7.4) are respectively*

$$(7.5) \quad G_{\alpha\beta} = g_{ab} \gamma_\alpha^a \gamma_\beta^b + A_\alpha^E A_\beta^E,$$

$$(7.6) \quad B^\alpha = \gamma_\alpha^a b^a + A_E^\alpha A_\beta^E B^\beta.$$

and

$$(7.7) \quad B_\alpha = \gamma_\alpha^a b_a + A_\alpha^E A_\beta^E B_\beta.$$

It is immediate from (7.6) and (7.7) that any vectors in the exceptional manifold of V_m are in one-one correspondence with V_n vectors.

8. **Postulates for the determination of the linear connection.** — A linear connection $L_{\beta\alpha}^\alpha(x)$ will be defined in terms of the tensors γ_α^a , A_E^α , $G_{\alpha\beta}$, g_{ab} , and $m - n$ arbitrarily chosen V_n tensors F_{ac}^E by means of the following tensor equations.

Postulate I

$$G_{\alpha\beta;c} \equiv \frac{\partial G_{\alpha\beta}}{\partial x^c} - L_{\alpha c}^\lambda G_{\lambda\beta} - L_{\beta c}^\lambda G_{\alpha\lambda} = 0.$$

Postulate II (*)

$$\gamma_{a;c}^z \equiv \frac{\partial \gamma_a^z}{\partial x^c} + L_{\lambda c}^z \gamma_a^\lambda - \left\{ \begin{matrix} h \\ ac \end{matrix} \right\} \gamma_h^z = A_E^z F_{ac}^E$$

where $\left\{ \begin{matrix} h \\ ac \end{matrix} \right\}$ are the CHRISTOFFEL symbols of V_n .

Postulate III (*)

$$A_{z;c}^E A_D^z \equiv \left(\frac{\partial A_z^E}{\partial x^c} - L_{zc}^\lambda A_\lambda^E \right) A_D^z = 0.$$

It is to be noted that if $m - n = 1$, Postulate III is not independent but is a consequence of definitions and Postulate I.

THEOREM 18. — *The linear connection as determined by the above postulates is*

$$(8.1) \quad L_{zc}^\mu = \left(\frac{\partial A_z^D}{\partial x^c} + F_{ac}^D \gamma_z^a \right) A_D^\mu + \left(\frac{\partial \gamma_{za}^\alpha}{\partial x^c} - \gamma_{zh}^\alpha \left\{ \begin{matrix} h \\ ac \end{matrix} \right\} - A_z^D F_{ac}^D \right) \gamma^{\mu\alpha}.$$

Proof: Since

$$\gamma_a^\alpha A_z^E = 0$$

then

$$(8.2) \quad \gamma_a^\alpha A_{\alpha;c}^E = - \gamma_{\alpha;c}^\alpha A_z^E.$$

Multiply (8.2) by γ_β^α . With the aid of Theorem 16, postulates II and III, and the relation

$$(8.3) \quad A_\alpha^E A_D^z = \delta_D^E,$$

the equations resulting from (8.2) become

$$(8.4) \quad A_{\beta;c}^E \equiv \frac{\partial A_\beta^E}{\partial x^c} - L_{\beta c}^\lambda A_\lambda^E = - F_{ac}^E \gamma_\beta^a.$$

By postulates I and II we have

$$(8.5) \quad \gamma_{za;c} \equiv \frac{\partial \gamma_{za}}{\partial x^c} - L_{zc}^\lambda \gamma_{\lambda a} - \left\{ \begin{matrix} h \\ ac \end{matrix} \right\} \gamma_{zh} = A_z^E F_{ac}^E.$$

Multiply (8.5) by $\gamma^{\mu\alpha}$. With the aid of Theorem 16 we obtain

$$(8.6) \quad L_{zc}^\mu = L_{zc}^\lambda A_D^D A_\lambda^\mu + \left(\frac{\partial \gamma_{za}^\alpha}{\partial x^c} - \left\{ \begin{matrix} h \\ ac \end{matrix} \right\} \gamma_{zh} - A_z^D F_{ac}^D \right) \gamma^{\mu\alpha}.$$

From (8.4) we have

$$(8.7) \quad L_{zc}^\lambda A_D^D A_\lambda^\mu = \left(\frac{\partial A_z^D}{\partial x^c} + F_{ac}^D \gamma_z^a \right) A_D^\mu,$$

(*) We consider these postulates with reference to a fixed set of exceptional directions A_E^z .

and hence (8.6) establishes the validity of the formula (8.1) for the linear connection $L_{\alpha c}^{\mu}$.

