Contact Totally Umbilical Submanifolds of a Sasakian Space Form (*).

IKUO ISHIHARA - MASAHIRO KON (Shinjuku-ku, Tokyo)

Summary. – We define a notion of contact totally umbilical submanifolds of Sasakian space forms corresponds to those of totally umbilical submanifolds of complex space forms. We study a contact totally umbilical submanifold M of a Sasakian space form $\bar{M}(c)$ ($c \neq -3$) and prove that M is an invariant submanifold or an anti-invariant submanifold. Furthermore we study a submanifold M with parallel second fundamental form of a Sasakian space form $\bar{M}(c)$ ($c \neq 1$) and prove that M is invariant or anti-invariant.

Introduction.

Let $\bar{M}$ be a $(2m+1)$ -dimensional Sasakian manifold with structure tensors $(\varphi, \xi, \eta, \bar{g})$. Then the structure tensors of $\bar{M}$ satisfy

$$\begin{aligned}\varphi^2 X &= -X + \eta(X)\xi, & \varphi\xi &= 0, & \eta(\varphi X) &= 0, & \eta(\xi) &= 1, \\ \bar{g}(\varphi X, \varphi Y) &= \bar{g}(X, Y) - \eta(X)\eta(Y), & \eta(X) &= \bar{g}(X, \xi),\end{aligned}$$

for any vector fields X and Y on $\bar{M}$. Let $\bar{\nabla}$ be the operator of covariant differentiation with respect to $\bar{g}$. Then we have

$$\bar{\nabla}_X \xi = \varphi X, \quad (\bar{\nabla}_X \varphi)Y = \bar{R}(X, \xi)Y = -\bar{g}(X, Y)\xi + \eta(Y)X$$

for any vector fields X and Y on $\bar{M}$, where $\bar{R}$ denotes the Riemannian curvature tensor of $\bar{M}$.

Let M be an $(n+1)$ -dimensional submanifold tangent to the structure vector field ξ of $\bar{M}$. We denote by g the induced metric tensor field on M . Let ∇ be the operator of covariant differentiation with respect to g . Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y) \quad \text{and} \quad \bar{\nabla}_X N = -A_N(X) + D_X N$$

for any vector fields X, Y tangent to M and any vector field N normal to M , where D is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^\perp$. Both A and B are called the second fundamental

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forms of M and they satisfy $\bar{g}(B(X, Y), N) = g(A_N(X), Y)$. A normal vector field N is said to be *parallel* if $D_X N = 0$ for any vector field X tangent to M . The *mean curvature vector* μ of M in $\bar{M}$ is defined as $\mu = (\text{Tr } B)/(n+1)$, where $\text{Tr } B = \sum_{i=1}^{n+1} B(e_i, e_i)$ for an orthonormal frame $\{e_i\}$ of M . If the second fundamental form B of M is of the form $B(X, Y) = g(X, Y)\mu$, then M is said to be *totally umbilical*. In particular, if the second fundamental form B vanishes identically, M is said to be *totally geodesic*. If $\mu = 0$, then M is said to be *minimal*. If the second fundamental form B of M is of the form

$$B(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha + \eta(X)B(Y, \xi) + \eta(Y)B(X, \xi)$$

for any vector fields X and Y tangent to M , where α denotes a normal vector field to M , then M is said to be *contact totally umbilical*. The notion of contact totally umbilical submanifolds of Sasakian manifolds corresponds to those of totally umbilical submanifolds of Kählerian manifolds (See [8]).

A submanifold M of a Sasakian manifold $\bar{M}$ is said to be *invariant* if the structure vector field ξ of $\bar{M}$ is tangent to M and each tangent space $T_x(M)$ of M is mapped into itself under the action of φ , i.e., $\varphi T_x(M) \subset T_x(M)$ for all $x \in M$. On the other hand, if each tangent space of M is mapped into the normal space of M by φ , i.e., $\varphi T_x(M) \subset T_x(M)^\perp$ for all $x \in M$, M is said to be *anti-invariant* in $\bar{M}$.

A Sasakian manifold $\bar{M}$ of constant φ -sectional curvature c is called a *Sasakian space form*, denote it by $\bar{M}(c)$, and a Riemannian manifold of constant curvature is called a *real space form*.

CARTAN [2] proved that an n -dimensional totally umbilical submanifold of a euclidean m -space is either an n -plane or an n -sphere. Furthermore CHEN and OGIUE [4] proved a corresponding result for totally umbilical submanifolds of a Kählerian manifold of constant holomorphic sectional curvature $c \neq 0$.

