On Wave Solutions of Gauge-Invariant Generalization of Field Theories with Asymmetric Fundamental Tensor in a Generalized Peres Space-Time (*) (**).

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Summary. – *The gauge invariant generalization of field theories with asymmetric fundamental tensor developed by Buchdahl has been considered and its plane wave-like solutions in the sense of Takeno are investigated in generalized Peres space-time, recently considered by the author. It has been shown that under certain conditions these solutions become identical with those of strong field equations of Einstein in the same space-time. It has been also shown that this space-time satisfying the field equations of Buchdahl admits a parallel null vector field and is gravitationally null which further, transforms to other well known forms of space-time under a new time coordinate $Z = z - t$.*

I. – Introduction.

Combining the unified field theories of WEYL [1] ⁽¹⁾ and EINSTEIN [2], BUCHDAHL ([3], [4]) has developed a gauge-invariant theory based on an asymmetric covariant tensor g_{ij} and a covariant vector K_i relating these to an asymmetric linear connection L_{jk}^i in such a way that this new theory can be regarded equivalently as « gauge-invariant generalization of Einstein's non-symmetric theory » or « the asymmetric generalization of Weyl's theory ».

In Buchdahl's theory [4] the asymmetric tensor $g_{ij} = h_{ij} + f_{ij}$ (where h_{ij} is the symmetric part of g_{ij} which coincides with the fundamental tensor of the metric space and f_{ij} is skew-symmetric parts of g_{ij}), a covariant vector K_i and the linear connection L_{jk}^i are defined by

$$(1.1) \quad g_{ij} \Gamma_{ik}^l + g_{il} \Gamma_{kj}^l - g_{ij,k} = 0,$$

$$(1.2) \quad g_{ij} \gamma_{ik}^l + g_{il} \gamma_{kj}^l - g_{ij} K_k = 0,$$

$$(1.3) \quad L_{jk}^i = \Gamma_{jk}^i - \gamma_{jk}^i,$$

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⁽¹⁾ Numbers in brackets refer to the references at the end of the paper.

where Γ_{jk}^i is the linear connection of Einstein's non-symmetric theory [2]. The indices $i, j, k, \dots$ take the values 1, 2, 3, 4 and a comma (,) before an index k denotes its partial differentiation with respect to x^k . The simplest field equations based upon this theory, as given by BUCHDAHL [4], are

$$(1.4) \quad L_i \equiv L_{[ij]}^j = 0,$$

$$(1.5) \quad G_{ij} \equiv P_{ij} + \gamma_{ij,k}^k - (K_{i;j} + K_{j;i}) + \gamma_{il}^k \gamma_{kj}^l - 2\gamma_{ij}^k K_k = 0,$$

where P_{ij} is defined by

$$(1.6) \quad P_{ij} = -\Gamma_{ij,k}^k + \frac{1}{2} \{ \Gamma_{(ij),i}^k + \Gamma_{(kj),i}^k \} + \Gamma_{il}^k \Gamma_{kj}^l - \Gamma_{ij}^k \Gamma_{(kj)}^l.$$

A semicolon (;) stands for covariant differentiation with respect to Γ_{jk}^i , the sign « + » or « - » or « 0 » below an index fixes the position of covariant index k in connection as $\Gamma_0^0 k$, Γ_{k0}^0 , $\Gamma_{(0k)}^0$ and parenthesis () and bracket [] including two indices denote the symmetry and antisymmetry respectively between them. Tensors G_{ij} and P_{ij} are the hermitian tensors of gauge invariant and Einstein's usual asymmetric theory respectively.

Further BUCHDAHL [4] has considered the field eq. (1.4) and (1.5) in a static spherically symmetric space-time and the possible physical significance of such field equations is briefly discussed. LAL and KHARE [5] have obtained the solutions of these field equations in a $V_2 \times V_2$ space-time. Recently, LAL and SINGH ([6], [7]) have investigated the cylindrical wave solutions of Buchdahl's field equations. Wave solutions of Buchdahl's field equations have been also considered in a generalized Takeno space-time by LAL and SRIVASTAVA [8].

