Convergence of Solutions of Nonlinear Differential Equations (*).

K. E. SWICK (Flushing, New York, U.S.A.)

Summary. — *Solutions of $\dot{x} = f(t, x_t)$ are said to converge if every pair of solutions $x(t), y(t)$ satisfy $x(t) - y(t) \rightarrow 0$ as $t \rightarrow \infty$. An invariance principle of LaSalle is used to determine conditions under which the solutions of $\dot{x} = F(t, x) + G(t, x) + e(t)$ converge. In certain cases the approach used does not require boundedness of solutions as has been required in most previous results on convergence of solutions. The results of this investigation are applied to a number of nonlinear second order differential equations. Sufficient conditions are also found for the convergence of solutions of certain functional differential equations.*

1. — We shall be concerned with the differential equation

$$(1.1) \quad \dot{x} = F(t, x) + G(t, x) + e(t) \quad \left(\dot{x} = \frac{dx}{dt} \right),$$

where $F: I \times R^n \rightarrow R^n$, $I = [0, \infty)$, $G: I \times R^n \rightarrow R^n$, and where F , G and e are continuous for $(t, x) \in I \times R^n$ and $t \in I$ respectively. It will be assumed that for every $t_0 \geq 0$, the solution $x(t; x_0, t_0)$ of (1.1) exists for $t \geq t_0$ and that F and G satisfy:

- (1) $|F(t, x)|$ is bounded for all t on every compact set $K \subset R^n$,
- (2) $\int_t^\infty |G(x, x(s))| ds \rightarrow 0$ as $t \rightarrow \infty$, uniformly in t over solutions $x(t)$ of (1.1).

Solutions of (1.1) are said to *converge* if, for every pair of solutions $x(t)$ and $y(t)$ of (1.1),

$$\lim_{t \rightarrow \infty} [x(t) - y(t)] = 0.$$

We seek conditions on F , G and e which will imply that the solutions of (1.1) converge.

The present work is a continuation of an earlier investigation by this author [11]. There, we considered the equation

$$(1.2) \quad \dot{x} = f(x) + e(t),$$

and it was assumed that all solutions of equation (1.2) were ultimately bounded. It was shown there, under appropriate restrictions on f and e , that if there exists

(*) Entrata in Redazione il 10 febbraio 1976.

a nonnegative Liapunov function $V(x, y)$, whose derivative with respect to the product system, $\dot{x} = f(x) + e(t)$ $\dot{y} = f(y) + e(t)$, is negative definite with respect to a closed set $\Omega \subset R^n$; then the solutions of (1.2) converge if the largest invariant set, contained in Ω , of $\dot{x} = f(x)$ $x \in \Omega$, is the origin. The results of [11] are extended here to equation (1.1), and it is shown that the boundedness required in [11] and most other results can, in appropriate situations, be removed. In [13] and [12], convergence of solutions of the equations $\ddot{x} + f(x)\dot{x} + g(x) = e(t)$ and $\ddot{x} + a\ddot{x} + g(x)\dot{x} + h(x) = e(t)$ was established without the boundedness restriction, but the rather special technique used there does not appear to be useful in most other applications.

The approach used here is similar to that used by a number of authors [in examining the extent of asymptotic stability [6, Chap. 2] or [15, Chap. 3].

In section 4 we exhibit application of our results to a number of nonlinear second order differential equations.

In section 5 convergence of solutions is examined for the functional differential equation

$$(1.3) \quad \dot{x}(t) = F(t, x_t)$$

where x and F are n -vectors. Numerous results, such as the ones described above, have been established for convergence of solutions of ordinary differential equations. In most cases these results were obtained by means of the construction of an appropriate Liapunov-type function; indeed, the necessary and sufficient conditions for convergence of solutions in terms of these Liapunov-type functions parallels that for asymptotic stability. Although this theory can be extended completely to (1.3), see e.g. [5, Chap. 7], very few applications of this theory have been obtained due to the difficulty of constructing the appropriate Liapunov-type function. A technique developed by KRASOVSKIĬ [4, Chap 6] for the study of asymptotic stability of functional differential equations is modified and used to establish sufficient conditions for the convergence of solutions of the equations

$$(1.4) \quad \dot{x}(t) = f(t, x(t-h(t))) + e(t)$$

$$(1.5) \quad \dot{x}(t) = g(t, x(t)) + f(t, x(t-h(t))) + e(t),$$

where $0 \leq h(t) \leq r$, $r \geq 0$, and e, f, g and h are continuous real valued functions for $t \geq 0$ and real x .

2. — We begin by stating some definitions and notation which will be used in the remainder of the paper.

We denote by R^n real Euclidean n -space and by $|x|$ the Euclidean norm of $x \in R^n$. If A and B are subsets of R^n , we denote by $A \times B$ their cartesian product.

Let U be an open subset of R^n containing the origin, and let V be a continuous scalar function on $S = R^1 \times U \times U$. We say that $V \in C_0(x, y)$ on S if for each $(x_0, y_0) \in$

$\in U \times U$ there exists $\delta > 0$ and $L > 0$ such that for $(x, y) \in U \times U$, $|x - x_0| < \delta$, $|y - y_0| < \delta$ we have $|V(t, x, y) - V(t, x_0, y_0)| \leq L[|x - x_0| + |y - y_0|]$. We will call V a Liapunov function on S if V is a continuous scalar function on S and $V \in C_0(x, y)$ on S . If V is a Liapunov function on S , we define $\dot{V}_{(2.1)}(t, x, y)$ by

$$\dot{V}_{(2.1)}(t, x, y) = \limsup_{h \rightarrow 0} \frac{1}{h} \{V[t+h, x+hF(t, x)+hG(t, x)+he(t), \\ y+hF(t, y)+hG(t, y)+he(t)] - V[t, x, y]\} ,$$

where (2.1) is the product system

$$(2.1) \quad \begin{aligned} \dot{x} &= F(t, x) + G(t, x) + e(t) , \\ \dot{y} &= F(t, y) + G(t, y) + e(t) . \end{aligned}$$

If ε is a positive number, p a point in R^n and Ω a subset of R^n , then

$$d(p, \Omega) = \text{glb} \{ |x - p| \mid x \in \Omega \}$$

and

$$N(\varepsilon, \Omega) = \{x \in R^n \mid d(x, \Omega) < \varepsilon\} .$$

If $\{P_n\}$ is a sequence of points in R^n , then $P_n \rightarrow \Omega$ as $n \rightarrow \infty$ if $\lim_{n \rightarrow \infty} d(P_n, \Omega) = 0$.

A set $S \subset R^n$ will be called a *weakly invariant set* of (1.1) if every solutions of (1.1) passing through a point of S at time $t_0 \geq 0$ remains in S for all future time.

If Ω is a closed subset of R^n , then we will say that a scalar function $W: U \times U \rightarrow R^1$ is *positive definite with respect to* Ω if $W(x, y) = 0$ when $(x, y) \in U \times U$ and $x - y \in \Omega$ and if corresponding to each $\varepsilon > 0$ there is a positive number δ such that $W(x, y) \geq \delta$ whenever $(x, y) \in U \times U$ and $x - y \notin N(\varepsilon, \Omega)$.

Let A_k , $k = 1, \dots, n$, be selected such that every solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ of (1.1), starting in a prescribed set, ultimately satisfies

$$|x_k(t)| < A_k, \quad k = 1, \dots, n.$$

We include here the possibility that some or all of the A_k 's are infinite. If we set $I_k = (-A_k, A_k)$, $J_k = (-2A_k, 2A_k)$ for A_k finite, and $I_k = J_k = R^1$ otherwise, and $Q_1 = \prod_{k=1}^n I_k$, $Q_2 = \prod_{k=1}^n J_k$ and $Q = Q_1 \times Q_2$; then for every pair of solutions $x(t)$ and $y(t)$ of (1.1), there is a positive constant T such that $x(t) - y(t) \in Q_2$ and $(x(t), y(t)) \in Q$ for $t \geq t_0 + T$.

3. - THEOREM 1. - Let $V(t, x, y)$ be a nonnegative Liapunov function defined on $I \times Q$ and assume that

$$(3.1) \quad - \dot{V}_{(2.1)}(t, x, y) \geq W(x, y)$$

where W is positive definite with respect to a closed set $\Omega \subset R^n$. If R and G satisfy (1) and (2) and if $|F(t, \theta(t)) - F(t, \varphi(t))|$ is bounded for $t \geq 0$ for every pair of solutions $\theta(t)$, $\varphi(t)$ of (1.1), whose difference is bounded for $t \in I$, then every pair of solutions $\theta(t)$, $\varphi(t)$ of (1.1), whose difference is bounded for $t \in I$, satisfies $\theta(t) - \varphi(t) \rightarrow \Omega$ as $t \rightarrow \infty$.

PROOF. — Let $\theta(t)$, $\varphi(t)$ be a pair of solutions of (1.1) whose difference is bounded for $t \in I$, then there is $B(\theta, \varphi) > 0$ such that

$$(3.2) \quad |F(t, \theta(t)) - F(t, \varphi(t))| \leq B(\theta, \varphi)$$

for $t \in I$.