The purpose of the present paper is to prove the following

THEOREM 1. — *Let M be an $(n+1)$ -dimensional submanifold tangent to the structure vector field ξ of a $(2m+1)$ -dimensional Sasakian space form $\bar{M}^{2m+1}(c)$ of constant φ -sectional curvature $c \neq 1$. If the second fundamental form B of M is parallel, then M is one of the following submanifolds:*

(a) *a Sasakian space form immersed in $\bar{M}^{2m+1}(c)$ as an invariant and totally geodesic submanifold, or*

(b) *an anti-invariant submanifold immersed in $\bar{M}^{2n+1}(c)$ in $\bar{M}^{2m+1}(c)$, where $\bar{M}^{2n+1}(c)$ is an invariant and totally geodesic submanifold of $\bar{M}^{2m+1}(c)$ of constant φ -sectional curvature $c \neq 1$.*

THEOREM 2. — *Let M be an $(n+1)$ -dimensional ($n \geq 2$) contact totally umbilical submanifold tangent to the structure vector field ξ of a $(2m+1)$ -dimensional Sasakian*

space form $\bar{M}^{2m+1}(c)$ of constant φ -sectional curvature $c \neq -3$. Then M is one of the following submanifolds:

(a) a Sasakian space form immersed in $\bar{M}^{2m+1}(c)$ as an invariant and totally geodesic submanifold, or

(b) an anti-invariant submanifold immersed in $\bar{M}^{2m+1}(c)$ as locally a Riemannian direct product $M^n \times M^1$, where M^n is a real space form, a totally geodesic hypersurface of M and a totally geodesic submanifold of $\bar{M}^{2m+1}(c)$ and M^1 is a 1-dimensional space generated by ξ , or

(c) an anti-invariant submanifold immersed in $\bar{M}^{2m+1}(c)$ as locally a Riemannian direct product $M^n \times M^1$, where M^n is a real space form and a totally geodesic hypersurface of M with nonzero parallel mean curvature vector in $\bar{M}^{2m+1}(c)$ and M^1 is a 1-dimensional space generated by ξ .

Case (b) occurs only when $m \geq n$ and case (c) occurs only when $m > n$.

1. – Preliminaries.

Let M be an $(n+1)$ -dimensional submanifold tangent to the structure vector field ξ of a $(2m+1)$ -dimensional Sasakian manifold $\bar{M}$. For the second fundamental form B of M we define its covariant derivative, denoted by $\nabla_x B$, to be

$$(1.1) \quad (\nabla_x B)(Y, Z) = D_x(B(Y, Z)) - B(\nabla_x Y, Z) - B(Y, \nabla_x Z)$$

for any vector fields X, Y and Z tangent to M . If $\nabla_x B = 0$ for any vector field X tangent to M , then the second fundamental form B is said to be *parallel*. We denote by R the Riemannian curvature tensor of M and by $R^\perp$ the curvature tensor of the normal bundle $T(M)^\perp$, which is defined to be $R^\perp(X, Y) = [D_x, D_y] - D_{[X, Y]}$. Then the equations of Gauss, Codazzi and Ricci are given respectively by

$$(1.2) \quad \begin{aligned} \bar{g}(\bar{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) + \bar{g}(B(X, Z), B(Y, W)) \\ &\quad - \bar{g}(B(X, W), B(Y, Z)), \end{aligned}$$

$$(1.3) \quad (\bar{R}(X, Y)Z)^\perp = (\nabla_x B)(Y, Z) - (\nabla_y B)(X, Z)$$

and

$$(1.4) \quad \bar{g}(\bar{R}(X, Y)N, V) = \bar{g}(R^\perp(X, Y)N, V) - g([A_N, A_V](X), Y)$$

for any vector fields X, Y, Z, W tangent to M and any vector fields N, V normal to M , where $\perp$ in (1.3) denotes the normal projection.

A Sasakian manifold $\bar{M}$ is of constant φ -sectional curvature c if and only if

$$(1.5) \quad \begin{aligned} \bar{R}(X, Y)Z &= \frac{1}{4}(c+3)[\bar{g}(Y, Z)X - \bar{g}(X, Z)Y] \\ &+ \frac{1}{4}(c-1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \bar{g}(X, Z)\eta(Y)\xi - \bar{g}(Y, Z)\eta(X)\xi \\ &+ \bar{g}(\varphi Y, Z)\varphi X - \bar{g}(\varphi X, Z)\varphi Y - 2\bar{g}(\varphi X, Y)\varphi Z] \end{aligned}$$

for any vector fields X , Y and Z on $\bar{M}$.