The consistency of the postulates has been shown by long but straightforward calculations, which will be omitted here for the sake of brevity.

THEOREM 19. — *The components of the V_m curvature tensor P_{abc}^{μ} based on the linear connection (8.1) is given in terms of the Riemann-Christoffel tensor R_{bcd}^a , the tensors g_{ij} , γ_{α}^a , A_{α}^E and F_{ac}^E , by*

$$(8.8) \quad \left\{ \begin{aligned} P_{\alpha\beta c}^{\mu} = & \gamma^{\mu\alpha} [\gamma_{\alpha h} (F_{Dc}^h F_{ab}^D - F_{Db}^h F_{ac}^D - R_{abc}^h) + \\ & + A_{\alpha}^D (F_{ac;b}^D - F_{ab;c}^D)] + A_D^{\mu} \gamma_{\alpha a} (F_{Db;c}^a - F_{Dc;b}^a) + \\ & + A_D^{\mu} A_{\alpha}^E (F_{Db}^a F_{ac}^E - F_{Dc}^a F_{ab}^E), \end{aligned} \right.$$

where F_{Dc}^a is given by (8.10).

Proof: (8.4) may be written as

$$(8.9) \quad A_{\alpha;b}^E = -F_{Eb}^a \gamma_{\alpha a}$$

where

$$(8.10) \quad F_{Eb}^a = g^{ah} F_{hb}^E.$$

From (8.5) and (8.9) we have by calculation

$$(8.11) \quad A_{\alpha;b;c}^E = -F_{Eb;c}^a \gamma_{\alpha a} - F_{Eb}^a A_{\alpha}^D F_{ac}^D$$

and

$$(8.12) \quad \gamma_{\alpha a;b;c} = -F_{Dc}^h \gamma_{\alpha h} F_{ab}^D + A_{\alpha}^D F_{ab;c}^D.$$

By the generalized RICCI identity (2.3) we have

$$(8.13) \quad A_{\alpha;b;c}^D - A_{\alpha;c;b}^D = -P_{abc}^{\lambda} A_{\lambda}^D$$

and

$$(8.14) \quad \gamma_{\alpha a;b;c} - \gamma_{\alpha a;c;b} = -P_{abc}^{\lambda} \gamma_{\lambda a} - R_{abc}^h \gamma_{\alpha h}.$$

Multiply (8.14) by $\gamma^{\mu\alpha}$, apply Theorem 16, and employ relation (8.13). This yields

$$(8.15) \quad \begin{aligned} P_{abc}^{\mu} = & -[A_D^{\mu} (A_{\alpha;b;c}^D - A_{\alpha;c;b}^D) + \\ & + (\gamma_{\alpha a;b;c} - \gamma_{\alpha a;c;b} + R_{abc}^h \gamma_{\alpha h}) \gamma^{\mu\alpha}]. \end{aligned}$$

Substitution from (8.11) and (8.12) gives (8.8) as was to be proved.

Note that the last parenthesis in the explicit form (8.8) of the V_m curvature tensor P_{abc}^{μ}

$$(F_{Db}^a F_{ac}^E - F_{Dc}^a F_{ab}^E).$$

vanishes identically if $m - n = 1$.

9. Theorems on parallel vector fields. — The first three of the following four theorems provide an alternative set of postulates for the determination of the connection.