In the present paper we have considered gauge-invariant generalization of Einstein's field equations as given by Buchdahl in a generalized Peres space-time and it has been exhibited that under certain conditions solutions of these field equations exist.

The author [9] has investigated the plane wave-like solutions of field equations of non-symmetric unified field theories of Einstein, Bonnor and Schrödinger by taking the non-symmetric fundamental tensor $g_{ij} = h_{ij} + f_{ij}$, given by

$$(1.7) \quad g_{ij} = \begin{bmatrix} -A & 0 & \varrho & -\varrho \\ 0 & -B & \sigma & -\sigma \\ -\varrho & -\sigma & -(1-E) & -E \\ \varrho & \sigma & -E & 1+E \end{bmatrix},$$

where A , B , E , ϱ and σ are functions of (x, y, Z) , $Z = (z - t)$ and

$$(1.8) \quad h_{ij} = \begin{bmatrix} -1/A & 0 & 0 & 0 \\ 0 & -1/B & 0 & 0 \\ 0 & 0 & -(1+E) & 0 \\ 0 & 0 & -E & 1-E \end{bmatrix}.$$

We use h_{ij} and its conjugate h^{ij} to lower and raise the indices of a tensor.

2. – Solutions of equations (1.1) and (1.2) and evaluation of connections L_{ij}^k .

Equation (1.1) is the first equation of Einstein non-symmetric theory and it has been solved in [9].

We shall use HLAVATY [10] method to solve eq. (1.2). MISHRA [11] has proved that γ_{ij}^k can be put into the form

$$(2.1) \quad \gamma_{ij}^k = H_{ij}^k + S_{ij}^k + U_{ij}^k,$$

where

$$(2.2) \quad U_{ij}^k = 2h^{km} S_{m(ij)l}^l,$$

$$(2.3) \quad H_{ij}^k = \frac{1}{2} h^{kl} (K_i h_{jl} + K_j h_{il} - K_l h_{ij}),$$

$$(2.4) \quad S_{ij}^k = \gamma_{[ij]}^k = h^{kl} (K'_{ijl} + 2U_{l[ij]m}^m),$$

$$(2.5) \quad \begin{aligned} a) \quad K'_{ijk} &= K_{ijk} + 2f_{n[i} H_{j]k}^n, \\ b) \quad K_{ijk} &= \frac{1}{2} (K_j f_{ik} + K_k f_{ij} - K_i f_{jk}). \end{aligned}$$

As f_{ij} given by (1.7) are of the third class in the sense of HLAVATY [10], we have solution of (2.4) as

$$(2.6) \quad S_{ij}^k = h^{kl} (K'_{ijl} - 2f_{[ij}^m K_{l]mn} f_l^n).$$

Using (1.7) in (2.3) and (2.5b), we have the components of H_{ji}^k ($= H_{ij}^k$) and the non-zero components of K_{ijk} ($= -K_{jki}$) as follows:

$$H_{11}^k = [\frac{1}{2} K_1 - \frac{1}{2} A K_2 / B, \{-\frac{1}{2} A K_3 - \frac{1}{2} (K_3 + K_4) A E\}, \{\frac{1}{2} A K_4 - \frac{1}{2} (K_3 + K_4) A E\}],$$

$$H_{22}^k = [-B K_1 / 2A, \frac{1}{2} K_2, \{-\frac{1}{2} B K_3 - \frac{1}{2} (K_3 + K_4) B E\}, \{\frac{1}{2} B K_4 - \frac{1}{2} (K_3 + K_4) B E\}],$$

$$\begin{aligned}
H_{33}^k &= [-(1-E)K_1/2A, -(1-E)K_2/2B, \{\tfrac{1}{2}(1+E^2)K_3 - \tfrac{1}{2}E(1-E)K_4\}, \\
&\quad \{-\tfrac{1}{2}E(1-E)K_3 + \tfrac{1}{2}(1-E)^2K_4\}], \\
(2.7) \quad H_{44}^k &= [(1+E)K_1/2A, (1+E)K_2/2B, \{\tfrac{1}{2}(1+E)^2K_3 + \tfrac{1}{2}E(1+E)K_4\}, \\
&\quad \{\tfrac{1}{2}E(1+E)K_3 + \tfrac{1}{2}(1+E^2)K_4\}],
\end{aligned}$$

