Plane Sections of the Tangent Surface of a Space Curve.

BY BUCHIN SU (Taiho, Kiangsi, China).

1. In the study of projective invariants of a space curve it is often convenient to consider the tangent surface of the curve. WILCZYNSKI ([1]) was the first one who utilized the plane section of the tangent surface T of a space curve Γ made by the osculating plane at an ordinary point P of Γ and called the osculating conic of the section at P the osculating conic of Γ. Certainly, P is also an ordinary point of the section.

On the contrary, if we consider the other plane section made by a plane π passing through P, then P is an inflexion or a cusp of the section according as the tangent of Γ at P is or is not contained in π. In the former case we have obtained ([2]) a projectivity $\mathfrak{B}$ between a plane π through the tangent t and a point P_1 on t such that the plane section of T made by π has P_1 for its BOMPIANI osculant O_4 ([3]).

It seems of some interest to investigate the remaining case when the plane section has P for a cusp. For this purpose we have to represent the neighbourhoods of various orders of the curve at P by means of POPA osculants ([4]) and then arrive at certain correspondences which are intimately related to the projectivity $\mathfrak{B}$ and the polarity $\mathfrak{S}$ with respect to the osculating conic of Γ at P.

In a recent paper BOMPIANI ([5]) has enriched the projective differential geometry of a space curve in the neighbourhood of an ordinary point by

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([1]) E. J. WILCZYNSKI, *Projective differential geometry of curves and ruled surfaces*, (Leipzig, Teubner, 1906).

([2]) B. SU, *On certain quadratic cones projectively connected with a space curve and a surface*, « Tôhoku Math. Journ. », **38** (1933), 233-244.

([3]) Concerning BOMPIANI osculants of a plane curve with an inflexion cfr. E. BOMPIANI, *Per lo studio proiettivo-differenziale della singolarità*, « Boll. dell' Unione Mat. Italiana », 5 (1926), 118-120. See also my paper: *Note on the projective differential geometry of space curves*, « Journ. Chin. Math. Soc. », 2 (1937), 98-137.

([4]) I. POPA, *Geometria proiettivo-differenziale delle singolarità delle curve piane*, « Rend. dei Lincei », (VI), **25** (1937), 220-222.

([5]) E. BOMPIANI, *Sulle curve sghembe*, « Scritti matematici offerti a Luigi Berzolari », Pavia (1936), 515-552.

introducing the plane section of the tangent surface T made by a general plane. As a supplement of his investigation we will show that the two correspondences obtained by BOMPIANI just coincide with $\mathfrak{B}$ and $\mathfrak{S}$.

The remaining part of this note is devoted to certain configurations projectively connected with a space curve and a surface. These results were obtained during August, 1937 and the publication has been delayed by war.

2. Let P be an ordinary point of an analytic space curve Γ; the non-homogeneous coordinates x, y, z of P are analytic functions of x so that, taking the tangent t of Γ at P for the x-axis and the osculating plane at the same point for xy-plane with origin P, we have

$$\left\{\begin{aligned} y &= ax^2 + bx^3 + cx^4 + dx^5 + (6), \\ z &= rx^3 + sx^4 + tx^5 + ux^6 + (7), \end{aligned}\right. \tag{1}$$

where $ar \neq 0$ and (n) denotes the terms of degree $\geqq n$ in x.

The tangents of Γ describe a developable surface T, namely, the tangent surface. The equations to T are evidently of the form

$$\left\{\begin{aligned} \xi &= x + \mu, \\ \eta &= y + \mu y', \\ \zeta &= z + \mu z', \end{aligned}\right. \tag{2}$$

where $y' = \dfrac{dy}{dx}$, $z' = \dfrac{dz}{dx}$; μ denotes another parameter and ξ, η, ζ the current coordinates of a point.