Assume that $\theta(t) - \varphi(t)$ does not approach Ω as $t \rightarrow \infty$. Then there is an $\eta > 0$ and a sequence $\{t_k\}$, $t_k \rightarrow \infty$ as $k \rightarrow \infty$, such that

$$d(\theta(t_k) - \varphi(t_k), \Omega) \geq \eta$$

$k = 1, 2, \dots$. We will assume that $\eta \leq \varepsilon(K)$ and, using (2) that

$$\int_{t_1}^{\infty} |G(s, \theta(s))| ds < \frac{\eta}{16} \quad \text{and} \quad \int_{t_1}^{\infty} |G(s, \varphi(s))| ds < \frac{\eta}{16}.$$

Consider the function $d(t) = d(\theta(t) - \varphi(t), \Omega)$. We wish to show that $d(t) > \eta/2$ for $t_k \leq t \leq t_k + \eta/8B(K)$ $k = 1, 2, \dots$, so for a fixed k assume that $d(t) \leq \eta/2$ at some point in this interval.

Setting $\bar{t}_k = \text{glb}\{t \in [t_k, t_k + \eta/8B(K)] | d(t) \leq \eta/2\}$ and $t_k = \text{lub}\{t \in [t_k, \bar{t}_k] | d(t) \geq \varepsilon\}$, we have

$$(3.3) \quad \left\{ \begin{array}{l} d(\bar{t}_k) = \eta/2, \quad d(\hat{t}_k) \geq \eta, \\ \text{and} \\ \theta(t) - \varphi(t) \in \overline{N(\varepsilon(K), \Omega)}, \end{array} \right.$$

for $\hat{t}_k \leq t \leq \bar{t}_k$. On the interval $[\hat{t}_k, \bar{t}_k]$

$$\begin{aligned} |\theta(t) - \varphi(t) - [\theta(\hat{t}_k) - \varphi(\hat{t}_k)]| &= \left| \int_{\hat{t}_k}^t [\dot{\theta}(s) - \dot{\varphi}(s)] ds \right| \\ &\leq \int_{\hat{t}_k}^t |F(s, \theta(s)) - F(s, \varphi(s))| ds + \int_{\hat{t}_k}^t |G(s, \theta(s))| ds + \int_{\hat{t}_k}^t |G(s, \varphi(s))| ds \\ &\leq \eta B(K)/8B(K) + 2\eta/16 = \eta/4. \end{aligned}$$

This last inequality implies that $d(\bar{t}_k) \geq 3\eta/4$, which is in contradiction to (3.3). Hence, $d(\theta(t) - \varphi(t), \Omega) > \eta/2$ for $t_k \leq t \leq t_k + \eta/8B(K)$ $k = 1, 2, \dots$.

We can assume, by taking a subsequence of the sequence $\{t_k\}$ if necessary, that this sequence of intervals is disjoint. Now since $\theta(t) - \varphi(t) \in K_1$ for $t \geq 0$, we can find, using (3.1), a constant $\delta(\eta) > 0$ such that

$$\dot{V}_{(2.1)}(t, \theta(t), \varphi(t)) < \alpha(t)$$

for $t \geq t_0$, where

$$\alpha(t) = \begin{cases} 0 & t_0 < t < t_1 \text{ and } t_k + \eta/8B(K) < t < t_{k+1}, \\ -\delta(\eta) & t_k < t < t_k + \eta/8B(K), \quad k = 1, 2, \dots \end{cases}$$

Hence, by the comparison theorem [7, Lemma 1],

$$V(t_k, \theta(t_k), \varphi(t_k)) - V(t_0, \theta(t_0), \varphi(t_0)) < -\frac{\delta(\eta)\eta k}{8B(K)} \rightarrow -\infty$$

as $k \rightarrow \infty$. This is, of course, impossible since V is nonnegative.

The following theorem is useful in determining when the difference between pairs of solutions of (1.1) is bounded.

DEFINITION. – System (1.1) is *uniform-distance-bounded*, if for each $\alpha > 0$ there exists a $\beta(\alpha) > 0$ such that if $x(t)$ and $y(t)$ are solutions of (1.1) such that $|x(t_0) - y(t_0)| < \alpha$, then $|x(t) - y(t)| < \beta(\alpha)$ for $t \geq t_0$.

THEOREM 2 [15, Thm. 15.4]. – Suppose that there exists a Liapunov function $V(t, x, y)$ defined on $I \times Q$ satisfying the following conditions:

- (i) $a(|x - y|) \leq V(t, x, y) \leq b(|x - y|)$, where $a(r)$ and $b(r)$ are nonnegative continuous functions and $a(r) \rightarrow \infty$ as $r \rightarrow \infty$,
- (ii) $\dot{V}_{(2.1)}(t, x, y) < 0$.

Then system (1.1) is uniform-distance-bounded.

Let $\theta(t)$ and $\varphi(t)$ be an arbitrary pair of solutions of (1.1) and define

$$I_{\theta\varphi}^+ = \left\{ \gamma \in R^n \left| \begin{array}{l} \text{there is a sequence } \{t_k\}, t_k \rightarrow \infty \text{ as } k \rightarrow \infty, \\ \text{such that } \lim_{k \rightarrow \infty} [\theta(t_k) - \varphi(t_k)] = \gamma. \end{array} \right. \right\}$$

THEOREM 3. – If $\theta(t)$ and $\varphi(t)$ are a pair of solutions of (1.1) such that $\theta(t) - \varphi(t)$ is bounded by all t , and if $\theta(t) - \varphi(t)$ approaches a closed set $\Omega \subset R^n$, then $I_{\theta\varphi}^+$ is a nonempty compact connected subset of Ω .

The proof of Theorem 3 is similar to the proof of [11, Theorem 2] and will not be repeated here.

THEOREM 4. — Let $\theta(t)$ and $\varphi(t)$ be a pair of solutions of (1.1) such that $\theta(t) - \varphi(t)$ approaches a closed set $\Omega \subset Q_2$ as $t \rightarrow \infty$. If the solutions of (1.1) are uniform-bounded and if F and G satisfy (1), (2):

(3) $F(t, x)$ converges to a function $H(x)$ for $x \in \Omega$ as $t \rightarrow \infty$ and on any compact set in Ω this convergence is uniform,

(4) $F(t, x) - F(t, y) = F(t, x - y)$ whenever $(x, y) \in Q$ and $(x - y) \in \Omega$,

then $\Gamma_{\theta\varphi}^+ \subset S$ where S is a weakly invariant set contained in Ω of the equation

$$(3.4) \quad \dot{x} = H(x) \quad x \in \Omega.$$

PROOF. — Let $J = [0, T]$ where $T > 0$ is an arbitrary but fixed real number. We will show first that for each $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that if

$$(3.5) \quad \begin{aligned} |\theta(t^*) - x_0| &< \delta \quad \text{where } x_0 - y_0 = \alpha \in \Omega, \\ |\varphi(t^*) - y_0| &< \delta \end{aligned}$$

and

$$(3.6) \quad [\theta(t^* + t) - \varphi(t^* + t)] \in N(\delta, \Omega) \quad \text{for } t \in J,$$

then there is a pair of solutions $\theta^*(t)$ and $\varphi^*(t)$ of (1.1), passing through x and y respectively at $t = t^*$, for which

$$(3.7) \quad \begin{cases} |\theta(t^* + t) - \varphi(t^* + t) - [\theta^*(t^* + t) - \varphi^*(t^* + t)]| < \varepsilon \\ [\theta^*(t^* + t) - \varphi^*(t^* + t)] \in \Omega \end{cases}$$

for $t \in J$.

Suppose that this is not the case. Then there is an $\varepsilon > 0$ and a sequence of positive numbers $\{\delta_k\}$, $\delta_k \rightarrow 0$ as $k \rightarrow \infty$, and sequences of functions $\{\theta_k\}$ and $\{\varphi_k\}$, solutions of (1.1), which satisfy (3.5) and (3.6), but for which for every pair of solutions $x(t)$ and $y(t)$ of (1.1), passing through x and y respectively at $t = t^*$, either

$$(3.8) \quad \begin{cases} \text{lub}_{t \in J} \{|\theta_k(t^* + t) - \varphi_k(t^* + t) - [x(t^* + t) - y(t^* + t)]|\} \geq \varepsilon, \\ \text{or} \\ [x(t) - y(t)] \notin \Omega \quad \text{for some } t \in J. \end{cases}$$

We note that for each k , θ_k and φ_k can be written as

$$(3.9) \quad \begin{cases} \theta_k(t^* + t) = \theta_k(t^*) + \int_{t^*}^{t^*+t} F(s, \theta_k(s)) ds + \int_{t^*}^{t^*+t} G(s, \theta_k(s)) ds + \int_{t^*}^{t^*+t} e(s) ds \\ \varphi_k(t^* + t) = \varphi_k(t^*) + \int_{t^*}^{t^*+t} F(s, \varphi_k(s)) ds + \int_{t^*}^{t^*+t} G(s, \varphi_k(s)) ds + \int_{t^*}^{t^*+t} e(s) ds \end{cases}$$

for $t \geq 0$.

Since the solutions of (1.1) are uniform-bounded, there is $B_1 > 0$ such that if the solution $z(t; z_0, t_0)$ of (1.1) satisfies $|z_0| \leq \max(|x_0| + 1, |y_0| + 1)$ then $|z(t; z_0, t_0)| \leq B_1$ for $t \geq t_0$. Thus, with $\delta < 1$, we have

$$(3.10) \quad |\theta_k(t^* + t)| \leq B_1 \quad \text{and} \quad |\varphi_k(t^* + t)| \leq B_1$$

for $t \in J$ and $k = 1, 2, \dots$.