$$H_{12}^k = [\tfrac{1}{2}K_2, \tfrac{1}{2}K_1, 0, 0],$$

$$H_{13}^k = [\tfrac{1}{2}K_3, 0, \tfrac{1}{2}K_1, 0],$$

$$H_{14}^k = [\tfrac{1}{2}K_4, 0, 0, \tfrac{1}{2}K_1],$$

$$H_{23}^k = [0, \tfrac{1}{2}K_3, \tfrac{1}{2}K_2, 0],$$

$$H_{24}^k = [0, \tfrac{1}{2}K_4, 0, \tfrac{1}{2}K_2],$$

$$\begin{aligned}
H_{34}^k &= [-EK_1/2A, -EK_2/2B, \{-\tfrac{1}{2}E(1+E)K_3 + \tfrac{1}{2}(1-E^2)K_4\}, \\
&\quad \{\tfrac{1}{2}(1-E^2)K_3 + \tfrac{1}{2}E(1-E)K_4\}],
\end{aligned}$$

and

$$\begin{aligned}
K_{123} &= -K_{124} = \tfrac{1}{2}(\varrho K_2 - \sigma K_1), \quad K_{131} = -K_{141} = \varrho K_1, \\
K_{132} &= -K_{142} = K_{231} = -K_{241} = \tfrac{1}{2}(\varrho K_3 + \sigma K_1), \\
(2.8) \quad K_{133} &= \varrho K_3, \quad K_{144} = -\varrho K_4, \quad K_{134} = K_{143} = -\tfrac{1}{2}\varrho(K_3 - K_4), \\
K_{232} &= -K_{242} = \sigma K_2, \quad K_{233} = \sigma K_3, \quad K_{244} = -\sigma K_4, \\
K_{234} &= K_{243} = -\tfrac{1}{2}\sigma(K_3 - K_4), \quad K_{341} = -\tfrac{1}{2}\varrho(K_3 + K_4), \quad K_{342} = -\tfrac{1}{2}\sigma(K_3 + K_4)
\end{aligned}$$

When use is made of (1.7), (2.7) and (2.8) in equation (2.5a), we find the values of K'_{ijk} ($= -K'_{jik}$) as follows:

$$\begin{aligned}
K'_{131} &= -K'_{141} = K_2\sigma A/2B, \quad K'_{132} = -K'_{142} = -\tfrac{1}{2}K_2\varrho, \\
K'_{133} &= -K'_{134} = \tfrac{1}{2}\{\varrho K_4 - \varrho E(K_3 + K_4)\}, \\
K'_{143} &= -K'_{144} = \tfrac{1}{2}\{\varrho K_3 + \varrho E(K_3 + K_4)\}, \\
K'_{231} &= -K'_{241} = -\tfrac{1}{2}\sigma K_1, \quad K'_{232} = -K'_{242} = K_2\varrho B/2A, \\
(2.9) \quad K'_{233} &= -K'_{234} = \tfrac{1}{2}\{\sigma K_4 - \sigma E(K_3 + K_4)\}, \\
K'_{243} &= -K'_{244} = \tfrac{1}{2}\{\sigma K_3 + \sigma E(K_3 + K_4)\}, \\
K'_{343} &= -K'_{344} = -\tfrac{1}{2}(\varrho K_1/A + \sigma K_2/B), \\
K'_{121} &= -\tfrac{1}{2}\sigma A(K_3 + K_4), \quad K'_{122} = \tfrac{1}{2}\varrho B(K_3 + K_4), \\
K'_{341} &= K'_{342} = K'_{123} = K'_{124} = 0.
\end{aligned}$$