In order to obtain the plane section $\bar{\Gamma}$ of T made by the osculating plane of Γ at P we have merely to put $\zeta = 0$ in (2), which gives the value for μ:

$$\mu = -\frac{x}{3r}\left[r - \frac{1}{3}sx + \left(\frac{4}{9}\frac{s^2}{r} - \frac{2}{3}t\right)x^2 + (3)\right].$$

It follows that the expansion of $\bar{\Gamma}$ at P takes the form

$$\begin{aligned} \eta = {} & \frac{3}{4}a\xi^2 + \frac{3}{8}\frac{as}{r}\xi^3 \\ & + \left\{\frac{27}{16}\left(\frac{bs}{r} - c\right) - \frac{69}{64}a\left(\frac{s}{r}\right)^2 + \frac{9}{8}\frac{at}{r}\right\}\xi^4 + (5). \end{aligned} \tag{3}$$

The neighbourhood of the 4th order of Γ at P determines a polarity $\mathfrak{S}$ between a point P_1 on the tangent t and the line t_n through P and in the osculatig plane $\zeta = 0$ such that the corresponding elements are pole and polar with respect to any four-point conic of $\bar{\Gamma}$ at P. In virtue of (3) we easily obtain the equation to the four-point conic, namely,

$$\frac{3}{4}a\xi^2 + \frac{1}{2}\frac{s}{r}\xi\eta - \eta + k\eta^2 = 0, \tag{4}$$

k bing a parameter. Let the line t_n be given by

$$\zeta = \eta - n\xi = 0 \tag{5}$$

and P_1 of the coordinates $(x_0, 0, 0)$, the polarity $\mathfrak{F}$ is then given by the equation

$$\frac{1}{x_0} = \frac{1}{2}\left(\frac{s}{r} + \frac{3a}{n}\right). \tag{6}$$

3. Suppose now that a plane π through the tangent t be different from the osculating plane. The plane section of T made by π has an inflexion at P and consequently determines BOMPIANI osculant O_4 on the tangent t. Denoting π by the equation

$$\zeta + \lambda\eta = 0 \tag{7}$$

and O_4 by the coordinates $(x_0, 0, 0)$, we have [6]

$$\frac{1}{x_0} = 2\frac{s}{r} - 3\frac{b}{a} - 3\frac{r}{a}\frac{1}{\lambda}. \tag{8}$$

This projectivity will be denoted by the letter $\mathfrak{B}$. It shall be noted that the point P and the osculating plane of Γ at P correspond to each other.

As the definition or the equation (8) shows, *the projectivity $\mathfrak{B}$ is determined by the neighbourhood of the 4th order of* Γ *at* P, but it is connected with the osculating linear line complex of Γ, which is determined by the neighbourhood of the 5th order. In fact, we can show that *the null plane of any point* P_1 *on* t *with respect to the linear complex coincides with the correspondig plane* π *of* P_1 *under* $\mathfrak{B}$.

To prove this, let us denote the line coordinates of the join of two points (ξ, η, ζ) and (ξ', η', ζ') by

$$r_1 = \xi\eta' - \xi'\eta, \quad r_2 = \xi\zeta' - \xi'\zeta, \quad r_3 = \xi - \xi',$$
$$r_4 = \eta'\zeta - \eta'\zeta, \quad r_5 = \eta' - \eta, \quad r_6 = \zeta - \zeta';$$

the equation to the osculating linear complex of Γ at P is easily found to be [7]

$$3ar^3r_1 - arlr_2 + mr_4 + a^2r^2r_6 = 0, \tag{9}$$

where

$$l = 3br - 2as, \quad m = 5art - 6as^2 - 9r^2c + 9rbs. \tag{10}$$

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(6) For the details cfr. my paper: *On the intersection of two curves in space*, « Tôhoku Math Journ. », **39** (1934), 226-232.

(7) Cfr. e. g. my paper: *Invariants of intersection of two curves in space*, « Sci. Rep. Tôhoku Imp. Univ. », (I), **25** (1936), 22-33.

Consequently, we obtain the equation to the null plane at the point $P_1(x_0, 0, 0)$, namely,

$$(ar + lx_0)\zeta - 3r^2 x_0 \eta = 0. \tag{11}$$

Putting this in the form (7) we have the relation

$$\frac{1}{\lambda} = -\frac{a}{3r}\left(3\frac{b}{a} - 2\frac{s}{r} + \frac{1}{x_0}\right),$$

which is equivalent to (8).