Now we state some lemmas for later use.

LEMMA 1 ([16]). — *Let M be an invariant submanifold of a Sasakian manifold $\bar{M}$. Then M is totally geodesic if and only if the second fundamental form B of M is parallel.*

LEMMA 2 ([14]). — *Let M be an $(n+1)$ -dimensional anti-invariant submanifold tangent to the structure vector field ξ of a $(2m+1)$ -dimensional Sasakian manifold $\bar{M}$. Then the structure vector field ξ is parallel on M and M is locally a Riemannian direct product $M^n \times M^1$, where M^n is a totally geodesic hypersurface of M and M^1 is a 1-dimensional space generated by ξ .*

LEMMA 3 ([14]). — *Let M be an anti-invariant submanifold of a Sasakian manifold $\bar{M}$. If $\dim M \geq 2$, then M is not totally umbilical.*

Let M be an $(n+1)$ -dimensional submanifold tangent to the structure vector field ξ of a Sasakian manifold $\bar{M}$. Then the Gauss formula implies that

$$\varphi X = \bar{\nabla}_X \xi = \nabla_X \xi + B(X, \xi),$$

from which we have

$$(1.6) \quad (\varphi X)^T = \nabla_X \xi \quad \text{and} \quad (\varphi X)^\perp = B(X, \xi),$$

where $(\varphi X)^T$ and $(\varphi X)^\perp$ denote the tangential and normal parts of φX respectively. Since $\varphi \xi = 0$, the second equation of (1.6) implies that

$$(1.7) \quad B(\xi, \xi) = 0.$$

If we assume that M is totally umbilical, then we have $B(X, Y) = g(X, Y)\mu$, which implies that $0 = B(\xi, \xi) = \mu$. That is, M is minimal in $\bar{M}$ and hence M is totally geodesic. Therefore we have

PROPOSITION 1. — *Let M be a submanifold tangent to the structure vector field ξ of a Sasakian manifold $\bar{M}$. If M is totally umbilical, then M is totally geodesic.*

REMARK. — If M is totally geodesic, then the second equation of (1.6) shows that $(\varphi X)^\perp = 0$. Therefore φX is tangent to M and hence M is an invariant submanifold and a Sasakian manifold (See [12]).

Now we define a tensor field T on $\bar{M}$ by setting

$$(1.8) \quad T(X, Y)Z = \bar{R}(X, Y)Z + \bar{g}(\varphi Y, Z)\varphi X - \bar{g}(\varphi X, Z)\varphi Y - 2\bar{g}(\varphi X, Y)\varphi Z$$

for any vector fields X, Y and Z on $\bar{M}$.

Let M be a submanifold tangent to the structure vector field ξ of $\bar{M}$. If each tangent space $T_x(M)$ of M satisfies $(T(X, Y))T_x(M) \subset T_x(M)$ for any vector fields X and Y tangent to M , then M is said to be *T-invariant*. The notion of *T-invariant* corresponds to those of curvature-invariant of submanifolds in Kählerian manifolds in [4] (See (5.4) in [9]).

PROPOSITION 2. — *Let M be a submanifold tangent to the structure vector field ξ of a Sasakian space form $\bar{M}(c)$ ($c \neq -3$). Then M is T-invariant if and only if M is invariant or anti-invariant.*

PROOF. — From (1.5) and (1.8) we have

$$(1.9) \quad \begin{aligned} T(X, Y)Z &= \frac{1}{4}(c+3)[\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(\varphi Y, Z)\varphi X - \bar{g}(\varphi X, Z)\varphi Y - 2\bar{g}(\varphi X, Y)\varphi Z] \\ &\quad + \frac{1}{4}(c-1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X - g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi] \end{aligned}$$

for any vector fields X, Y and Z tangent to M . If M is invariant or anti-invariant, then (1.9) implies that M is *T-invariant*. Conversely if we assume that M is *T-invariant*, then M is invariant or anti-invariant. In fact, taking vectors X, Y and Z tangent to M and orthogonal to ξ and putting $X = Z$, we have

$$T(X, Y)X = \frac{1}{4}(c+3)[\bar{g}(Y, X)X - \bar{g}(X, X)Y + 3\bar{g}(X, \varphi Y)\varphi X].$$

Since $T(X, Y)X \in T_x(M)$, we have $\bar{g}(X, \varphi Y)\varphi X \in T_x(M)$ so that either $\varphi X \in T_x(M)$ or $g(X, \varphi Y) = 0$. This implies that either $\varphi(T_x(M)) \subset T_x(M)$ or $\varphi(T_x(M)) \subset T_x(M)^\perp$. Consequently M is invariant or anti-invariant.