Substituting (1.7) and (2.9) in (2.6) we find that the last term on the right hand side of (2.6) is identically zero and consequently equation (2.6) reduces to

$$(2.10) \quad S_{ij}^k = h^{kl} K'_{ijl}.$$

Using (1.8) and (2.9) in (2.10), the components of S_{ij}^k ($= -S_{ji}^k$) are found to be

$$(2.11) \quad \begin{aligned} S_{12}^1 &= \frac{1}{2}\sigma(K_3 + K_4), & S_{13}^1 &= -S_{14}^1 = -\sigma K_2/2B, \\ S_{12}^2 &= -\frac{1}{2}\varrho(K_3 + K_4), & S_{13}^2 &= -S_{14}^2 = \varrho K_2/2B, \\ S_{23}^1 &= -S_{24}^1 = \sigma K_1/2A, & S_{13}^3 &= S_{14}^3 = -\frac{1}{2}\varrho\{K_4 - E(K_3 + K_4)\}, \\ S_{23}^2 &= -S_{24}^2 = -\varrho K_1/2A, & S_{14}^3 &= S_{14}^4 = -\frac{1}{2}\varrho\{K_3 + E(K_3 + K_4)\}, \\ S_{23}^3 &= S_{23}^4 = -\frac{1}{2}\sigma\{K_4 - E(K_3 + K_4)\}, & S_{34}^3 &= S_{34}^4 = \frac{1}{2}(\varrho K_1/A + \sigma K_2/B), \\ S_{24}^3 &= S_{24}^4 = -\frac{1}{2}\sigma\{K_3 + E(K_3 + K_4)\}, & S_{34}^1 &= S_{34}^2 = S_{12}^3 = S_{12}^4 = 0. \end{aligned}$$

Equation (2.2) with the help of equations (1.7), (1.8) and (2.11) identically reduces to

$$(2.12) \quad U_{ij}^k = 0.$$

Thus we have solved (2.2), (2.3) and (2.4) in terms of g_{ij} and K_i . Using equations (2.7), (2.11) and (2.12) we find the components of γ_{ij}^k as follows:

$$(2.13) \quad \begin{aligned} \gamma_{11}^k &= [\frac{1}{2}K_1, -AK_2/2B, -\frac{1}{2}A\{K_3 + E(K_3 + K_4)\}, \frac{1}{2}\{K_4 - E(K_3 + K_4)\}], \\ \gamma_{22}^k &= [-BK_1/2A, \frac{1}{2}K_2, -\frac{1}{2}B\{K_3 + E(K_3 + K_4)\}, \frac{1}{2}\{K_4 - E(K_3 + K_4)\}], \\ \gamma_{33}^k &= [-K_1(1 - E)/2A, -K_2(1 - E)/2B, \frac{1}{2}\{K_3 - EK_4 + E^2(K_3 + K_4)\}, \\ &\quad \frac{1}{2}\{K_4 - EK_3 - 2EK_4 + E^2(K_3 + K_4)\}], \\ \gamma_{44}^k &= [K_1(1 + E)/2A, K_2(1 + E)/2B, \frac{1}{2}\{K_3 + 2EK_3 + EK_4 + E^2(K_3 + K_4)\}, \\ &\quad \frac{1}{2}\{K_4 + EK_3 + E^2(K_3 + K_4)\}], \\ \gamma_{12}^k &= [\frac{1}{2}K_2 \pm \frac{1}{2}\sigma(K_3 + K_4), \frac{1}{2}K_1 \mp \frac{1}{2}\varrho(K_3 + K_4), 0, 0], \\ \gamma_{13}^k &= [\frac{1}{2}K_3 \mp \frac{1}{2}\sigma K_2/B, \pm \varrho K_2/2B, \frac{1}{2}K_1 \mp \frac{1}{2}\varrho\{K_4 - E(K_3 + K_4)\}, \\ &\quad \mp \frac{1}{2}\varrho\{K_4 - E(K_3 + K_4)\}], \\ \gamma_{14}^k &= [\frac{1}{2}K_4 \pm \sigma K_2/2B, \mp \varrho K_2/2B, \mp \frac{1}{2}\varrho\{K_3 + E(K_3 + K_4)\}, \\ &\quad \frac{1}{2}K_1 \mp \frac{1}{2}\varrho\{K_3 + E(K_3 + K_4)\}], \\ \gamma_{23}^k &= [\pm \sigma K_1/2A, \frac{1}{2}K_3 \mp \varrho K_1/2A, \frac{1}{2}K_2 \mp \frac{1}{2}\sigma\{K_4 - E(K_3 + K_4)\}, \\ &\quad \mp \frac{1}{2}\sigma\{K_4 - E(K_3 + K_4)\}], \\ \gamma_{24}^k &= [\mp \sigma K_1/2A, \frac{1}{2}K_4 \pm \varrho K_1/2A, \mp \frac{1}{2}\sigma\{K_3 + E(K_3 + K_4)\}, \\ &\quad \frac{1}{2}K_2 \mp \frac{1}{2}\sigma\{K_3 + E(K_3 + K_4)\}], \\ \gamma_{34}^k &= [-EK_1/2A, -EK_2/2B, \frac{1}{2}\{-E(1 + E)K_3 + (1 - E^2)K_4 \pm \\ &\quad \pm (\varrho K_1/A + \sigma K_2/B)\}, \frac{1}{2}\{(1 - E^2)K_3 + E(1 - E)K_4 \pm (\varrho K_1/A + \sigma K_2/B)\}]. \end{aligned}$$