4. We come now to consider the plane section of T made by a plane π which passes through P but does not contain the tangent t of Γ at P. The equation of π is

$$\zeta + \lambda(\eta - n\xi) = 0, \tag{12}$$

where λ denotes a parameter. The line of intersection of π and the osculating plane of Γ at P is evidently t_n given by (5).

In order to obtain the plane section C in consideration we have to substitute (2) in (12) so as to determine μ. The result of computation is as follows:

$$\begin{cases} \eta = \dfrac{yz' - y'z + \lambda n(xy' - y)}{z' + \lambda(y' - n)}, \\ \zeta = \dfrac{\lambda\{y'z - yz' + n(xz' - z)\}}{z' + \lambda(y' - n)}. \end{cases} \tag{13}$$

These equations represent the projection of the curve C in the $\eta\zeta$-plane.

Substituting (1) in (13) and putting

$$X = -\frac{1}{a}\eta, \quad Y = -\zeta \tag{14}$$

for the sake of convenience, we obtain

$$\begin{aligned} X = x^2 &+ 2\left(\frac{b}{a} + \frac{a}{n}\right)x^3 + \left(3\frac{c}{a} + \frac{4r}{\lambda n} + \frac{7b}{n} + 4\frac{a^2}{n^2}\right)x^4 \\ &+ \left(4\frac{d}{a} + 6\frac{s}{\lambda n} + 10\frac{c}{n} + 14\frac{ar}{\lambda n^2} + 20\frac{ab}{n^2} + 6\frac{b^2}{an} + 6\frac{br}{\lambda an} + 8\frac{a^3}{n^3}\right)x^5 + (6) \end{aligned} \tag{15}$$

$$\begin{aligned} Y &= 2rx^3 + 3\left(s + \frac{ar}{n}\right)x^4 + \left(4\frac{as}{n} + 4t + 6\frac{a^2r}{n^2} + 6\frac{br}{n} + 6\frac{r^2}{\lambda n}\right)x^5 \\ &+ \left\{\frac{1}{n}(5at + 8bs + 9cr) + 5u + 8\frac{a^2s}{n^2} + 12\frac{a^3r}{n^3} + 21\frac{abr}{n^2} + 21\frac{ar^2}{\lambda n^2} + 17\frac{rs}{\lambda n}\right\}x^6 + (7). \end{aligned} \tag{16}$$

It follows that *the curve* C *has a cusp at* P and therefore that we can represent the neighbourhoods of various orders by means of Popa's method.

For this purpose it is convenient to rewrite the expansions (15) and (16) in the form used by POPA, namely,

$$(17)\qquad \begin{cases} X = t^2, \\ Y = \mathfrak{A}t^3 + \mathfrak{B}t^4 + \mathfrak{C}t^5 + \mathfrak{D}t^6 + (7). \end{cases}$$

In virtue of (15) there is no difficulty in expanding x into a power series of t. A simple calculation suffices to demostrate that

$$(18)\qquad x = t + At^2 + Bt^3 + Ct^4 + (5),$$

where

$$(19)\qquad \begin{cases} A = -\left(\dfrac{b}{a} + \dfrac{a}{n}\right), \\ B = \dfrac{5}{2}\dfrac{b^2}{a^2} - \dfrac{3}{2}\dfrac{c}{a} + \dfrac{3}{2}\dfrac{b}{n} + \dfrac{1}{2}\dfrac{a^2}{n^2} - 2\dfrac{r}{\lambda n}, \\ C = 9\dfrac{bc}{a^2} - 2\dfrac{d}{a} - 8\dfrac{b^3}{a^3} + 4\dfrac{c}{n} - 6\dfrac{b^2}{an} - \dfrac{ab}{n^2} + \dfrac{1}{\lambda}\left(5\dfrac{ar}{n^2} - 3\dfrac{s}{n}\right). \end{cases}$$