It now follows from (1), (2), (3.9) and (3.10) that the sequences $\{\theta_k\}$ and $\{\varphi_k\}$ are equi-continuous and uniformly bounded on $t^* \leq t \leq t^* + T$. From Ascoli's Theorem, there is a uniformly convergent subsequence $\{\theta_{k_i}\}$ of $\{\theta_k\}$ and a uniformly convergent subsequence $\{\varphi_{k_i}\}$ of $\{\varphi_k\}$. Now the subsequences $\{\theta_{k_{i_j}}\}$ and $\{\varphi_{k_{i_j}}\}$ are uniformly convergent subsequences of $\{\theta_k\}$ and $\{\varphi_k\}$ respectively.

If $\hat{\theta}(t)$ and $\hat{\varphi}(t)$ are the limit functions of these subsequences, then it is clear from (3.5) and (3.9) that $\hat{\theta}$ and $\hat{\varphi}$ are solutions of (1.1) satisfying $\hat{\theta}(t^*) = x_0$ and $\hat{\varphi}(t^*) = y_0$ and from (3.6) that

$$(3.11) \quad \hat{\theta}(t^* + t) - \hat{\varphi}(t^* + t) \in \Omega \quad \text{for } t \in J.$$

Since $\hat{\theta}$ and $\hat{\varphi}$ are the uniform limits of the sequences $\{\theta_{k_{i_j}}\}$ and $\{\varphi_{k_{i_j}}\}$, there is a positive integer J_0 , such that for $j \geq J_0$,

$$|\theta_{k_{i_j}}(t^* + t) - \hat{\theta}(t^* + t)| < \varepsilon/2$$

and

$$|\varphi_{k_{i_j}}(t^* + t) - \hat{\varphi}(t^* + t)| < \varepsilon/2$$

for $t \in J$. But this implies that if $j > J_0$, then

$$(3.12) \quad |\theta_{k_{i_j}}(t^* + t) - \varphi_{k_{i_j}}(t^* + t) - [\hat{\theta}(t^* + t) - \hat{\varphi}(t^* + t)]| < \varepsilon$$

for $t \in J$. Now combining (3.11) and (3.12), we have arrived at a contradiction of (3.8).

For a given $\varepsilon > 0$, let $\delta(\varepsilon) > 0$ be such that if (3.5) and (3.6) are satisfied, then there is a pair of functions, θ^* and φ^* , such that (3.7) is satisfied. Since θ^* and φ^*

are solutions of (1.1) and $\theta^*(t^* + t) - \varphi^*(t^* + t) \in \Omega$ for $t \in J$, it follows from (4) that

$$\begin{aligned} \dot{\theta}(t^* + t) - \dot{\varphi}(t^* + t) &= F(t^* + t, \theta^*(t^* + t) - \varphi^*(t^* + t)) \\ &\quad + (G(t^* + t, \theta^*(t^* + t)) - G(t^* + t, \varphi^*(t^* + t))) \quad \text{for } t \in J. \end{aligned}$$

Setting $x(t) = \theta^*(t) - \varphi^*(t)$, we have

$$\begin{aligned} \dot{x}(t^* + t) &= F(t^* + t, x(t^* + t)) + G(t^* + t, \theta^*(t^* + t)) \\ &\quad - G(t^* + t, \varphi^*(t^* + t)) \quad \text{and} \quad x(t^* + t) \in \Omega \quad \text{for } t \in J. \end{aligned}$$

Since the solutions of (1.1) are uniform-bounded, we can find a compact set $K \subset Q$ such that $x(t) \in K$ for $t \geq 0$. Thus, as a result of (1), there is a positive constant B such that $|F(t, x(t))| \leq B$ for $t \geq 0$. Now since $F(t, x)$ converges uniformly to $H(x)$ on $K \cap \Omega$, it follows that $H(x)$ is continuous and bounded on $K \cap \Omega$, and thus, that there is a continuous bounded function $H^*(x)$, defined on R^n , such that $H^*(x) = H(x)$ for $x \in K \cap \Omega$.

Consider now the systems

$$(3.13) \quad \dot{y} = H^*(y)$$

and

$$(3.14) \quad \dot{y} = H^*(y) + \gamma(t),$$

where

$$\gamma(t) = F(t^* + t, x(t^* + t)) - H^*(x(t^* + t)) + G(t^* + t, \theta^*(t^* + t)) - G(t^* + t, \varphi^*(t^* + t)).$$

We note that $x(t^* + t)$ is a solution of (3.14) over the interval J .

Since $x(t^* + t) \in K \cap \Omega$ for $t \in J$, and since $H^*(x) = H(x)$ on Ω , it follows from (3) that for each $\varepsilon > 0$, there is $T_1(\varepsilon) > 0$ such that

$$(3.15) \quad |F(t^* + t, x(t^* + t)) - H(x(t^* + t))| < \frac{\varepsilon}{2T}$$

for $t \in J$ if $t^* \geq T_1(\varepsilon)$.

Using (2) we can select $T_2(\varepsilon) > 0$ such that

$$(3.16) \quad \int_0^t |G(t^* + s, \theta^*(t^* + s)) - G(t^* + s, \varphi^*(t^* + s))| ds < \varepsilon/2$$

for $t \in J$ if $t^* \geq T_2(\varepsilon)$.

Setting $t^* = \max(T_1, T_2)$ and using (3.15) and (3.16) we have

$$(3.17) \quad \int_0^t |\gamma(s)| ds < \varepsilon$$

for $t \in J$.

Writing $H^*(y) = [H^*(y) + \gamma(t)] - \gamma(t)$, it follows from (3.17) and [15, Theorem 5.1], that every $\eta > 0$ there is a solution $y(t)$ of (3.13) such that

$$(3.18) \quad \begin{cases} y(0) = \gamma \\ |y(t) - x(t^* + t)| < \eta \end{cases}$$

for $t \in J$, if t^* is sufficiently large.

For each k , $k = 1, 2, \dots$, corresponding to $\varepsilon_k = 1/2k$ let δ_k be a positive constant such that if (3.5) and (3.6) are satisfied, then there are functions satisfying (3.7).

Now let $\gamma \in I_{\theta\varphi}^+$, then there is a sequence $\{t_k\}$, $t_k \rightarrow \infty$ as $k \rightarrow \infty$, such that

$$(3.19) \quad |\theta(t_k) - \varphi(t_k) - \gamma| < \delta_k$$

$k = 1, 2, \dots$

Since $\theta(t) - \varphi(t) \rightarrow \Omega$ as $t \rightarrow \infty$, we can also assume, by taking a subsequence of the sequence $\{t_k\}$ if necessary, that

$$(3.20) \quad d(\theta(t_k + t) - \varphi(t_k + t), \Omega) < \delta_k$$

for $t \in J$, $k = 1, 2, \dots$:

It now follows from (3.5)-(3.7), (3.19) and (3.20) that for $k = 1, 2, \dots$, there are solutions $\theta_k(t)$ and $\varphi_k(t)$ of (1.1) satisfying

$$(3.21) \quad \begin{cases} \theta_k(t_k) - \varphi_k(t_k) = \gamma, \\ |\theta(t_k + t) - \varphi(t_k + t) - [\theta_k(t_k + t) - \varphi_k(t_k + t)]| < \frac{1}{2k}, \\ [\theta_k(t_k + t) - \varphi_k(t_k + t)] \in \Omega, \end{cases}$$

for $t \in J$. While from (3.18) we can find a subsequence $\{t_{k_i}\}$ of the sequence $\{t_k\}$ and solutions $y_i(t)$ of (3.13), $i = 1, 2, \dots$, satisfying

$$(3.22) \quad \begin{cases} y_i(0) = \gamma, \\ |\theta_{k_i}(t_{k_i} + t) - \varphi_{k_i}(t_{k_i} + t) - y_i(t)| < \frac{1}{2i}, \end{cases}$$

for $t \in J$.

Combining (3.21) and (3.22) we have

$$(3.23) \quad d(y_i(t), \Omega) < \frac{1}{i}$$

for $t \in J$, $i = 1, 2, \dots$:

Since each $y_i(t)$ satisfies

$$(3.24) \quad y_i(t) = \gamma + \int_0^t H^*(y_i(s)) ds,$$

and since H^* is continuous and bounded on R^n , it follows that the sequence of functions $\{y_i\}$ is equicontinuous and uniformly bounded on J . Ascoli's theorem now implies the existence of a uniformly convergent subsequence of the sequence $\{y_i\}$ on J . If $y(t)$ is the limit of this subsequence, then it is clear from (3.22)-(3.24) that $y(t)$ is a solution of (3.13), passing through γ at $t = 0$, and remaining in Ω for $t \in J$.

Since $y(t) \in \Omega$ for $t \in J$ and $H^*(y) = H(y)$ on Ω , $y(t) = \gamma + \int_0^t H(y(s)) ds$ for $t \in J$, and thus it follows that $y(t)$ is a solution of (3.4) on J .

Finally, since T is arbitrary, it is clear that there is a solution of (3.4) passing through γ and remaining in Ω for $t \geq 0$; i.e., $\Gamma_{\theta\varphi}^+ \subset S$ where S is a weakly invariant set of (3.4) contained in Ω .

Our next objective is to remove the requirement that solutions of (1.1) be bounded. To do this it is necessary to require that F also satisfy:

- (5) Corresponding to each $\varepsilon > 0$ and each compact set $K \subset \Omega \cap Q_2$ there exists $\delta(\varepsilon, K) > 0$ and $T(\varepsilon, K) > 0$ such that if $(x, y) \in Q$, $|x - y - z| < \delta(\varepsilon, K)$, $z \in K$, and $t \geq T(\varepsilon, K)$, then $|F(t, x) - F(t, y) - F(t, z)| < \varepsilon$.