Substituting the values of Γ_{ij}^k from [9] and γ_{ij}^k from (2.13) into (1.3) we get the components of L_{ij}^k .

3. – Solution of equation (1.4).

Using the components of Γ_{ij}^k from [9] and γ_{ij}^k from (2.13) in equation (1.4) we find that when $i = 1$ and 2 equations (1.4) give

$$(3.1) \quad K_3 + K_4 = 0,$$

while when $i = 3$ and 4 equations (1.4) give

$$(3.2) \quad 2(\varrho K_1 B + \sigma K_2 A) - 2(B\varrho_x + A\sigma_y) + B\varrho\{\log(A/B)\}_x + A\sigma\{\log(B/A)\}_y = 0.$$

From hereonwards, the lower suffixes x and y attached with any function indicate partial differentiation with respect to x and y respectively while an overhead bar stands for partial differentiation with respect to Z .

Thus (3.1) and (3.2) are necessary conditions in order that the gauge-invariant field equation (1.4) be satisfied.

4. – Tensors G_{ij} and P_{ij} .

The gauge-invariant Einstein tensor G_{ij} , as given by BUCHDAHL [4], is

$$(4.1) \quad G_{ij} = P_{ij} - (K_{i,j} + K_{j,i} - 2K_1 \Gamma_{ij}^l) - 2\gamma_{ij}^l K_1 + \gamma_{ij,l}^l + \\ I_{(ml)}^l \gamma_{ij}^m - I_{il}^m \gamma_{mj}^l - I_{ij}^m \gamma_{im}^l + \gamma_{il}^m \gamma_{mj}^l.$$

Using the components of Γ_{ij}^k from [9] we can compute from (1.6) the components of P_{ij} as follows:

$$(4.2) \quad \begin{aligned} P_{11} &= L/B, \quad P_{12} = P_{21} = 0, \quad P_{22} = L/A, \\ P_{13} &= -P_{14} = -N/B - H_x - (H - I)B_x/2B + b_y/2B + (A_y/A - B_y/B)b/4B, \\ P_{31} &= -P_{41} = -N/B + H_x + (H - I)B_x/2B - b_y/2B - (A_y/A - B_y/B)b/4B, \\ P_{23} &= -P_{24} = M/A - I_y + (H - I)A_y/2A + b_x/2A - (A_x/A - B_x/B)b/4A, \\ P_{32} &= -P_{42} = M/A + I_y - (H - I)A_y/2A - b_x/2A + (A_x/A - B_x/B)b/4A, \\ P_{33} &= -P_{34} = -P_{43} = P_{44} = (\bar{A} - \bar{A}^2/2A)/2A + (\bar{B} - \bar{B}^2/2B)/2B - \\ &\quad S_x - T_y - H^2 - I^2 + \bar{H} + \bar{I} - b^2/2AB - \frac{1}{2}S(A_x/A + B_x/B) - \\ &\quad \frac{1}{2}T(A_y/A + B_y/B), \end{aligned}$$