Substituting (18) in the right-hand side of (16) and reducing, we obtain that

$$(20)\qquad \begin{cases} \mathfrak{A} = 2r, \quad \mathfrak{B} = 6Ar + 3\left(s + \dfrac{ar}{n}\right), \\ \mathfrak{C} = 6r(B + A^2) + 12\left(s + \dfrac{ar}{n}\right)A + 4\dfrac{as}{n} + 4t + 6\dfrac{a^2r}{n^2} + \dfrac{6br}{n} + 6\dfrac{r^2}{\lambda n}, \\ \mathfrak{D} = 2r(3C + 6AB + A^3) + 3\left(s + \dfrac{ar}{n}\right)(4B + 6A^2) \\ \qquad + 5A\left(4\dfrac{as}{n} + 4t + 6\dfrac{a^2r}{n^2} + 6\dfrac{br}{n} + 6\dfrac{r^2}{\lambda n}\right) \\ \qquad + \dfrac{1}{n}(5at + 8bs + 9cr) + 5u + 8\dfrac{a^2s}{n^2} + 12\dfrac{a^3r}{n^3} \\ \qquad + 21\dfrac{abr}{n^2} + 21\dfrac{ar^2}{\lambda n^2} + 17\dfrac{rs}{\lambda n}. \end{cases}$$

5 We need to recall some notions due to POPA for a plane curve C with a cusp P ([8]). If the curve be represented in the form (17), then ∞^3 cusped cubics determined by six consecutive points of P on C are representable by the equation

$$(21)\qquad (1 - uX - vY)Y^2 - \mathfrak{A}^2(X - \mu Y)^3 = 0,$$

u, v and μ being parameters. The value of μ and consequently the line (d) joining P and the only inflexion of the cubic (21) will be determined when we impose the condition in order that the cubic should pass through a new

([8]) I. POPA, loc. cit, § 1.

consecutive point of C. In fact, we have that

$$\mu = -\frac{2\mathfrak{B}}{3\mathfrak{A}^2}$$

and therefore that the line (d) has the equation

$$3\mathfrak{A}^2 X + 2\mathfrak{B} Y = 0. \tag{22}$$

That is to say, *the neighbourhood of the 6th order of* C *at* P *determines the covariant line* (d).

In virtue of successive neighbourhoods of C we can easily determine u and v and, in consequence, the cusped cubic (21) which osculates C at P. Thus

$$\left\{\begin{aligned} u &= 2\frac{\mathfrak{C}}{\mathfrak{A}} - \frac{7}{3}\left(\frac{\mathfrak{B}}{\mathfrak{A}}\right)^2, \\ v &= 2\frac{\mathfrak{D}}{\mathfrak{A}^2} - 4\frac{\mathfrak{B}\mathfrak{C}}{\mathfrak{A}^3} + \frac{46}{27}\frac{\mathfrak{B}^3}{\mathfrak{A}^4}. \end{aligned}\right. \tag{23}$$

In particular, we obtain the inflexional tangent

$$uX + vY - 1 = 0, \tag{24}$$

a covariant line determined by the neighbourhood of the 8th order of C *at* P. This line intersects the line (d) and the cuspidal tangent at *the principal points* O_1 and O_2 respectively, the former being the only inflexion of the osculating cusped cubic.

6. Let us now apply the above result to the configuration considered in § 4. From (14), (19), (20) and (22) it follows that the covariant line (d) for the plane section C of T made by the plane (12) is given by the equations

$$\zeta + \lambda(\eta - n\xi) = 0, \tag{12}$$

$$\frac{r}{a}\eta + \left(\frac{1}{2}\frac{s}{r} - \frac{b}{a} - \frac{a}{2n}\right)\zeta = 0. \tag{25}$$

Since the latter does not contain λ, we have the following theorem:

Let t_n *be any line in the osculating plane* π *of a space curve* Γ *at* P *and through* P, *but distinct from the tangent* t. *If we consider all the plane sections of the tangent surface* T *made by planes through* t_n, *then every section has a cusp at* P *and the covariant line* (d) *of* POPA *describes a plane* $\tilde{\omega}_n$. *The correspondence between* t_n *and* $\tilde{\omega}_n$ *is projective.*

This projectivity can also be defined by the following method.