If each tangent space $T_x(M)$ of M satisfies $(\bar{R}(X, Y))T_x(M) \subset T_x(M)$ for all vector fields X and Y tangent to M , then M is said to be *$\bar{R}$ -invariant*.

PROPOSITION 3. — *Let M be a submanifold tangent to the structure vector field ξ of a Sasakian space form $\bar{M}(c)$ ($c \neq 1$). Then M is $\bar{R}$ -invariant if and only if M is invariant or anti-invariant.*

PROOF. — If M is invariant or anti-invariant, by (1.5) we easily see that M is $\bar{R}$ -invariant. Let us assume that M is $\bar{R}$ -invariant. Then (1.5) implies that

$$\bar{g}(\varphi Y, Z)\varphi X - \bar{g}(\varphi X, Z)\varphi Y - 2\bar{g}(\varphi X, Y)\varphi Z$$

is tangent to M . Putting $X = Z$, we see that $\bar{g}(\varphi Y, X)\varphi X \in T_x(M)$. Therefore M is invariant or anti-invariant.

2. — Proof of Theorem 1.

Let M be an $(n+1)$ -dimensional submanifold tangent to the structure vector field ξ of a Sasakian space form $\bar{M}^{2m+1}(c)$ ($c \neq 1$) with parallel second fundamental form. Then the Codazzi equation (1.3) reduces to $(R(X, Y)Z)^\perp = 0$ for any vector fields X, Y and Z tangent to M . Therefore M is $\bar{R}$ -invariant. Thus Proposition 3 implies that M is invariant or anti-invariant. Let M be an invariant submanifold. Then Lemma 1 proves (a) of Theorem 1. In the following we assume that M is an anti-invariant submanifold of $\bar{M}^{2m+1}(c)$. Then the normal space $T_x(M)^\perp$ is decomposed as

$$T_x(M)^\perp = \varphi(T_x(M)) \oplus N_x(M),$$

where $N_x(M)$ is the complement of $\varphi(T_x(M))$ in $T_x(M)^\perp$. Then we see $\varphi(N_x(M)) = N_x(M)$. Since M is anti-invariant, by (1.6), we have $\varphi X = B(X, \xi)$. On the other hand, by (1.1) and Lemma 2, we have that $D_X(\varphi Y) = \varphi \nabla_X Y$ for any vector fields X and Y tangent to M . Using the Gauss and Weingarten formulas, we obtain

$$\eta(Y)X - \bar{g}(X, Y)\xi + \varphi B(X, Y) + A_{\varphi Y}(X) = 0.$$

Therefore we have

$$\eta(Y)\varphi X - B(X, Y) + \varphi A_{\varphi Y}(X) = 0.$$

From this, for any vector $N \in N_x(M)$, we have that $\bar{g}(B(X, Y), N) = 0$. Thus we have

$$(2.1) \quad A_N(X) = 0 \quad \text{for } N \in N_x(M).$$

Now our assertion (b) of Theorem 1 reduces to the following

PROPOSITION 4. — *Let M be an $(n+1)$ -dimensional anti-invariant submanifold tangent to the structure vector field ξ of a Sasakian space form $\bar{M}^{2m+1}(c)$. If $A_N(X) = 0$ for any vector $N \in N_x(M)$, then M is immersed in $\bar{M}^{2n+1}(c)$ and $\bar{M}^{2m+1}(c)$, where $\bar{M}^{2n+1}(c)$ is an invariant and totally geodesic submanifold of $\bar{M}^{2m+1}(c)$.*

PROOF. – First of all, we prove that $\varphi T(M)$ is parallel with respect to the normal connection, that is, for every local section φY in $\varphi T(M)$, $D_x \varphi Y$ is also a local section in $\varphi T(M)$.