where

$$\begin{aligned}
 2L &= A_{yy} + B_{xx} - (A_y^2 + A_x B_x)/2A - (B_x^2 + A_y B_y)/2B, \\
 2M &= \bar{A}_y - \frac{1}{2}A_y(\bar{A}/A + \bar{B}/B), \\
 2N &= -\bar{B}_x + \frac{1}{2}B_x(\bar{A}/A + \bar{B}/B), \\
 H &= (-\varrho_x + \varrho A_x/2A - \sigma A_y/2B)/A, \\
 I &= (-\sigma_y + \sigma B_y/2B - \varrho B_x/2A)/B, \\
 b &= \sigma_x + \varrho_y - \varrho A_y/A - \sigma B_x/B, \\
 S &= \{\frac{1}{2}E_x + 2\varrho(-\varrho_x + \varrho A_x/2A)/A - \sigma(b + \varrho A_y/A)/B\}/A, \\
 T &= \{\frac{1}{2}E_y + 2\sigma(-\sigma_y + \sigma B_y/2B)/B - \varrho(b + \sigma B_x/B)/A\}/B.
 \end{aligned}
 \tag{4.3}$$

Again using Γ_{ij}^k from [9], (2.13) and (3.1) in (4.1) the components of G_{ij} are given by

$$\begin{aligned}
 G_{11} &= P_{11} - \frac{3}{2}(K_1)_{,1} - \frac{1}{2}(AK_2/B)_{,2} - \frac{1}{2}A(K_{3,3} - K_{4,4}) + \\
 &\quad 3K_1(A_x/4A) - K_1(B_x/4B) + AK_2^2/2B - AK_2(A_y/A + B_y/B)/4B, \\
 G_{22} &= P_{22} - \frac{3}{2}(K_2)_{,2} - \frac{1}{2}(BK_1/A)_{,1} - \frac{1}{2}B(K_{3,3} - K_{4,4}) + \\
 &\quad 3K_2(B_y/4B) - K_2(A_y/4A) + B(K_1^2/2A) - BK_1(A_x/A + B_x/B)/4A, \\
 G_{12} &= G_{21} = K_1(A_y/2A) + K_2(B_x/2B) - \frac{1}{2}K_1K_2 - \frac{1}{2}(K_{1,2} + K_{2,1}), \\
 G_{13} &= P_{13} - \frac{1}{2}(K_{1,3} + K_{3,1}) - \frac{1}{2}K_1K_3 - 3b(K_2/4B) - \varrho K_1^2/2A - \\
 &\quad \varrho K_2^2/2B - \frac{1}{2}(\sigma K_2/B)_{,1} + \frac{1}{2}(\varrho K_2/B)_{,2} - \frac{1}{2}(\varrho K_4)_{,3} - \\
 &\quad \frac{1}{2}(\varrho K_4)_{,4} + K_1(\bar{A}/2A + H - \frac{1}{2}I) + K_1(\sigma A_y + \varrho B_x)/4AB + \\
 &\quad K_2(\varrho B_y - \sigma B_x)/4B^2 + \xi K_2/2B, \\
 G_{31} &= P_{31} - \frac{1}{2}(K_{1,3} + K_{3,1}) - \frac{1}{2}K_1K_3 + 3b(K_2/4B) + \varrho K_1^2/2A + \\
 &\quad \varrho K_2^2/2B + \frac{1}{2}(\sigma K_2/B)_{,1} - \frac{1}{2}(\varrho K_2/B)_{,2} + \frac{1}{2}(\varrho K_4)_{,3} + \frac{1}{2}(\varrho K_4)_{,4} + \\
 &\quad K_1(\bar{A}/2A - H + \frac{1}{2}I) - K_1(\sigma A_y + \varrho B_x)/4AB - \\
 &\quad K_2(\varrho B_y - \sigma B_x)/4B^2 - \xi K_2/2B, \\
 &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
 \end{aligned}
 \tag{4.4}$$

and the expressions for G_{14} , G_{41} , G_{23} , G_{32} , G_{24} , G_{42} , G_{33} , G_{34} , G_{44} are omitted for brevity's

sake. Here a comma followed by an index i after a small and curly bracket denotes partial differentiation with respect to x^i .