THEOREM 20. — *Let X^z be any vector field parallel along an arbitrarily chosen curve. Then necessary and sufficient conditions that X_z , defined by*

$$X_\alpha = G_{\alpha\beta} X^\beta,$$

be a parallel vector field along that curve, for all curves and all such X^z , are that

$$G_{\alpha\beta;c} = 0.$$

THEOREM 21. — *Let B^z be any vector in the exceptional manifold V_m . i. e. there exists b^a such that $B^z = \gamma_a^z b^a$, and let b^a be displaced parallel to itself along any curve. Then necessary and sufficient conditions that δB^z so defined be orthogonal to the exceptional manifold are that*

$$\gamma_{a;c}^z = A_E^z F_{ac}^E.$$

THEOREM 22. — *A necessary and sufficient condition that δA_E^z be a vector of the exceptional manifold of V_m is that $A_F^z A_{z;c}^E$ be zero.*

THEOREM 23. — *Let the parallel displacement of b^a be in the direction $dx^a = \rho b^a$. Then necessary and sufficient conditions that δB^a ($B^z = \gamma_a^z b^a$) be zero are that the tensors F_{ac}^E be skewsymmetric,*

$$F_{ac}^E = -F_{ca}^E.$$

10. Special representations. — The following two theorems will be found useful in the next paragraph.

THEOREM 24. — *The determinant $|M_\beta^\alpha|$ whose elements are given by*

$$(10.1) \quad M_\beta^\alpha = \delta_\beta^b \gamma_b^\alpha + \delta_\beta^B A_B^\alpha$$

is not zero. Furthermore the inverse matrix of $\|M_\beta^\alpha\|$ is $\|\tilde{M}_\beta^\alpha\|$ where $\tilde{M}_\beta^\alpha$ is given by

$$(10.2) \quad \tilde{M}_\beta^\alpha = \delta_\alpha^a \gamma_\beta^a + \delta_\beta^A A_A^\alpha.$$

The first statement in the theorem is proved by reductio ad absurdum; the second statement is verified by direct calculation.

DEFINITION. — The representation derived from a given representation by the change of vector coordinates alone as determined by (10.1) will be referred to as a special representation.

THEOREM 25. — *In a special representation the components of the fundamental quantities are*

$$(10.3) \quad \gamma_a^\alpha = \delta_a^\alpha, \quad \gamma_\alpha^a = \delta_\alpha^a, \quad A_E^\alpha = \delta_E^\alpha, \quad A_\alpha^E = \delta_\alpha^E, \quad G_{\alpha\beta} = \delta_\alpha^a \delta_\beta^b g_{ab} + \delta_\alpha^A \delta_\beta^A.$$

$$(10.4) \quad L_{\beta c}^\alpha = \delta_\alpha^z \delta_\beta^b \left\{ \begin{matrix} \alpha \\ b c \end{matrix} \right\} + \delta_A^z \delta_\beta^b F_{bc}^A - \delta_\alpha^z \delta_\beta^B F_{Bc}^\alpha.$$

(10.3) is calculated from the law of transformation of composite tensors. (10.4) is more simply calculated from the expressions (7.5) and (8.1) for $G_{\alpha\beta}$ and $L_{\beta c}^\alpha$.

The components of P_{abc}^μ in a special representation may be calculated from (8.8). The results are embodied in the following theorem.

THEOREM 26. — *The V_m curvature tensor P_{abc}^μ has the following components in a special representation:*

$$\begin{aligned} P_{abc}^\mu = & \delta_\alpha^s \delta_p^\mu (F_{sc}^D F_{Db}^p - F_{sb}^D F_{Dc}^p + R_{sbc}^p) + \delta_M^\mu \delta_\alpha^s (F_{sb;c}^M - F_{se;b}^M) + \\ & + \delta_p^\mu \delta_\alpha^A (F_{Ac;b}^p - F_{Ab;c}^p) + \delta_M^\mu \delta_\alpha^A (F_{ab}^M F_{Ac}^A - F_{ac}^M F_{Ab}^A). \end{aligned}$$

11. Reduction theorem for tensor differential invariants with only Latin indices. — Besides the general replacement theorem, Theorem 12, it is possible to state a reduction theorem applicable to certain tensor differential invariants of the geometry under consideration in this part of the paper.

A prime (') before components of tensors, connection, etc. will be used to indicate that those components are referred to a special representation with Riemannian normal coordinates y^i .