Property (5) is a modified form of a hypothesis used by YOSHIZAWA in [15, Theorem 14.1], « condition (b) », where he discusses the asymptotic behavior of solutions of (1.1). It is interesting to note that this condition can be dropped in [15, Theorem 14.1] if one replaces the requirement that solutions of (1.1) be bounded by the requirement that solutions be uniform-bounded. The approach used in the proof of Theorem 4 can be used to establish this result.

THEOREM 5. — Let $\theta(t)$, $\varphi(t)$ be a pair of solutions of (1.1) such that $\theta(t) - \varphi(t)$ is bounded, $(\theta(t) - \varphi(t)) \in Q_2$ for $t \geq t_0$ and $\theta(t) - \varphi(t)$ approaches a closed set Ω as $t \rightarrow \infty$. If F and G satisfy (1)-(5), then $\Gamma_{\theta\varphi} \subset S$ where S is a weakly invariant set contained in Ω of the equation

$$\dot{x} = H(x).$$

PROOF. — Since the method of the proof is similar to the proof of [15, Theorem 13.1], we only sketch the details here.

Since $\psi(t) = \theta(t) - \varphi(t)$ is bounded for $t \geq t_0$ we can select a compact set $Q_2^* \subset Q_2$ such that $\psi(t) \in Q_2^*$ for $t \geq t_0$, and thus $\Gamma_{\theta\varphi} \subset \Omega \cap Q_2^* = \Omega_1$. We can also select a continuous bounded function $H^*(x)$ on R^n such that $H^*(x) = H(x)$ on Ω_1 . For $\gamma \in \Gamma_{\theta\varphi}$ there is a sequence $\{t_k\}$ such that $\psi(t_k) \rightarrow \gamma$, $t_k \rightarrow \infty$ as $k \rightarrow \infty$.

Consider the systems

$$(3.25) \quad \frac{dy}{dt} = H^*(y),$$

$$(3.26) \quad \frac{dy}{dt} = H^*(y) + \eta(t),$$

where

$$\begin{aligned} \eta(t) = & F(t_k + t, \theta(t_k + t)) - F(t_k + t, \varphi(t_k + t)) - H^*(\psi(t_k + t)) \\ & + G(t_k + t, \theta(t_k + t)) - G(t_k + t, \varphi(t_k + t)) . \end{aligned}$$

It is clear that $\psi(t_k + t)$ is a solution of (3.26) for $t \geq 0$ and since $H^*(y)$ is bounded, all solutions of (3.25) are defined for $t \geq 0$.

Let $\varepsilon > 0$ and $T > 0$ be given. It follows from [15, Theorem 5.1] that there is $\delta > 0$ such that if $|\psi(t_k) - \gamma| < \delta$ and $\int_0^T |\eta(t)| dt < \delta$ then there exists a solution $z(t)$ of (3.25) such that $|\psi(t_k + t) - z(t)| < \varepsilon$ for $0 \leq t \leq T$.

Using (2), we can select $k_1 \geq 0$ such that

$$(3.27) \quad \int_0^\infty |G(t_k(t_k + t, \theta(t_k + t)) - G(t_k + t, \varphi(t_k + t))| dt < \frac{\delta}{2}$$

if $k \geq k_1$, and since $\psi(t_k) \rightarrow \gamma$, there is $k_2 \geq k_1$ such that

$$(3.28) \quad |\psi(t_k) - \gamma| < \delta$$

for $k \geq k_2$. Using (1), (5), the continuity of $H^*(y)$ and the assumption that $d(\psi(t), \Omega) \rightarrow 0$ as $t \rightarrow \infty$, we can select $k_3 \geq k_2$ such that

$$(3.29) \quad |F(t_k + t, \theta(t_k + t)) - F(t_k + t, \varphi(t_k + t)) - H^*(\psi(t_k + t))| < \frac{\delta}{2T}$$

for $k \geq k_3$.

Combining (3.27)-(3.29), we conclude that there is a solution $z(t)$ of (3.25) such that $z(0) = \gamma$ and $|z(t) - \psi(t_k + t)| < \varepsilon$ for $0 \leq t \leq T$. Let $\{\varepsilon_k\}$ be a sequence of positive numbers such that $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$ and let $\{y_p(t)\}$ be a sequence of solutions of (3.25) such that

$$y_p(t) = \gamma + \int_0^t H^*(y_p(s)) ds \quad \text{and} \quad d(y_p(t), I_{\theta\varphi}^+) < \varepsilon_k$$

for $0 \leq t \leq T$, $k = 1, 2, 3, \dots$. Let $y(t)$ be the uniform limit of the sequence $\{y_p(t)\}$ on $[0, T]$, then $y(t)$ is a solution of (1.1) satisfying $y(0) = \gamma$ and $y(t) \in I_{\theta\varphi}^+$ $0 \leq t \leq T$. Since $T > 0$ is arbitrary, the proof is completed.

THEOREMS 1-5 can be summerized in the following theorems.

THEOREM 6. – Suppose that the solutions of (1.1) are uniform-bounded and that there is a nonnegative Liapunov function $V(t, x, y)$ defined on $I \times Q$ such that $-\dot{V}_{(2.1)}(t, x, y) \geq W(x, y)$ where W is positive definite with respect to a closed set

$\Omega \subset R^n$. If F and G satisfy conditions (1)-(4), then every pair of solutions $\theta(t)$, $\varphi(t)$ of equation (1.1) satisfy

$$\theta(t) - \varphi(t) \rightarrow S \quad \text{as} \quad t \rightarrow \infty$$

where S is the largest weakly invariant set contained in Ω of the equation

$$\dot{x} = H(x) \quad x \in \Omega.$$

THEOREM 7. - If, in Theorem 6, the solutions of (1.1) are « uniform-distance bounded » rather than « uniform-bounded » and if F also satisfies condition (5), then the conclusion of Theorem 6 remains valid.

4. - For the applications of Theorems 6 or 7, one usually looks for a function $V(t, x, y)$ whose derivative, $\dot{V}_{(2.1)}(t, x, y)$, is zero, or possibly negative for some values of t , on a linear subspace Ω of R^n and negative on $R^n - \Omega$. If Ω is a linear subspace, then conditions (4) implies that $F(t, x)$ is linear in x on Ω .

We consider first the equation

$$(4.1) \quad \ddot{x} + a(t)f(\dot{x}) + b(t)x = e(t),$$

where a , b , e and f are continuous real valued functions of a real variable. Eq. (4.1) is equivalent to the system

$$(4.2) \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= -a(t)f(y) - b(t)x + e(t) \end{aligned}$$

and we will also need the product system

$$(4.3) \quad \begin{aligned} \dot{x} &= y & \dot{u} &= v \\ \dot{y} &= -a(t)f(y) - b(t)x + e(t) & \dot{v} &= -a(t)f(v) - b(t)u + e(t). \end{aligned}$$

THEOREM 8. - Suppose that there is a solution $x(t)$ of (4.1) which exists for $t \geq 0$ and for which $\dot{x}(t)$ is bounded for $t \geq 0$. If $f(y)$ is strictly monotonic increasing for all y , $f(0) = 0$ and $0 < a_1 \leq a(t) \leq a_2$, $0 < b_1 \leq b(t) \leq b_2$ and either $b'(t) \leq 0$ or $b'(t) > 0$; then the solutions of (4.1) converge.

PROOF. - We consider first the case $b'(t) \leq 0$. If we select $V = b(t)X^2 + Y^2$, where $X = x - u$, $Y = y - v$, then $\dot{V}_{(4.3)} = b'(t)X^2 - 2a(t)[f(y) - f(v)]Y \leq -F(y, v)$ where $F(y, v) = +2a_1[f(y) - f(v)]Y$. Since $F(y, v) \geq 0$, it is clear that system (4.2) is uniform distance bounded. Combining this with the existence of a solution of (4.1) which exists for $t \geq 0$, it follows that all solutions of (4.2) exist for $t \geq 0$. Since (4.1)

has a solution with a bounded derivative, it follows from the uniform distance boundedness that for every $\alpha > 0$ there is $B(\alpha) > 0$ such that every solution $(x(t), y(t))$ of (4.2), satisfying $x^2(t_0) + y^2(t_0) \leq \alpha^2$, satisfies $|y(t)| \leq B(\alpha)$ $t \geq t_0$:

For a fixed $\alpha > 0$ we will apply Theorem 7 with $Q_1 = R \times [-B(\alpha), B(\alpha)]$. Since $-\dot{V} \geq F(y, v)$ and $F(y, v) = 0$ for $y = v$, we select $\Omega = \{(x, y) \in R^2: y = 0\}$. Since $f(y)$ is strictly increasing, $F(y, v) > 0$ for $(y, v) \in S_1 = [-B(\alpha), B(\alpha)] \times [-B(\alpha), B(\alpha)]$ and $y \neq v$. For $\delta > 0$, the set $S_2 = S_1 - \{(y, v) \in S_1: |y - v| < \delta\}$ is compact, and since F is continuous on S_2 , there is $\varepsilon(\delta) > 0$ such that $F(y, v) \geq \varepsilon(\delta)$ on S_2 . Setting $W = F(y, v)$, W is positive definite with respect to Ω . On Ω , (4.2) becomes, dropping $e(t)$,

$$(4.4) \quad \begin{aligned} \dot{x} &= 0 \\ \dot{y} &= -b(t)x \end{aligned}$$

and since $b'(t) \leq 0$ and $b(t) \geq b_1$, there is $b_0 \geq b_1$ such that $b(t) \rightarrow b_0$ as $t \rightarrow \infty$. Setting

$$H(x, y) = \begin{pmatrix} 0 \\ -b_0 x \end{pmatrix},$$

it is easily seen that (1)-(4) are satisfied and since f is strictly increasing, $|f(y) - f(v)| \leq f(B(\alpha)) - f(-B(\alpha))$ on S_1 , and, using the upper bounds for a and b , it follows that (5) will be satisfied by taking $\delta = (\varepsilon/3) \min\{1, 1/b_2, 1/a_2(f(B(\alpha)) - f(-B(\alpha)))\}$. Thus the solutions of (4.2), starting in R^2_α , converge if the largest weakly invariant set of (4.4), contained in Ω , is the origin: and this is clearly the case.