There are various possibilities under which the solution of (1.5) may be considered. However, in this paper we consider the case when $K_1 = K_2 = 0$ and $K_3 = -K_4 = \psi(z-t)$.

5. - Solution of equation (1.5).

If $K_1 = K_2 = 0$ and $K_3 = -K_4 = \psi(z-t)$, then the field equation (1.5) with the help of (3.2) gives

$$\begin{aligned}
 (5.1) \quad & a) \quad L = 0, \\
 & b) \quad N/B + H_x + HB_x/B - b_y/2B - (A_y/A - B_y/B)b/4B = 0, \\
 & c) \quad N/B - H_x - HB_x/B + b_y/2B + (A_y/A - B_y/B)b/4B = 0, \\
 & d) \quad M/A + H_y + HA_y/A + b_x/2A - (A_x/A - B_x/B)b/4A = 0, \\
 & e) \quad M/A - H_y - HA_y/A - b_x/2A + (A_x/A - B_x/B)b/4A = 0, \\
 & f) \quad (\bar{A} - \bar{A}^2/2A)/2A + (\bar{B} - \bar{B}^2/2B)/2B - S_x - T_y - 2H^2 - b^2/2AB - \\
 & \quad \frac{1}{2}S(A_x/A + B_x/B) - \frac{1}{2}T(A_y/A + B_y/B) - \tilde{\psi} - \frac{1}{2}\psi^2 = 0.
 \end{aligned}$$

From equations (5.1b) and (5.1c), we get

$$(5.2) \quad a) \quad N = 0 \quad \text{and} \quad b) \quad 2(BH)_x - b_y - \frac{1}{2}b\{\log(A/B)\}_y = 0.$$

From equations (5.1d) and (5.1e), we have

$$(5.3) \quad a) \quad M = 0 \quad \text{and} \quad b) \quad 2(AH)_y + b_x - \frac{1}{2}b\{\log(A/B)\}_x = 0.$$

Equation (5.1a) can be written as

$$(5.4) \quad (B_x/\sqrt{AB})_x + (A_y/\sqrt{AB})_y = 0.$$

Equations (5.2a) and (5.3a), after integration, reduce to

$$(5.5) \quad a) \quad B_x/\sqrt{AB} = C_1 \quad \text{and} \quad b) \quad A_y/\sqrt{AB} = C_2,$$

where C_1 and C_2 are some functions of (x, y) . Now we have to find the solutions of equations (5.1f), (5.2b), (5.3b), (5.4) and (5.5) only. These equations are non-linear partial differential equations and their solutions are complicated. Hence let us assume

$$(5.6) \quad A = Bf, \quad (f = f(Z)).$$

From equations (5.5) we get

$$(5.7) \quad B_x A_y = ABC_1 C_2.$$

With the help of (5.6), equation (5.7) reduces to

$$(5.8) \quad pq = \theta, \quad (p = B_x, q = B_y, \theta = K_1 K_2 B^2),$$

which is a standard form of differential equation which can be solved by using the standard technique of CHARPIT [12].

By virtue of (5.6), equations (5.1f), (5.2b), (5.3b) and (5.4) reduce to

$$(5.9) \quad \begin{aligned} a) \quad & 2\bar{B} - (\bar{B}/B)^2 + \bar{B}\bar{f} + \bar{f}/f - \frac{1}{2}(\bar{f}/f) - 2S'B_x/B - 2T'B_y/B - \\ & (S'_x + T'_y + 2H'^2) - b'^2/B^2 f - \bar{\psi} - \frac{1}{2}\psi^2 = 0, \\ b) \quad & 2(BH')_x - b'_y = 0, \\ c) \quad & 2f(BH')_y + b'_x = 0, \\ d) \quad & (\log B)_{xx} + f(\log B)_{yy} = 0, \end{aligned}$$

where S' , T' , H' and b' are the values of S , T , H and b when $A = Bf$ is used in expressions of S , T , H and b respectively. Hence we have:

The g_{ij} given by (1.7) with (5.6) is a solution of Buchdahl's field equations under condition (5.9).