From the Gauss and Weingarten formulas we have

$$\begin{aligned} D_x \varphi Y &= \bar{\nabla}_x \varphi Y + A_{\varphi Y}(X) \\ &= (\bar{\nabla}_x \varphi) Y + \varphi \bar{\nabla}_x Y + A_{\varphi Y}(X) \\ &= \eta(Y)X - \bar{g}(X, Y)\xi + \varphi \nabla_{\varphi X} Y + \varphi B(X, Y) + A_{\varphi Y}(X). \end{aligned}$$

Since $A_N = 0$ for any $N \in N_x(M)$, we have

$$\begin{aligned} \bar{g}(D_x \varphi Y, N) &= \bar{g}(\varphi B(X, Y), N) \\ &= -\bar{g}(B(X, Y), \varphi N) = -g(A_{\varphi N}(X), Y) = 0 \end{aligned}$$

because of φN is also in $N_x(M)$. Consequently, $\varphi T(M)$ is parallel with respect to the normal connection. Moreover $\varphi T(M)$ is of constant dimension on M . Therefore, from the fundamental theorem in submanifold theory, there exists a totally geodesic Sasakian submanifold $\bar{M}^{2n+1}(c)$ in $\bar{M}^{2m+1}(c)$, where M is immersed in $\bar{M}^{2n+1}(c)$ as an anti-invariant submanifold. This completes our assertion.

EXAMPLE 1. – Let E^{2n+1} be a euclidean space with cartesian coordinates $(x^1, \dots, x^n, y^1, \dots, y^n, z)$. Then a Sasakian structure on E^{2n+1} is defined by φ, ξ, η and $\bar{g}$ such that

$$\xi = (0, \dots, 0, 2), \quad 2\eta = (-y^1, \dots, -y^n, 0, \dots, 0, 1),$$

$$(\bar{g}_{AB}) = \begin{pmatrix} \frac{1}{4}(\delta_{ij} + y^i y^j) & 0 & -\frac{1}{4}y^i \\ 0 & \frac{1}{4}\delta_{ij} & 0 \\ -\frac{1}{4}y^j & 0 & \frac{1}{4} \end{pmatrix},$$

$$(\varphi_B^A) = \begin{pmatrix} 0 & \delta_j^i & 0 \\ -\delta_j^i & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}.$$

Then E^{2n+1} is of constant φ -sectional curvature -3 and denoted by $E^{2n+1}(-3)$ (cf. [10] and [11]).

Now we can consider the following natural imbedding of E^{n+1} into $E^{2n+1}(-3)$:

$$(x^1, \dots, x^n, z) \mapsto (x^1, \dots, x^n, 0, \dots, 0, z).$$

Then the second fundamental forms of E^{n+1} in $E^{2n+1}(-3)$ are given by

$$A_{x^*} = \begin{matrix} & x \\ & \begin{pmatrix} 0 & 0 \dots 0 & 1 & 0 & \dots & 0 \\ \hline 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \\ x & 1 & & 0 & & \\ & 0 & & & & \\ & \vdots & & & & \\ & 0 & & & & \end{pmatrix} \end{matrix}, \quad x = 1, \dots, n.$$

Thus the second fundamental form of E^{n+1} is parallel and E^{n+1} is obviously anti-invariant in $E^{2n+1}(-3)$. Since $E^{2n+1}(-3)$ is a totally geodesic and invariant submanifold of $E^{2m+1}(-3)$ ($m > n$), E^{n+1} is an anti-invariant submanifold of $E^{2m+1}(-3)$ with parallel second fundamental form.

Let $S^1(r_i) = \{z_i \in C : |z_i|^2 = r_i^2\}$ be circles of radius r_i , $i = 1, \dots, n$. Now we consider

$$M^{n+1} = S^1(r_1) \times \dots \times S^1(r_n) \times E^1.$$

Then we can consider an imbedding M^{n+1} into $E^{2n+1}(-3)$ given by

$$(r_1 \cos u^1, \dots, r_n \cos u^n, r_1 \sin u^1, \dots, r_n \sin u^n, z).$$

Then M^{n+1} is an anti-invariant submanifold of $E^{2n+1}(-3)$ and hence of $E^{2m+1}(-3)$. The second fundamental forms of M^{n+1} in $E^{2n+1}(-3)$ are given by

$$A_{x^*} = \begin{matrix} & x \\ & \begin{pmatrix} 0 & 0 \dots 0 & 1 & 0 & \dots & 0 \\ \hline 0 & 0 & & & & \\ \vdots & \ddots & & & & \\ 0 & & 0 & & & \\ x & 1 & \text{---} & \lambda_x & & \\ & 0 & & & 0 & \\ & \vdots & & & \ddots & \\ & 0 & & & & 0 \end{pmatrix} \end{matrix}, \quad x = 1, \dots, n.$$

And the second fundamental form of M^{n+1} is parallel.