Since α is an arbitrary positive number, it follows that all solutions of (4.2) converge.

When $b'(t) \geq 0$, we select $V = X^2 + (1/b(t)) Y^2$, then

$$\dot{V}_{(4.3)} = (-2a(t)/b(t)) [f(y) - f(v)] Y - (b'(t)/b^2(t)) Y^2 \leq (-2a_1/b_2) [f(y) - f(v)] Y,$$

and the remainder of the proof follows as in the first case.

COROLLARY. — If $b > 0$, $f(0) = 0$ and $f(y)$ is strictly monotonic increasing for all y , then every solution, $x(t)$, of

$$(4.5) \quad \ddot{x} + f(\dot{x}) + bx = ct + d$$

satisfies $x(t) - [ct/b - (f(c/b)/b + d/b)] \rightarrow 0$ and $\dot{x}(t) \rightarrow c/b$ as $t \rightarrow \infty$.

PROOF. — Since $\varphi(t) = ct/b - f(c/b)/b + d/b$ is a solution of (4.5), the result follows from Theorem 8.

YOSHIZAWA [15] showed that the solutions of

$$(4.6) \quad \ddot{x} + \dot{x}^3 + x = t + 1$$

are uniform distance bounded, and as a result, since $x = t$ is a solution, all solutions become unbounded as $t \rightarrow \infty$. The preceding corollary shows that if $x(t)$ is any solution of (4.6) then $x(t) - t - 1 \rightarrow 0$ and $\dot{x}(t) \rightarrow 1$ as $t \rightarrow \infty$.

The requirement in Theorem 8 that (4.1) have a solution with a bounded derivative can be removed if it is further assumed that f' is bounded above and below by positive constants.

THEOREM 9. - If f and b are differentiable, $f(0) = 0$ and

$$(i) \quad 0 < f_1 \leq f'(y) \leq f_2 \quad y \in R^1,$$

$$(ii) \quad 0 < a_1 \leq a(t) \leq a_2 \quad t \geq 0,$$

$$(iii) \quad 0 < b_1 \leq b(t) \leq b_2 \quad t \geq 0,$$

$$(iv) \quad \text{Either } b'(t) \leq 0 \text{ or } b'(t) \geq 0 \quad t \geq 0, \text{ then the solutions of (4.1) converge.}$$

PROOF. - Using the functions V, F, H and W and the set Ω defined in the proof of Theorem 8, it must be shown that (5) is satisfied and that W is positive definite with respect to Ω .

Since f is differentiable it follows from (i) and the mean value theorem that (5) will be satisfied by taking $\delta = (\varepsilon/3) \min [1, 1/b_2, 1/a_2 f_2]$. The function W will be positive definite with respect to Ω if $\inf \{[f(y) - f(v)][y - v] : |y - v| > \delta\} > 0$ for every $\delta > 0$, and this again follows from (i) and the mean value theorem. The remainder of the proof is identical to the proof of Theorem 8.

If we assume a, b and e are periodic of period $\omega > 0$, then the preceding results can be used to establish the following characterization of solutions of (4.1).

THEOREM 10. - Assume that a and e are periodic of period $\omega > 0$, $0 < a_1 \leq a(t) \leq a_2$, $b(t) = b_0 > 0$ and let $I_1 = \min e(t)/a(t)$, $I_2 = \max e(t)/a(t)$ $0 \leq t \leq \omega$. Assume further that $f(0) = 0$, f is strictly monotonic increasing and that there is $K > 0$ such that $f(K) > -I_1$ and $f(-K) < -I_2$. Then (4.1) has a periodic solution to which all other solutions converge as $t \rightarrow \infty$.

PROOF. - We show first that solutions of (4.1) are uniform bounded. Since $f(K) > -I_1$, $f(-K) < -I_2$ there is $K_1 > 0$ such that $f(K) + e(t)/a(t) - K/2a(t) K_1 \geq 0$, $f(-K) + e(t)/a(t) + K/2a(t) K_1 \leq 0$ $0 \leq t \leq \omega$. Let

$$K_2 = (1/b) \max [(2K + 1)|e(t)| + a_2 f(K) - a_2 f(-K)] \quad 0 \leq t \leq \omega$$

and set $K = \max (K_1, K_2)$. Define $V = V(x, y)$ by

$$V = \begin{cases} bx^2 + y^2 + (\operatorname{sgn} xy) K & |x| \geq K, |y| \geq K \\ bx^2 + y^2 + (\operatorname{sgn} x) y & |x| \geq K, |y| < K \\ bx^2 + y^2 + (\operatorname{sgn} y) Kx/K & |x| < K, |y| \geq K. \end{cases}$$

Clearly there is $R > 0$ and $c > 0$ such that $V \geq c(x^2 + y^2)$ for $x^2 + y^2 \geq R$.

For $|x| \geq K$, $|y| \geq K$, $-\dot{V} = 2a(t)[f(y) + e(t)/a(t)]y > 0$. For $|x| \geq K$, $|y| < K$,

$$\begin{aligned} -\dot{V} &= 2a(t)[f(y) + e(t)/a(t)]y + (\operatorname{sgn} x)a(t)f(y) + (\operatorname{sgn} x)bx - (\operatorname{sgn} x)e(t) \\ &\geq b|x| - a(t)|f(y)| - 2|e(t)y| - |e(t)| \\ &\geq 0 \end{aligned}$$

since $|x| \geq K_0$. For $|x| < K_0$, $|y| \geq K$,

$$-\dot{V} = 2a(t)[f(y) + e(t)/a(t) - (\operatorname{sgn} y)K/2a(t)K_0]y \geq 0$$

since $|y| \geq K$. Summarizing, we have $\dot{V} \leq 0$ where V is defined. It follows that solutions of (4.1) are uniform bounded and thus that there exists a periodic solution of (4.1). The convergence of all other solutions of (4.1) to this periodic solution follows from Theorem 8.

Consider next the equation

$$(4.7) \quad \ddot{x} + g(x)\dot{x} + b(t)h(x) = e(t).$$

The question of convergence of solutions of (4.7) was studied first by LEVINSON [8] for the case $b(t) = 1$ and $h(x) = x$, later by CARTWRIGHT and LITTLEWOOD [1], REUTER [10], WALTMAN and BRIDGLAND [13] and YOSHIZAWA [14] under the assumption that $b(t) = 1$.

We will examine here the system

$$(4.8) \quad \begin{aligned} \dot{x} &= y - a(t)G(x) \\ \dot{y} &= -b(t)h(x) + e(t), \end{aligned}$$

where $G(x) = \int_0^x g(s)ds$, a , b , e , g and h are continuous functions of a real variable.

If $a(t) \equiv 1$ then (4.8) is equivalent to (4.7). We will require first that the solutions of (4.8) be ultimately bounded, as was required in each of the results [1], [8], [10] and [14] and these results will be extended to (4.8) without further restrictions on g , h or e .

THEOREM 11. — Assume that $h(0) = 0$ and that there are positive constants A and B such that every solution $(x(t), y(t))$ of (4.8) ultimately satisfies $|x(t)| \leq A$ and $|\dot{x}(t)| \leq B$. Assume that, for $|x| \leq A$,

- (i) $0 < a_1 \leq g(x) \leq a_2$;
- (ii) $0 < a_3 \leq h'(x) \leq a_4$;
- (iii) $h''(x)$ exists and $|h''(x)| \leq \gamma(A)$; and that for $t \geq 0$;

$$(iv) \ 0 < a(t) \leq a_6 \text{ and } \inf_{t \geq 0} a(t) > \frac{\gamma(A)B}{a_1 a_2};$$

$$(v) \ 0 < a_6 \leq b(t) \leq a_7, \ b'(t) \text{ exists and either } b'(t) < 0 \text{ or } b'(t) > 0 \text{ for } t > 0,$$

then the solutions of (4.8) converge.