THEOREM 5.1. — If $K_1 = K_2 = 0$ and $K_3 = K_4 = \psi(z - t)$ then the necessary and sufficient condition that the solution of gauge-invariant generalized field equations of Einstein be a solution of strong field equations of Einstein's unified field theory in the generalized Peres space-time is that ψ must be of the form

$$(5.10) \quad \psi = (C_3 + \frac{1}{2}Z)^{-1}, \quad C_3 \text{ is some function of } (x, y).$$

PROOF. — We have shown that if $K_1 = K_2 = 0$ and $K_3 = -K_4 = \psi(z - t)$ then gauge-invariant generalized field equation of Einstein, i.e. $G_{ij} = 0$ reduces to (5.1).

If the solution of gauge-invariant generalized field equations of Einstein are those of strong field equations of Einstein's unified field theory then $G_{ij} = P_{ij} = 0$ which by virtue of equation (5.1) gives $\bar{\psi} + \frac{1}{2}\psi^2 = 0$ which on integration gives $\psi = (C_3 + \frac{1}{2}Z)^{-1}$.

Conversely if (5.10) holds then by differentiating it we get $\bar{\psi} + \frac{1}{2}\psi^2 = 0$, which reduces (5.1) to $G_{ij} = 0$, i.e. $G_{ij} = P_{ij} = 0$. Hence the condition is sufficient.

6. – Special case.

If we set $C_1 = C_2 = 0$ then from equations (5.5) we find that $B_x = A_y = 0$ and in this case the g_{ij} given by (1.7) where $A = A(x, Z)$, $B = B(y, Z)$ and $E = E(x, y, Z)$ constitute the wave solutions of Buchdah's field equations provided the following relations hold:

$$(6.1) \quad \begin{aligned} a) \quad & 2(BH^*)_x - \sigma_{xy} - \varrho_{yy} + (\sigma_x + \varrho_y)B_y/2B = 0, \\ b) \quad & 2(AH^*)_x - \sigma_{xx} - \varrho_{xy} - (\sigma_x + \varrho_y)A_x/2A = 0, \\ c) \quad & (\bar{A} - \bar{A}^2/2A)/A - (\bar{B} - \bar{B}^2/2B)/B - S^*A_x/A - T^*B_y/B - \\ & 2(S_x^* + T_y^* + 2H^{*2}) - (\sigma_x + \varrho_y)^2 - \bar{\psi} - \frac{1}{2}\psi^2 = 0, \end{aligned}$$

where S^* , T^* and H^* are the values of S , T , and H respectively when $A = A(x, Z)$ and $B = B(y, Z)$.

REMARK. – Here we add some words concerning the solution obtained in Section 6. When $C_1 = C_2 = 0$ then the space-time h_{ij} representing the gravitational potential takes the form

$$(6.2) \quad \begin{aligned} ds^2 = & -A dx^2 - B dy^2 - (1 - E) dz^2 - 2E dz dt + (1 + E) dt^2, \\ & A = A(x, Z), \quad B = B(y, Z), \quad E = E(x, y, Z). \end{aligned}$$

The space-time defined by (6.2) being gravitationally null admits a parallel null vector field $v^i = (0, 0, 1, 1)$ (or cv^i , c being constant). Here it should be mentioned that the space-time represented by the metric (6.2) is a generalization of Peres space-time [13], and it also becomes H space-time studied by TAKENO [14], BONDI-PIRANI-ROBINSON [15] and SHIBATA [16] when we assume $A = A(Z)$, $B = B(Z)$ and $E = E(Z)$.

The solutions found out in this paper can be called plane wave-like solutions following TAKENO [14] and the wave is propagating in the positive direction of z -axis with unit velocity.

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