REMARK. — Let S^{2n+1} be a $(2n+1)$ -dimensional unit sphere, *i.e.*, $S^{2n+1} = \{z \in C^{n+1}: |z| = 1\}$. For any point $z \in S^{2n+1}$, put $\xi = Jz$, where J is the almost complex structure of C^{n+1} . We consider the orthogonal projection

$$\pi: T_z(C^{n+1}) \rightarrow T_z(S^{2n+1}).$$

Putting $\varphi = \pi \circ J$, we have a Sasakian structure $(\varphi, \xi, \eta, \bar{g})$ on S^{2n+1} , where η is a 1-form dual to ξ and $\bar{g}$ the standard metric tensor field on S^{2n+1} . We see that S^{2n+1} is of constant φ -sectional curvature 1, that is, of constant curvature 1 (cf. [10] and [11]).

Now we consider the Clifford hypersurface $M_{p,q}$ defined by

$$M_{p,q} = S^{2p+1} \left(\sqrt{\frac{p}{2n}} \right) \times S^{2q+1} \left(\sqrt{\frac{q}{2n}} \right), \quad p + q = n - 1.$$

Then $M_{p,q}$ is a minimal hypersurface of S^{2n+1} tangent to the structure vector field ξ of S^{2n+1} and $M_{p,q}$ has the parallel second fundamental form. But $M_{p,q}$ is neither invariant nor anti-invariant in S^{2n+1} . Therefore the assumption in Theorem 1 on the φ -sectional curvature $c \neq 1$ of the ambient manifold $\bar{M}^{2m+1}(c)$ is essential.

On the other hand, there exists an example of anti-invariant submanifold with parallel second fundamental form of $\bar{M}^{2m+1}(1)$. Indeed, if we consider

$$M^{n+1} = S^1(r_1) \times \dots \times S^1(r_{n+1}), \quad r_1^2 + \dots + r_{n+1}^2 = 1$$

in S^{2n+1} , then M^{n+1} is an anti-invariant submanifold of S^{2n+1} with parallel second fundamental form (See [14]).

3. — Proof of Theorem 2.

Let M be an $(n+1)$ -dimensional contact totally umbilical submanifold tangent to the structure vector field ξ of a Sasakian space form $\bar{M}^{2m+1}(c)$ ($c \neq -3$). Then we have

$$(3.1) \quad B(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]H + \eta(X)B(Y, \xi) + \eta(Y)B(X, \xi)$$

for any vector fields X and Y tangent to M , where H is given by $H = (\text{Tr } B)/n$. From (1.1) and (3.1) we have

$$\begin{aligned} (\nabla_X B)(Y, Z) &= [g(Y, Z) - \eta(Y)\eta(Z)]D_X H + \bar{g}(\varphi X, Y)B(Z, \xi) \\ &\quad + \bar{g}(\varphi X, Z)B(Y, \xi) - \bar{g}(\varphi X, Y)\eta(Z)H - \bar{g}(\varphi X, Z)\eta(Y)H \\ &\quad + \eta(Y)B(Z, \nabla_X \xi) + \eta(Z)B(Y, \nabla_X \xi) \\ &\quad + \eta(Y)(\nabla_X B)(Z, \xi) + \eta(Z)(\nabla_X B)(Y, \xi). \end{aligned}$$

If X , Y and Z are orthogonal to ξ , using (1.6), we have

$$(3.2) \quad (\nabla_x B)(Y, Z) = g(Y, Z)D_x H + \bar{g}(\varphi X, Y)(\varphi Z)^\perp + \bar{g}(\varphi X, Z)(\varphi Y)^\perp.$$