PROOF. — We will first assume that $b'(t) \leq 0$ for $t \geq 0$ and consider the function $V = V(t, x, y, u, v)$ defined by $V = b(t)\varphi(x, u)X^2 + Y^2$ where $X = x - u$, $Y = y - v$ and

$$\varphi(x, u) = \begin{cases} \frac{h(x) - h(u)}{x - u} & x \neq u, \\ h'(x) & x = u. \end{cases}$$

To evaluate $\dot{V}$ we need the product system

$$(4.9) \quad \begin{aligned} \dot{x} &= y - a(t)G(x) & \dot{u} &= v - a(t)G(u) \\ \dot{y} &= -b(t)h(x) + e(t) & \dot{v} &= -b(t)h(u) + e(t). \end{aligned}$$

Along solutions of (4.9) we have

$$\begin{aligned} \dot{V}_{(4.9)} &= b(t)\Phi X^2 - 2a(t)b(t)\varphi(x, u)\psi(x, u)X^2 + b'(t)\varphi(x, u)X^2 \\ &\leq -b(t)[2a(t)\varphi(x, u)\psi(x, u) - \Phi]X^2, \end{aligned}$$

where

$$\Phi = \frac{\partial \varphi}{\partial x} \dot{x} + \frac{\partial \varphi}{\partial u} \dot{u}, \quad \psi(x, u) = \begin{cases} \frac{G(x) - G(u)}{x - u} & x \neq u, \\ g(x) & x = u. \end{cases}$$

Since $|\dot{x} - \dot{u}| \leq 2B$ and $|h''(x)| > \gamma(A)$, it follows that $|\Phi| \leq 2B\gamma(A)$, and, setting $k = \inf_{t \geq 0} a(t)$, that $2a(t)\varphi\psi - \Phi \geq 2ka_1a_2 - 2B\gamma(A)$. This last quantity, call it M , is positive since $\inf_{t \geq 0} a(t) > \gamma(A)B/a_1a_2$. It follows that $-\dot{V} \geq W(x, y, u, v) \equiv a_6MX^2$ is positive definite with respect to the set $\Omega = \{(x, y) \in R^2: x = 0\}$.

It is clear from conditions (1)-(4) that the function $\begin{pmatrix} y - a(t)G(x) \\ -b(t)h(x) \end{pmatrix}$ satisfies (1)-(4).

On the set Ω , (4.8), excluding $e(t)$, takes the form

$$(4.10) \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= 0, \end{aligned}$$

and it is clear that the largest weakly invariant set contained in Ω of (4.10) is the origin. It follows from Theorem 6 that every pair of solutions $\theta_1(t), \theta_2(t)$ of (4.8) satisfy $\theta_1(t) - \theta_2(t) \rightarrow 0$ as $t \rightarrow \infty$.

For the case $b'(t) \geq 0$ we consider the function $U = X^2 + (1/b(t)) Y^2$, and find

$$\begin{aligned} -\dot{U}_{(4.9)} &= [2a(t)\varphi\psi - \Phi] X^2 + (b'(t)/b^2(t)) Y^2 \\ &\geq [2a(t)\varphi\psi - \Phi] X^2, \end{aligned}$$

where X , Y , φ , ψ and Φ are defined as in the first part of the proof. The same argument as used in the first case can now be used to complete the proof.

If conditions (i)-(iii) of Theorem 11 are satisfied for all x and if every solution $(x(t), y(t))$ of (4.8) exists and $\dot{x}(t)$ is bounded for $t \geq 0$, then the boundedness restriction can be removed from $x(t)$ and $y(t)$.

THEOREM 12. - Assume that $h(0) = 0$ and that there is a positive constant B such that every solution $(x(t), y(t))$ of (4.8) exists for $t \geq 0$ and ultimately satisfies $|\dot{x}(t)| \leq B$. If (i)-(iii) of Theorem 11 are satisfied for all x and (iv), (v) are satisfied, then the solutions of (4.8) converge.

PROOF. - Following the proof of the preceding theorem, we need only establish condition (5) to be able to apply Theorem 7. If $(w, z) \in \Omega$ then $w = 0$ and

$$F(t, x, y) - F(t, u, v) - F(t, w, z) = \begin{pmatrix} y - v - z - a(t)(G(x) - G(u)) \\ -b(t)(h(x) - h(u)) \end{pmatrix}.$$

Condition (5) now follows from (i), (ii) and the mean value theorem and the result follows from Theorem 7.

In [13], convergence of solutions of (4.8), with $b(t) \equiv 1$, was obtained with no restriction on the boundedness of solutions. In the following we will show that the technique used there can be viewed as a Liapunov function of the type discussed in this paper. It will be shown, in fact, that the results of [13] can be extended to (4.8) without further restrictions on g , h or e .

THEOREM 13. - Suppose $h \in C^1(R^1)$ and that solutions of (4.8) exist for $t \geq t_0$. If there exist positive numbers A , B , ε , $A > B^2$ such that

$$(4.11) \quad [Ba(t)g(x) - b(t)h'(x) - A]^2 \leq 4(A - B^2)b(t)h'(x) - \varepsilon$$

for $t \geq t$ and all x , then the solutions of (4.8) converge.

PROOF. - If we define $V = AX^2 - 2BXY + Y^2$ where $X = x - u$, $Y = y - v$, then, since $A > B^2$, there are positive constants a_1 , a_2 such that

$$(4.12) \quad a_1(X^2 + Y^2) \leq V \leq a_2(X^2 + Y^2).$$

Along solutions of (4.9)

$$\begin{aligned} -\dot{V}_{(4.9)} &= 2[Aa(t)\Phi - Bb(t)\Psi]X^2 + 2BY^2 + [b(t)\Psi - Ba(t)\Phi - A]XY \\ &= (X, Y)C(X, Y)^T \end{aligned}$$

where

$$\Phi = \begin{cases} \frac{G(x) - G(u)}{x - u} & x \neq u, \\ g(x) & x = u, \end{cases} \quad \Psi = \begin{cases} \frac{h(x) - h(u)}{x - u} & x \neq u, \\ h'(x) & x = u, \end{cases}$$

and

$$C = \begin{pmatrix} 2(Aa\Phi - Bb\Psi) & b\Psi - Ba\Phi - A \\ b\Psi - Ba\Phi - A & 2B \end{pmatrix}.$$

Noting that $\Phi = \int_0^1 g(sX + u) ds$ and $\Psi = \int_0^1 h'(sX + u) ds$, we can also write $-\dot{V} = \int_0^1 (X, Y)C(s)(X, Y)^T ds$ where

$$C(s) = \begin{pmatrix} 2[Aag(sX + u) - Bbh'(sX + u)] & bh'(sX + u) - Bag(sX + u) - A \\ bh'(sX + u) - Bag(sX + u) - A & 2B \end{pmatrix}$$

and V will be negative definite if the eigenvalues of C are bounded below by a positive constant for $0 \leq s \leq 1$, $t \geq t$ and all x, u . This will be the case if there exist positive constants γ, η such that $Aag - Bbh' \geq \eta$ and $H = 4B(Aag - Bbh') - (bh' - Bag - A)^2 \geq \gamma$ for $t \geq t$ and all x . Using (4.11) we have, writing H as $H = 4(A - B^2)bh' - (bh' - Bag - A)^2 \geq \varepsilon$, and as a result $Aag - Bbh' \geq [(bh' - Bag - A)^2 + \varepsilon]/4B \geq \varepsilon/4B$. It follows that there is a positive constant a_3 such that

$$(4.13) \quad \dot{V} \leq -a_3(X^2 + Y^2)$$

and the result follows from (4.12) and (4.13).

We consider next the equation

$$(4.14) \quad \ddot{x} + a(t)\dot{x} + b(t)h(x) = e(t).$$

For $a(t) \equiv a$ and $b(t) \equiv 1$, LOUD [9] showed that if $a > 0$, $0 < h \leq h'(x)$ and every solution of (4.14) ultimately satisfies $|x(t)| \leq A$, then the solutions of (4.14) converge provided $\sup h'(x) < a_0^2/2$ $|x| \leq A$. EZEILO [2] showed that the right hand side of this inequality can be replaced by a_0^2 .

We extend this result to the nonautonomous equation (4.14) and show, further more, that the boundedness required in [9] and [2] is not necessary if the inequality for h' is satisfied for all x .

It is assumed that a, b, e and h are continuous and that every solution defined at $t = t_0$ exists for $t \geq t_0$. For each M , either a positive constant or $+\infty$, let $G(M)$

be a subset of R^2 such that if $x(t)$ is a solution of (4.14) satisfying $(x(t), \dot{x}(t)) \in G(M)$, then there is $T \geq 0$ such that $|x(t)| \leq M$ for $t \geq T + t$.

THEOREM 14. — Suppose that a and h are differentiable, $0 < a_1 \leq a(t) \leq a_2$, $0 < b_1 \leq b(t) \leq b_2$ and $2a'(t) < a^2(t)$ $t \geq 0$. If

$$(4.15) \quad \sup_{t \geq 0} \frac{a^2(t) + a'(t) - a(t) \sqrt{a^2(t) - 2a'(t)}}{2b(t)} < \inf_{-M < x < M} h'(x)$$

and

$$(4.16) \quad \sup_{-M < x < M} h'(x) < \inf_{t \geq 0} \frac{a^2(t) + a'(t) + a(t) \sqrt{a^2(t) - 2a'(t)}}{2b(t)}$$

then the solutions of (4.13), originating in $G(M)$, converge.

NOTE. — It is easily seen that $\sup [a^2(t) + a'(t) - a(t) \sqrt{a^2(t) - 2a'(t)}] / 2b(t) \geq 0$ and is equal to 0 if and only if $a'(t) \leq 0$ for $t \geq 0$. If $a(t) \equiv a_0$ and $b(t) \equiv 1$, then the left hand side of the inequality in (4.15) is 0 and the right hand side of the inequality in (4.16) is a_0^2 and the hypotheses of Theorem 14 reduce to those required by EZEILO [2].

PROOF. — Eq. (4.14) is equivalent to the system

$$(4.17) \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= -a(t)y - b(t)h(x) + e(t), \end{aligned}$$

and we need also the product system

$$(4.18) \quad \begin{aligned} \dot{x} &= y & \dot{u} &= v \\ \dot{y} &= H a(t)y - b(t)h(x) + e(t) & \dot{v} &= -a(t)v - b(t)h(u) + e(t). \end{aligned}$$

Define the function $V = V(t, x, y, u, v)$ as $V = Y^2 + \frac{1}{2}a^2(t)X^2 + a(t)XY$ where $X = x - u$ and $Y = y - v$. Since $0 < a_1 \leq a(t) \leq a_2$, there are positive constants p and q such that

$$(4.19) \quad p(X^2 + Y^2) \leq V \leq q(X^2 + Y^2)$$

for $t \geq 0$ and all x, y, u and v .