Eliminating x_0 from (6) and (8) we have

$$\frac{1}{2}\left(\frac{s}{r} + 3\frac{a}{n}\right) = 2\frac{s}{r} - 3\frac{b}{a} - 3\frac{r}{a}\frac{1}{\lambda}.$$

The plane (7) has then the equation

$$\frac{r}{a}\eta + \left(\frac{1}{2}\frac{s}{r} - \frac{b}{a} - \frac{a}{2n}\right)\zeta = 0,$$

which is precisely (25). In other words: *The projectivity between the line* t_n *and the plane* $\bar{\omega}_n$ *is the product of* $\mathfrak{S}$ *and* $\mathfrak{B}$.

7. Before we proceed to study the loci of other POPA osculants it is convenient to give here some remarkable properties of plane sections of T made by a general plane α, as BOMPIANI ([9]) has shown.

Suppose that the plane α does not pass through P so that its equation is of the form

$$\alpha_1\xi + \alpha_2\eta + \alpha_3\zeta + \alpha_4 = 0 \qquad (\alpha_4 \neq 0). \tag{26}$$

In order to express the representation for the section Γ_α of T made by α we have as before to substitute (2) in (26) so as to determine μ:

$$\mu = -\frac{x + \bar{\alpha}_2 y + \bar{\alpha}_3 z + \bar{\alpha}_4}{1 + \bar{\alpha}_2 y' + \bar{\alpha}_3 z'},$$

where $\bar{\alpha}_k = \alpha_k/\alpha_1$ $(k = 2, 3, 4)$.

Therefore the projection of Γ_α in the $\eta\zeta$-plane is representable in the form

$$\left\{\begin{aligned} \eta &= \frac{y - xy' + \bar{\alpha}_3(yz' - y'z) - \bar{\alpha}_4 y'}{1 + \bar{\alpha}_2 y' + \bar{\alpha}_3 z'},\\ \zeta &= \frac{z - xz' + \bar{\alpha}_2(y'z - yz') - \bar{\alpha}_4 z'}{1 + \bar{\alpha}_2 y' + \bar{\alpha}_3 z'}. \end{aligned}\right. \tag{27}$$

In virtue of (1) the right-hand sides of (27) are expansible in power series of x. The result of carrying out the computation is as follows:

$$\begin{aligned} \eta &= -2a\bar{\alpha}_4 x + (4a^2\bar{\alpha}_2\bar{\alpha}_4 - a - 3b\bar{\alpha}_4)x^2 + (3),\\ \zeta &= -3r\bar{\alpha}_4 x^2 + 2\{3ar\bar{\alpha}_2\bar{\alpha}_4 - (r + 2s\bar{\alpha}_4)\}x^3 + (4). \end{aligned}$$

Expanding further ζ in power series of η we have

$$\zeta = a_0\eta^2 + a_1\eta^3 + (4), \tag{28}$$

where

$$\left\{\begin{aligned} a_0 &= -\frac{3}{4}\frac{1}{a^2\bar{\alpha}_4},\\ a_1 &= \frac{1}{8a^4\bar{\alpha}_4^3}\{6a^2r\bar{\alpha}_2\bar{\alpha}_4 - ar + (4as - 9br)\bar{\alpha}_4\}. \end{aligned}\right. \tag{29}$$

([9]) E. BOMPIANI, *Sulle curve sghembe*, loc. cit.

Now, let us project the curve Γ from P in the same plane α, the projection being denoted by C_α. If X and Y be the points on α at which α intersects the tangent of Γ at its generic point M and the line PM respectively, then they describe the curves Γ_α and C_α, and X must lie on the tangent of C_α at Y. When M approaches P along Γ both X and Y approach the point P_α, i. e. the intersection of α and t. Hence C_α and Γ_α touch at P_α and the common tangent is the intersection of α and the osculating plane of Γ at P.