Define the functions $\Lambda_{\beta c i_1 \dots i_k}^\alpha(x)$ by the equations

$$(11.1) \quad \begin{aligned} \Lambda_{\beta c}^\alpha(q) &= ('L_{\beta c}^\alpha)_0 \\ \Lambda_{\beta c i_1 \dots i_k}^\alpha(q) &= \left(\frac{\partial^{k'} L_{\beta c}^\alpha}{\partial y^{i_1} \dots \partial y^{i_k}} \right)_0 \end{aligned}$$

where $x^i = q^i$ is the origin of the Riemannian normal coordinate system. Hence, by (10.4)

$$(11.2) \quad \begin{aligned} \Lambda_{\beta c}^\alpha &= \delta_A^z \delta_\beta^b F_{bc}^A - \delta_\alpha^z \delta_\beta^B F_{Bc}^\alpha \\ \Lambda_{\beta c i_1 \dots i_k}^\alpha &= \delta_\alpha^z \delta_\beta^b A_{bc i_1 \dots i_k}^\alpha + \delta_A^z \delta_\beta^b F_{bc, i_1 \dots i_k}^A - \delta_\alpha^z \delta_\beta^B F_{Bc, i_1 \dots i_k}^\alpha \end{aligned}$$

where $A_{bc i_1 \dots i_k}^\alpha$ are affine normal tensors, and $F_{bc, i_1 \dots i_k}^A$ and $F_{Bc, i_1 \dots i_k}^\alpha$ are extensions of F_{bc}^A and F_{Bc}^α based on the CHRISTOFFEL symbols. From Theorem 17 we see that

$$(11.3) \quad \left(\frac{\partial^{k'} G_{\alpha\beta}(y)}{\partial y^{i_1} \dots \partial y^{i_k}} \right)_0 = \delta_\alpha^z \delta_\beta^b (g_{ab, i_1 \dots i_k}(x))_q.$$

THEOREM 27. Any tensor differential invariant with no Greek indices

$$(11.4) \quad T_{k \dots l}^{i \dots j} \left(L_{\beta\alpha}^\alpha, \dots, \frac{\partial^s L_{\beta\alpha}^\alpha}{\partial x^{c_1} \dots \partial x^{c_s}}; \gamma_a^\alpha, \dots, \frac{\partial^i \gamma_a^\alpha}{\partial x^{c_1} \dots \partial x^{c_i}}; A_E^\alpha, \dots, \frac{\partial^j A_E^\alpha}{\partial x^{c_1} \dots \partial x^{c_j}}; \right. \\ \left. g_{ab}, \dots, \frac{\partial^t g_{ab}}{\partial x^{c_1} \dots \partial x^{c_t}}; G_{\alpha\beta}, \dots, \frac{\partial^u G_{\alpha\beta}}{\partial x^{c_1} \dots \partial x^{c_u}} \right)$$

may be expressed as the same functions with arguments that are linear combinations of V_n tensors; that is, it may be put in the form

$$(11.5) \quad T_{k \dots l}^{i \dots j} (\Lambda_{\beta\alpha}^\alpha, \dots, \Lambda_{\beta\alpha c_1 \dots c_s}^\alpha; \delta_a^\alpha, \dots, 0; \delta_E^\alpha, \dots, 0; \\ g_{ab}, \dots, g_{ab, c_1 \dots c_t}; \delta_\alpha^a \delta_\beta^b g_{ab} + \delta_\alpha^A \delta_\beta^A, \dots, \delta_\alpha^a \delta_\beta^b g_{ab, c_1 \dots c_u})$$

by replacing $L_{\beta\alpha}^\alpha$ by $\Lambda_{\beta\alpha}^\alpha$, $\frac{\partial^k L_{\beta\alpha}^\alpha}{\partial x^{c_1} \dots \partial x^{c_k}}$ by $\Lambda_{\beta\alpha c_1 \dots c_k}^\alpha$, γ_a^α by δ_a^α , derivatives of γ_a^α by zero, A_E^α by δ_E^α , derivatives of A_E^α by zero, g_{ab} by g_{ab} , $\frac{\partial^k g_{ab}}{\partial x^{c_1} \dots \partial x^{c_k}}$ (for $k > 1$) by the k^{th} extension of g_{ab} , $G_{\alpha\beta}$ by $\delta_\alpha^a \delta_\beta^b g_{ab} + \delta_\alpha^A \delta_\beta^A$, $\frac{\partial^k G_{\alpha\beta}}{\partial x^{c_1} \dots \partial x^{c_k}}$ (for $k > 1$) by $\delta_\alpha^a \delta_\beta^b$ times the k^{th} extension of g_{ab} .