Along solutions of (4.18) — $\dot{V}_{(4.18)} = ZCZ^T$ where $Z = (X, Y)$,

$$C = \begin{pmatrix} a(t)b(t)\varphi(x, u) - a(t)a'(t) & b(t)\varphi(x, u) - \frac{1}{2}a'(t) \\ b(t)\varphi(x, u) - \frac{1}{2}a'(t) & a(t) \end{pmatrix}$$

and

$$\varphi(x, u) = \begin{cases} \frac{h(x) - h(u)}{x - u} & x \neq u, \\ h'(x) & x = u. \end{cases}$$

Combining (4.19) with this formulation of $\dot{V}$, it follows that the solutions of (4.17) converge if the eigenvalues of C are bounded below by a positive constant for $t \geq 0$ and all x and u . This will be the case if there is a positive constant η such that for $t \geq 0$ and all x and u $a[b\varphi - a'] \geq \eta$, and

$$(4.20) \quad a^2[b\varphi - a'] - [b\varphi - \tfrac{1}{2}a']^2 \geq \eta.$$

Since the first inequality follows from (4.20), it remains to be shown that (4.20) is satisfied. Setting $L(\varphi) = a^2[b\varphi - a'] - [b\varphi - \tfrac{1}{2}a']^2 - \eta$ and expanding we have $L(\varphi) = -b^2\varphi^2 + (a^2b + ba')\varphi - (a^2a' + (\tfrac{1}{2}a')^2 + \eta)$. The roots of $L(\varphi) = 0$ are

$$(a^2 + a' \pm a\sqrt{a^2 - 2a' - 4\eta})/2b,$$

and since $L(\varphi) \rightarrow -\infty$ as $|\varphi| \rightarrow \infty$, it follows that $L(\varphi) \geq 0$ if the roots of $L(\varphi) = 0$ are real and

$$(4.21) \quad \frac{a^2 + a' - a\sqrt{a^2 - 2a' - 4\eta}}{2b} \leq \varphi \leq \frac{a^2 + a' + a\sqrt{a^2 - 2a' - 4\eta}}{2b}.$$

It follows from (4.15) and (4.16) that $\eta > 0$ can be selected such that (4.21) will hold for $t \geq 0$ and all x and u , completing the proof of Theorem 14.

The Liapunov function constructed in the proof of the preceding result can be used to examine the solutions of

$$(4.22) \quad \ddot{x} + a(t)\dot{x} + b(t)h(x) = R(t, x, \dot{x})$$

where R is continuous in each of its variables and satisfies sufficient additional conditions to imply that every solution of (4.22) defined at $t = t_0$ exists for $t \geq t_0$.

THEOREM 15. — Assume that the hypotheses of Theorem 14 are satisfied and that there are positive constants K_1 and K_2 such that

$$(4.23) \quad |R(t, x, y) - R(t, u, v)| \leq K_1 + K_2[|x - u| + |y - v|]$$

for $t \geq 0$ and all x, y, u and v . If K_2 is sufficiently small, then the solutions of (4.22) are uniform distance bounded.

PROOF. - Equation (4.22) is equivalent to the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -a(t)y - b(t)h(x) + R(t, x, y),\end{aligned}$$

and we also need the product system

$$(4.24) \quad \begin{aligned}\dot{x} &= y & \dot{u} &= v \\ \dot{y} &= -a(t)y - b(t)h(x) + R(t, x, y) & \dot{v} &= -a(t)v - b(t)h(x) + R(t, u, v).\end{aligned}$$

If $V = Y^2 + \frac{1}{2}a^2(t)X^2 + a(t)XY$, then it follows from the proof of Theorem 14 and (4.23) that there is $\lambda > 0$ such that

$$(4.25) \quad \begin{aligned}\dot{V}_{(4.24)} &\leq -\lambda[X^2 + Y^2] + [a(t)X + Y][R(t, x, y) - R(t, u, v)] \\ &\leq -\lambda[X^2 + Y^2] + K_1[a(t)|X| + |Y|] + 2K_2[(a(t) + 1)^2 X^2 + Y^2].\end{aligned}$$

If $2K_2[a(t) + 1]^2 < \lambda$, then there are positive constants α and c such that $\dot{V} \leq -c$ for $X^2 + Y^2 \geq \alpha$ completing the proof.

The following can be easily obtained from inequality (4.25).

THEOREM 16. - If the hypotheses of Theorem 15 are satisfied and if $K_1 = 0$ and K_2 is sufficiently small, then the solutions of (4.22) converge.

5. - In this section a technique developed by KRASOVSKIĬ [4, Chap. 6] for the study of asymptotic stability of solutions of (1.3) is modified and extended to give information about the convergence of solutions of (1.3). This result is then used to determine sufficient conditions on (1.4) and (1.5) for the convergence of solutions of these equations.

Let $J = [0, \infty)$ and $C^n = C[-r, 0], R^n$, $r \geq 0$, where the norm of an element $\varphi \in C^n$ is defined by $\|\varphi\| = \sup_{-r \leq s \leq 0} |\varphi(s)|$, $|\cdot|$ a convenient norm in R^n . It will be assumed that $F \in C[J \times C^n, R^n]$. For $x \in C[-r, \infty), R^n$, the symbol x_t is defined by $x_t(s) = x(t + s) - r \leq s \leq 0$, $t \in J$.

THEOREM 17. - Let $y(t)$ be a solution of (1.3) which exists and is bounded for $t \geq 0$. Suppose $\alpha(s) > 0$ for $s > 0$ is a continuous nondecreasing function on J and that there exists a function $V \in C[[-r, \infty) \times R^n \times R^n, R^1]$ such that

- (i) $a(|x - y(t)|) \leq V(t, x, y(t)) \leq b(|x - y(t)|)$ for $t \geq -r$ and $x \in R^n$ where $a, b \in CIP(S)$ and $a(s) \rightarrow \infty$ as $s \rightarrow \infty$,
- (ii) $\dot{V}(t, x(t), y(t)) \leq 0$ for $t \in J$ and every $x_t \in C^n$ such that

$$V(s, x(s), y(s)) < \alpha(V(t, x(t), y(t))) \quad t - r \leq s \leq t.$$

Then the solutions of (1.3) are uniform bounded and the solution $y(t)$ is uniform stable.

Note: $\dot{V}(t, x(t), y(t)) =$

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} \{V(t+h, x(t) + hF(t, x_t), y(t) + hF(t, y_t)) - V(t, x(t), y(t))\}.$$

PROOF OF THEOREM 16. - For an arbitrary function $x_t \in C^n$ and $t \geq 0$ we define

$$W(t, x_t, y_t) = \sup_{t-r \leq s \leq t} V(s, x(s), y(s)).$$

It is clear that

$$(5.1) \quad a(|x(t) - y(t)|) \leq W(t, x_t, y_t) \leq b(\|x_t - y_t\|)$$

for $t \geq 0$ and every $x_t \in C^n$.

If $V(s, x(s), y(s)) < \alpha(V(t, x(t), y(t)))$ for $t-r \leq s \leq t$ then, since α , V , x and y are continuous, there is $h > 0$ such that

$$V(t+\theta, x(t+\theta), y(t+\theta)) < \alpha(V(t+h, x(t+h), y(t+h)))$$

for $-r \leq \theta \leq h$ and $0 \leq h \leq h_0$. It follows from this inequality and (ii) that

$$\dot{V}(t+s, x(t+s), y(t+s)) \leq 0$$

for $0 \leq s \leq h$, and thus that

$$(5.2) \quad V(t+h, x(t+h), y(t+h)) \leq V(t, x(t), y(t))$$

for $0 \leq h \leq h_0$. Now from the definition of W and (5.2) we see that

$$W(t+h, x_{t+h}, y_{t+h}) \leq W(t, x_t, y_t)$$

for $0 \leq h \leq h_0$, and thus that

$$(5.3) \quad \dot{W}(t, x_t, y_t) \leq 0.$$

If $V(s_0, x(s_0), y(s_0)) \geq \alpha(V(t, x(t), y(t)))$ for some $s_0 \in [t-r, t]$, we can assume in fact that V takes its maximum value on $[t-r, t]$ at s , then

$$\begin{aligned} W(t, x_t, y_t) &= V(s_0, x(s_0), y(s_0)) \\ &\geq \alpha(V(t, x(t), y(t))) \\ &> V(t, x(t), y(t)). \end{aligned}$$

Since V is continuous, there is $h > 0$ such that

$$V(s_0 + h, x(s_0 + h), y(s_0 + h)) > V(t + h, x(t + h), y(t + h))$$

for $0 \leq h \leq h_0$. It follows from this last inequality that

$$W(t + h, x_{t+h}, y_{t+h}) \leq W(t, x_t, y_t)$$

for $0 \leq h \leq h_0$ and thus that

$$(5.4) \quad \dot{W}(t, x_t, y_t) \leq 0.$$

Combining (5.3) and (5.4) we see that

$$\dot{W}(t, x_t, y_t) \leq 0$$

for $t \geq 0$ and every $x_t \in C^n$, which, together with (5.1) completes the proof of Theorem 17.