From the Codazzi equation (1.3) and (3.2) we obtain

$$(3.3) \quad (\bar{R}(X, Y)Z)^\perp = g(Y, Z)D_x H - g(X, Z)D_Y H + \bar{g}(\varphi X, Z)(\varphi Y)^\perp \\ + \bar{g}(\varphi Z, Y)(\varphi X)^\perp - 2\bar{g}(\varphi Y, X)(\varphi Z)^\perp$$

for any vector fields X , Y and Z orthogonal to ξ on M . If $n \geq 3$, for any vector field X orthogonal to ξ on M , we can choose a unit vector field Y tangent to M which is orthogonal to X , φX and ξ . For such a choice it follows from (3.3) that

$$(3.4) \quad (\bar{R}(X, Y)Y)^\perp = D_x H.$$

On the other hand, (1.5) implies that $(\bar{R}(X, Y)Y)^\perp = 0$. Therefore we have

$$(3.5) \quad D_x H = 0$$

for any vector field X tangent to M and orthogonal to ξ . From (3.3) and (3.5) we have

$$(3.6) \quad (\bar{R}(X, Y)Z)^\perp = \bar{g}(\varphi X, Z)(\varphi Y)^\perp + \bar{g}(\varphi Z, Y)(\varphi X)^\perp - 2\bar{g}(\varphi Y, X)(\varphi Z)^\perp$$

for any tangent vector fields X , Y and Z orthogonal to ξ on M . Substituting (3.6) into (1.8) and taking the normal part, we have

$$(3.7) \quad (T(X, Y)Z)^\perp = 0.$$

Since $T(X, \xi)Z = 0$ for any vector fields X and Y tangent to M , (3.7) shows that $(T(X, Y)Z)^\perp = 0$ for any vector fields X , Y and Z tangent to M , which means that M is T -invariant.

If $n = 2$, put $M = M_1 \cup M_2$, where $M_1 = \{x \in M : \varphi(T_x(M)) \neq T_x(M) - \{\xi\}\}$ and $M_2 = \{x \in M : \varphi(T_x(M)) = T_x(M) - \{\xi\}\}$. We can see that M_1 is an open submanifold of M , where the preceding argument is available so that (3.7) holds on M_1 . Let M'_2 be the set of all interior points of M_2 . Then M'_2 is an invariant submanifold of $\bar{M}$ so that $H = 0$ on M'_2 and hence (3.7) holds on M'_2 . Since (3.7) is a tensorial equation, it holds on M . Therefore M is T -invariant when $n \geq 2$. Thus, by Proposition 2, M is invariant or anti-invariant.

If M is an invariant submanifold of $\bar{M}^{2m+1}(c)$, then M is minimal (cf. [6] and [12]). Then we have $\text{Tr } B = 0$ and hence $H = 0$. By the second equation of (1.6), we also have $B(X, \xi) = 0$ for any vector field X tangent to M . Therefore (3.1) implies

that $B(X, Y) = 0$ and hence M is totally geodesic. This completes the proof of (a) of Theorem 2.

In the sequel, we consider the case that M is an anti-invariant submanifold. Since $\varphi(T_x(M)) \subset T_x(M)^\perp$, we see that $m \geq n$. Moreover, by Lemma 2, M is locally a Riemannian direct product $M^n \times M^1$, where M^n is a totally geodesic hypersurface of M and M^1 is a 1-dimensional space generated by ξ . Since $B(\xi, \xi) = 0$, the mean curvature vector of M^n in $\bar{M}^{2m+1}(c)$ is just H . Moreover, we have $B(X, Y) = g(X, Y)H$ for any vector fields X and Y on M^n , which means that M^n is a totally umbilical submanifold of $\bar{M}^{2m+1}(c)$. From this and the Gauss equation (1.2) we see that M^n is a real space form of constant curvature $\frac{1}{4}(c + 3) + \bar{g}(H, H)$. Moreover, the mean curvature vector H of M^n in $\bar{M}^{2m+1}(c)$ is parallel with respect to the linear connection D' induced in the normal bundle of M^n in $\bar{M}^{2m+1}(c)$, which is proved by the following way.

First of all, by a straight forward computation, using the Gauss equation, we have (See [14])

$$(3.8) \quad A_{\varphi X}(Y) = A_{\varphi Y}(X)$$

for any vector fields X and Y tangent to M^n . Since $B(X, Y) = g(X, Y)H$, we have

$$(3.9) \quad g(A_{\varphi Z}(X), Y) = \bar{g}(B(X, Y), \varphi Z) = \bar{g}(X, Y)g(H, \varphi Z)$$

for any vector fields X, Y and Z tangent to M^n . From (3.8) and (3.9) we obtain

$$(3.10) \quad g(X, Y)\bar{g}(H, \varphi Z) = g(A_{\varphi X}(Z), Y) = g(Y, Z)\bar{g}(H, \varphi X).$$