Proof: By hypothesis we have

$$(11.6) \quad T_{k \dots l}^{i \dots j} \left(L_{\beta\alpha}^\alpha, \dots, \frac{\partial^u G_{\alpha\beta}}{\partial x^{c_1} \dots \partial x^{c_u}} \right) = \\ = T_{g \dots h}^{e \dots f} \left(*L_{\beta\alpha}^\alpha, \dots, \frac{\partial^u *G_{\alpha\beta}}{\partial y^{c_1} \dots \partial y^{c_u}} \right) \frac{\partial y^g}{\partial x^k} \dots \frac{\partial y^h}{\partial x^l} \frac{\partial x^i}{\partial y^e} \dots \frac{\partial x^j}{\partial y^f}$$

where (*) denotes evaluation in a Riemannian normal coordinate system with origin at $x^i = q^i$. Clearly

$$(11.7) \quad T_{g \dots h}^{e \dots f} \left(*L_{\beta\alpha}^\alpha, \dots, \frac{\partial^u *G_{\alpha\beta}}{\partial y^{c_1} \dots \partial y^{c_u}} \right) = T_{g \dots h}^{e \dots f} \left(L_{\beta\alpha}^\alpha, \dots, \frac{\partial^u G_{\alpha\beta}}{\partial y^{c_1} \dots \partial y^{c_u}} \right).$$

Since

$$\left(\frac{\partial x^i}{\partial y^j} \right)_0 = \left(\frac{\partial y^i}{\partial x^j} \right)_q = \delta_j^i$$

it follows from (10.3), (11.2), and (11.3) that

$$\left[T_{k \dots l}^{i \dots j} \left(L_{\beta\alpha}^\alpha, \dots, \frac{\partial^u G_{\alpha\beta}}{\partial x^{c_1} \dots \partial x^{c_u}} \right) \right]_{x=q} = \\ = \left[T_{k \dots l}^{i \dots j} \left(\Lambda_{\beta\alpha}^\alpha, \dots, \delta_\alpha^a \delta_\beta^b g_{ab, c_1 \dots c_u} \right) \right]_{x=q}.$$

THEOREM 28. — The components of any tensor differential invariant of the form (11.4) are expressible as functions of V_n tensors only; that is,

$$(11.8) \quad T_{k \dots l}^{i \dots j} = H_{k \dots l}^{i \dots j} (A_{ab c_1}^h, \dots, A_{ab c_1 \dots c_s}^h; F_{ab}^E, \dots, F_{ab, c_1 \dots c_s}^E; \\ g_{ab}, 0, g_{ab, c_1 c_2}, \dots, g_{ab, c_1 \dots c_q})$$

where q is the greatest of s , t , and u .

COROLLARY. — *The tensor differential invariant (11.4) may be expressed as*

$$T_{k\dots l}^{i\dots j} = K_{k\dots l}^{i\dots j}(R_{hab}^l, R_{hab;c_1}^l, \dots, R_{hab;c_1;c_2;\dots;c_s}^l; \\ F_{ab}^E, F_{ab;c_1}^E, \dots, F_{ab;c_1;c_2;\dots;c_s}^E; g_{ab})$$

where

- (1) R_{hab}^l is the Riemann-Christoffel tensor of V_n ,
- (2) a semicolon between indices indicates covariant differentiation based on the Christoffel symbols of V_n ,
- (3) n is the greatest of $s - 1$, $t - 2$, and $u - 2$.

Evident modifications of the formulas and theorems of this paper will result from substituting for Postulate II the more general postulate

$$\gamma_{a;c}^z = A_E^z F_{ac}^E + \gamma_r^z U_{ac}^r$$

where U_{ac}^r is a mixed V_n tensor.

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