THEOREM 18. – If the hypotheses of Theorem 17 are satisfied and if

(iii) $\dot{V}(t, x(t), y(t)) \leq -c(|x(t) - y(t)|)$ for $t \in J$ and every $x_t \in C^n$ such that

$$V(s, x(s), y(s)) < \alpha(V(t, x(t), y(t))) \quad t - r \leq s \leq t$$

where $c \in CP(J)$,

then the solutions of (1.3) converge.

PROOF. – Since much of the proof of Theorem 18 is similar to the proof of [3, Theorem 11.2], we only sketch the proof here.

It follows from Theorem 17 that the solutions of (1.3) are uniform-bounded and and that the solution $y(t)$ is uniform-stable. Let $x(t) = x(t; \varphi, t_0)$, $\varphi \in C^n$, $t_0 \geq 0$, be a solution of (1.3), then there is $H_1 > 0$ such that $|x(t) - y(t)| < H_1$ for $t \geq t_0$ and it remains to be shown that for any $H > 0$, $0 < H < H_1$, there is a positive $T = T(H, H_1)$ such that $|x(t) - y(t)| < H$ for $t \geq t_0 + T$.

Select $\beta > 0$ such that $\alpha(s) - s > \beta$ for $a(H) \leq s \leq b(H_1)$, a positive integer N such that $a(H) + N\beta \geq b(H_1)$ and let $\gamma = \inf_{H \leq s \leq H_1} b(s)$ and $T = Na(H)/\gamma$.

It can be shown by an induction argument that $V(t, x(t), y(t)) \leq a(H)$ for $t \geq t_0 + T + (N-1)r$, which implies that $|x(t) - y(t)| < H$ for $t \geq t_0 + T + (N-1)r$, completing the proof of Theorem 18.

With the aid of Theorems 17 and 18 we are now able to establish sufficient conditions for convergence of solutions of equations (1.4) and (1.5).

THEOREM 19. – Assume that there is a bounded solution $x_0(t) = x(t; \varphi_0, 0)$, $\varphi_0 \in C^1$, of (1.4) satisfying $|f(t, x(t))| \leq K$ for $t \geq -r$. Assume further that for $t \in J$ and all

real x and y that f , $\partial f/\partial x$ are continuous, $\partial^2 f/\partial x^2$ exists and that there are positive constants a , L and M such that $-L \leq \partial f/\partial x \leq -a$ and $|\partial^2 f/\partial x^2| \leq M$.

If there is an $\varepsilon > 0$ such that

$$(5.5) \quad 0 \leq h(t) \leq \frac{a}{L^2 + MK} - \varepsilon \quad t \in J,$$

then the solutions of (1.4) are uniform bounded, the solution $x_0(t)$ is uniform stable, and furthermore the solutions of (1.4) converge.

PROOF. — For $t > 2r$, (1.4) can be rewritten in the form

$$(5.6) \quad \dot{x}(t) = f(t, x(t)) - \int_{t-h(t)}^t \frac{\partial f}{\partial x}(t, x(s)) f(s, x(s-h(s))) ds.$$

If $x_0(t) = x(t; \varphi_0, 0)$ $\varphi_0 \in C^1$, is a solution of (1.4) satisfying $|f(t, x(t))| \leq K$ for $t \geq -r$, then we define V by $V(t, x, x_0(t)) = (x - x_0(t))^2$ for $t \geq -r$ and $-\infty < x < \infty$.

Along a solution $x(t) = x(t; \varphi, t_0)$ $\varphi \in C^1$, of (1.4) we have

$$(5.7) \quad \begin{aligned} \dot{V}(t, x(t), x_0(t)) &= 2[f(t, x(t)) - f(t, x_0(t))][x(t) - x_0(t)] - 2[x(t) - x_0(t)] \int_{t-h(t)}^t \Phi(t, s) ds \\ &\leq -2a[x(t) - x_0(t)]^2 - 2[x(t) - x_0(t)] \int_{t-h(t)}^t \Phi(t, s) ds \end{aligned}$$

where

$$\Phi(t, s) = \frac{\partial f}{\partial x}(t, x(s)) f(s, x(s-h(s))) - \frac{\partial f}{\partial x}(t, x_0(s)) f(s, x_0(s-h(s))).$$

Using the inequalities $|\partial f/\partial x| \leq L$ and $|\partial^2 f/\partial x^2| \leq M$ and the mean value theorem, we note that

$$\begin{aligned} |\Phi(t, s)| &= \left| \frac{\partial f}{\partial x}(t, x(s)) f(s, x(s-h(s))) - \frac{\partial f}{\partial x}(t, x_0(s)) f(s, x_0(s-h(s))) \right. \\ &\quad \left. + \frac{\partial f}{\partial x}(t, x(s)) f(s, x_0(s-h(s))) - \frac{\partial f}{\partial x}(t, x_0(s)) f(s, x_0(s-h(s))) \right| \\ &= \left| \frac{\partial f}{\partial x}(t, x(s)) \frac{\partial f}{\partial x}(s, \eta_1(s)) [x(s-h(s)) - x_0(s-h(s))] \right. \\ &\quad \left. + \frac{\partial^2 f}{\partial x^2}(t, \eta_2(s)) f(s, x_0(s-h(s))) [x(s) - x_0(s)] \right| \\ &\leq L^2 |x(s-h(s)) - x_0(s-h(s))| + MK |x(s) - x_0(s)|. \end{aligned}$$

Substituting this inequality into (5.7) we have

$$\dot{V} \leq -2[a - MKh(t)][x(t) - x_0(t)]^2 + 2L^2h(t)|x(t) - x_0(t)||x(s - h(s)) - x_0(s - h(s))|.$$

If we set $\alpha(s) = qs$ where $0 < q - 1 \leq \varepsilon/aMK$, then

$$\dot{V}(t, x(t), x_0(t)) \leq -\varepsilon L^2[x(t) - x_0(t)]^2$$

whenever $[x(t + s) - x_0(t + s)]^2 \leq q[x(t) - x_0(t)]^2$, $-r \leq s \leq 0$, $t \geq t_0$, and the result now follows from Theorems 17 and 18.

The following result can be established in a manner analogous to the proof of Theorem 19.

THEOREM 20. — Assume that there is a bounded solution $x_0(t) = x(t; \varphi_0, 0)$ $\varphi_0 \in C^1$, of (1.5) satisfying $|f(t, x_0(t))| \leq K$ and $|g(t, x_0(t))| \leq K_g$ for $t \geq -r$. Assume further that for $t \in J$ and all real x and y that f, g and $\partial f/\partial x$ are continuous, $\partial^2 f/\partial x^2$ and $\partial g/\partial x$ exist and that there are constants $a, b > 0$, $L \geq |a|$, L_g and M such that $-L \leq \partial f/\partial x \leq -a$, $-L_g \leq \partial g/\partial x \leq -b$ and $|\partial^2 f/\partial x^2| \leq M$.

If there is an $\varepsilon > 0$ such that

$$0 \leq h(t) \leq \frac{a + b}{M(K + K_g) + L(L + L_g)} - \varepsilon \quad t \in J,$$

then the solutions of (1.5) are uniform bounded, the solution $x_0(t)$ is uniform stable, and furthermore, the solutions of (1.5) converge.

REFERENCES

- [1] M. L. CARTWRIGHT - J. E. LITTLEWOOD, *On non-linear differential equations of the second order*, *Annals of Math.*, **48** (1947), pp. 472-494.
- [2] J. O. C. EZEILO, *A note on the convergence of solutions of certain non-linear differential equations*, *Portugal Math.*, **24** (1965), pp. 49-58.
- [3] J. HALE, *Functional Differential Equations*, Springer-Verlag, New York (1971).
- [4] N. N. KRASOVSKII, *Stability of Motion*, Stanford Univ. Press, Stanford, Calif. (1963).
- [5] V. LAKSHMIKANTHAM - S. LEELA, *Differential and Integral Inequalities*, vol. II, Academic Press, New York (1969).
- [6] J. P. LASALLE - S. LEFSCHETZ, *Stability by Liapunov's Direct Method, Method*, Academic Press, New York (1961).
- [7] J. P. LASALLE, *Stability theory for ordinary differential equations*, *J. Differential Equations*, **4** (1968), pp. 57-65.
- [8] N. LEVINSON, *On a non-linear differential equation of the second order*, *J. Math. and Phys.*, **22** (1942), pp. 181-187.
- [9] W. S. LOUD, *Boundedness and convergence of solutions of $x + cx + g(x) = e(t)$* , *Duke Math. J.*, **24** (1957), pp. 63-72.

- [10] G. E. H. REUTER, *On certain non-linear differential equations with almost periodic solutions*, J. London Math. Soc., **26** (1951), pp. 215-221.
 - [11] K. E. SWICK, *Invariant sets and convergence of solutions of nonlinear differential equations*, J. Differential Equations, **10** (1971), pp. 204-218.
 - [12] K. E. SWICK, *Convergence of solutions of third order differential equations*, Bull. Canad. Math. Soc., **12** (1969), pp. 779-792.
 - [13] P. WALTMAN - T. F. BRIDGLAND Jr., *On convergence of solutions of the forced Lienard equation*, J. Math. and Phys., **44** (1965), pp. 284-287.
 - [14] T. YOSHIKAWA, *On the convergence of solutions of the nonlinear differential equation*, Mem. Coll. Sci. Univ. Kyoto, Ser. A, **28** (1953), pp. 143-151.
 - [15] T. YOSHIKAWA, *Stability theory by Liapunov's second method*, J. Math. Soc. Japan, Tokyo (1966).
-