For any vector field Z tangent to M^n , we can choose a unit vector field X tangent to M^n which is orthogonal to Z and putting $X = Y$ in (3.10), we have

$$(3.11) \quad \bar{g}(H, \varphi Z) = 0.$$

Therefore the mean curvature vector H is orthogonal to $\varphi(T_x(M))$. Thus if $H \neq 0$, then $m > n$. On the other hand, by the Weingarten formula, we have

$$-A_H(X) + D_X H = -A'_H(X) + D'_X H$$

where A' denotes the second fundamental form of M^n in $\bar{M}^{2m+1}(c)$. Then (3.5) implies that

$$-A_H(X) = -A'_H(X) + D'_X H.$$

Thus we obtain that $\bar{g}(D'_X H, N) = 0$ for any vector field N orthogonal to M . More-

over we have

$$\bar{g}(D'_x H, \xi) = -g(A_H(X), \xi) = -\bar{g}(B(X, \xi), H) = -\bar{g}(\varphi X, H) = 0,$$

by using (1.6) and (3.11). Consequently we have

$$(3.12) \quad D'_x H = 0,$$

which means that the mean curvature vector H of M^n in $\bar{M}^{2m+1}(c)$ is parallel. Therefore we have (b) and (c) of Theorem 2.

EXAMPLE 2. — Let S^{2m+1} be a unit sphere of dimension $2m+1$ with standard Sasakian structure and let CP^m be a complex projective space of real dimension $2m$ of constant holomorphic sectional curvature 4. A real projective space RP^n of dimension n of constant curvature 1 is imbedded in CP^n in CP^m as an anti-invariant and totally geodesic submanifold (See [1]). Now we consider the following commutative diagram:

$$\begin{array}{ccc} (RP^n, S^1) & \xrightarrow{i} & S^{2m+1} \\ \pi \downarrow & & \downarrow \bar{\pi} \\ RP^n & \xrightarrow{i'} & CP^m, \end{array}$$

where (RP^n, S^1) denotes a circle bundle over RP^n . Then (RP^n, S^1) is an anti-invariant submanifold immersed in a S^{2n+1} in S^{2m+1} with parallel second fundamental form. Moreover, we see that (RP^n, S^1) is a contact totally umbilical submanifold of S^{2n+1} whose second fundamental forms are given by

$$A_{x*} = \begin{matrix} & x \\ \begin{matrix} x \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & \left(\begin{array}{c|cccc} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \hline & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{array} \right), \quad x = 1, \dots, n.$$

Let M^r be a totally umbilical submanifold of RP^n . Then M^r is a totally umbilical submanifold of CP^n . Therefore the circle bundle (M^r, S^1) over M^r is a contact totally umbilical submanifold of S^{2n+1} (See [8]). The second fundamental forms A_{x*} of an

anti-invariant contact totally umbilical submanifold M^{n+1} of a Sasakian manifold $\bar{M}^{2n+1}$ are given by

$$A_{x^*} = \begin{matrix} & x \\ \begin{matrix} x \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & \left(\begin{array}{c|cccc} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \hline 0 & \lambda & & & & & & \\ \vdots & & \ddots & & & & & \\ 0 & & & \ddots & & & & \\ 1 & & & & \ddots & & & \\ 0 & & & & & \ddots & & \\ \vdots & & & & & & \ddots & \\ 0 & & & & & & & \lambda \end{array} \right), \quad x = 1, \dots, n.$$

REMARK. — Let $E^{2n+1}(-3)$ be a Sasakian space form of constant φ -sectional curvature -3 as in Example 1. Now we consider $S^{2n-1} \times E^1 = M^{2n}$. Then M^{2n} is a submanifold of $E^{2n+1}(-3)$ tangent to the structure vector field ξ of $E^{2n+1}(-3)$ and M^{2n} is contact totally umbilical. But M^{2n} is neither invariant nor anti-invariant in $E^{2n+1}(-3)$. Therefore the assumption in theorem 2 on the φ -sectional curvature $c \neq -3$ of an ambient manifold $\bar{M}^{2m+1}(c)$ is essential.

REMARK. — Let M be an $(n+1)$ -dimensional submanifold tangent to the structure vector field ξ of a $(2m+1)$ -dimensional Sasakian manifold $\bar{M}$. If the second fundamental form B of M is of the form

$$B(X, Y) = \eta(X)B(Y, \xi) + \eta(Y)B(X, \xi)$$

for any vector field X and Y tangent to M , then M is said to be *contact totally geodesic*. The notion of contact totally geodesic submanifolds of Sasakian manifolds corresponds to those of totally geodesic submanifolds of Kählerian manifolds.

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