Let M have the coordinates (x, y, z); then those of Y are

$$\xi = \rho x, \quad \eta = \rho y, \quad \zeta = \rho z.$$

Since Y must lie on α, we have

$$\rho(x + \bar{\alpha}_2 y + \bar{\alpha}_3 z) + \bar{\alpha}_4 = 0.$$

Therefore the projection of C_α in the $\eta\zeta$-plane is given by the equations,

$$(30) \qquad \begin{cases} \eta = -\dfrac{\bar{\alpha}_4 y}{x + \bar{\alpha}_2 y + \bar{\alpha}_3 z}, \\[2ex] \zeta = -\dfrac{\bar{\alpha}_4 z}{x + \bar{\alpha}_2 y + \bar{\alpha}_3 z}, \end{cases}$$

Expanding the right-hand sides in power series of x by means of (1) and expressing then ζ in power serier of η, we have

$$(31) \qquad \zeta = \bar{a}_0 \eta^2 + \bar{a}_1 \eta^3 + (4),$$

where

$$(32) \qquad \begin{cases} \bar{a}_0 = -\dfrac{r}{a^2 \bar{\alpha}_4}, \\[2ex] \bar{a}_1 = \dfrac{1}{a^4 \bar{\alpha}_4^2} (a^2 r \bar{\alpha}_2 + as - 2br). \end{cases}$$

9. In virtue of (28) and (31) we can easily prove the two theorems due to BOMPIANI and appreciate his result by showing that *the two correspondences there obtained coincide with $\mathfrak{B}$ and $\mathfrak{S}$.*

From (29) and (32) it follows that

$$(33) \qquad \frac{a_0}{\bar{a}_0} = \frac{3}{4}.$$

That is, the invariant of SEGRE of Γ_α with respect to C_α is equal to $\frac{3}{4}$, as C. Servais and BOMPIANI have shown.

The neighbourhoods of the 3rd order of both curves C_α and Γ_α at the point P_α of contact determine a covariant line r_0 of the equation (26) and

(34) $$\eta + \frac{1}{2(a_0 - \bar{a}_0)}\left(\frac{a_1}{a_0} - \frac{\bar{a}_1}{\bar{a}_0}\right)\zeta = 0.$$

The latter can be written as

(35) $$2r^2\eta + \left(2as - 3br + \frac{ar}{\bar{\alpha}_4}\right)\zeta = 0$$

if we substitute the values of a_k, $\bar{a}_k$ given by (29) and (32). Noticing that the coordinates of P_α are

(36) $$x_0 = -\bar{\alpha}_4,\ 0,\ 0,$$

we can further put (35) in the form

$$\zeta + \lambda\eta = 0$$

provided that

$$\frac{1}{\lambda} = \frac{a}{3r}\left(2\frac{s}{r} - 3\frac{b}{a} - \frac{1}{x_0}\right)$$

or

$$\frac{1}{x_0} = 2\frac{s}{r} - 3\frac{b}{a} - 3\frac{r}{a}\frac{1}{\lambda}.$$

Thus we have the following theorem:

For a point P *on the curve* Γ *and* ∞^3 *planes* α *in space there are only* ∞^2 *covariant lines* r_0 *forming* ∞^1 *pencils, each of which has point* P_α *on the tangent* t *of* Γ *at* P *for its centre, and a plane* π *through* t *for its base. The correspondence between* P_α *and* π *is the projectivity* $\mathfrak{B}$.

The neighbourhoods of the 3rd order of C_α and Γ_α also determine a covariant point Q_α in such a manner that the ramaining common chord of any two four-point conics of C_α and Γ_α always passes through the fixed point Q_α. The coordinates of Q_α are given by

$$\xi + \bar{\alpha}_2\eta + \bar{\alpha}_3\zeta + \bar{\alpha}_4 = 0, \quad \zeta = 0, \quad \eta = -\frac{a_0 - \bar{a}_0}{\frac{a_1}{a_0}\bar{a}_0 - \frac{\bar{a}_1}{\bar{a}_0}a_0}$$

or

$$\xi = \frac{(2ar + as\bar{\alpha}_4)\bar{\alpha}_4}{3a^2r\bar{\alpha}_2\bar{\alpha}_4 - 2ar - as\bar{\alpha}_4},$$

$$\eta = \frac{3a^2r\bar{\alpha}_4^2}{3a^2r\bar{\alpha}_2\bar{\alpha}_4 - 2ar - as\bar{\alpha}_4},$$

$$\zeta = 0.$$

Denoting the join of this point and P by t_n:

$$\zeta = \eta - n\xi = 0$$

we obtain that

$$-\frac{1}{n} = \frac{2r + s\bar{\alpha}_4}{3ar\bar{\alpha}_4}$$

and therefore that the correspondence between P_α and t_n takes the form

$$\frac{1}{x_0} = \frac{1}{2}\left(\frac{s}{r} + 3\frac{a}{n}\right).$$

Thus we obtain the following theorem:

For a point P *on* Γ *and* ∞^3 *planes* α *in space all the covariant points* Q_α *lie in the osculating plane of* Γ *at* P. *The correspondence between* P_α, *the point of intersection of* α *and* t, *and the line* PQ_α *is the polarity* $\mathfrak{S}$.

From what was stated above it follows that the polarity $\mathfrak{S}$ and the projetctivity $\mathfrak{B}$ are fundamental in the study of plane sections of the tangent surface of a space curve.

9. We are now in a position to study the other POPA osculants of the plane section of T made by the plane (12). In order to simplify the calculation it is necessary to use some canonical expansions of Γ instead of (1). Take, for example, one of the fundamental tetrahedrons associated at P of Γ for the tetrahedron of reference, so that the expansions of Γ become [10]

$$(37) \qquad \begin{cases} y = \frac{1}{2}x^2 - \frac{2}{15}\theta_3 x^5 + (6), \\ z = \frac{1}{6}x^3 - \frac{1}{10}\theta_3 x^6 + (7). \end{cases}$$

Thus, putting

$$a = \frac{1}{2}, \quad b = c = 0, \quad d = -\frac{2}{15}\theta_3;$$

$$r = \frac{1}{6}, \quad s = t = 0, \quad u = -\frac{1}{10}\theta_3$$

in (19) and (20), we have

$$A = -\frac{1}{2n}, \quad B = \frac{1}{8n^2} - \frac{1}{3\lambda n}, \quad C = \frac{8}{15}\theta_3 + \frac{5}{12}\frac{1}{\lambda n^2};$$

$$\mathfrak{A} = \frac{1}{3}, \quad \mathfrak{B} = -\frac{1}{2n}, \quad \mathfrak{C} = \frac{1}{8n^2} - \frac{1}{6\lambda n},$$

$$\mathfrak{D} = \frac{1}{30}\theta_3 + \frac{7}{24}\frac{1}{\lambda n^2} - \frac{1}{24}\frac{1}{n^3}.$$

(10) Cfr. B. SU, *Note on the projective differential geometry of space curves*, loc cit., p. 113.

Therefore the plane (25) becomes

$$4n\eta - 3\zeta = 0 \tag{38}$$

which corresponds to the line t_n:

$$\zeta = \eta - n\xi = 0. \tag{39}$$

The quantities u and v given by (23) now take the values

$$\begin{cases} u = -\left(\dfrac{9}{16}\dfrac{1}{n^2} + \dfrac{1}{\lambda n}\right), \\ v = \dfrac{3}{5}\theta_3 + \dfrac{3}{4}\dfrac{1}{\lambda n^2} + \dfrac{15}{32}\dfrac{1}{n^3} \end{cases} \tag{40}$$

and consequently determine the line O_1O_2, namely, the inflexional tangent of the plane osculating cusped cubic of the section in consideration. From (12), (24) and (40) we obtain the equations of the line O_1O_2:

$$\begin{cases} \zeta + \lambda(\eta - n\xi) = 0, \\ -\left(\dfrac{9}{8}\dfrac{1}{n^2} + \dfrac{2}{\lambda n}\right)\eta + \left(\dfrac{3}{5}\theta_3 + \dfrac{3}{4}\dfrac{1}{\lambda n^2} + \dfrac{15}{32}\dfrac{1}{n^3}\right)\zeta + 1 = 0. \end{cases} \tag{41}$$

It follows that the principal point C_2 has the coordinates

$$\xi = \left(\frac{9}{8}\frac{1}{n} + \frac{2}{\lambda}\right)^{-1}, \quad \eta = n\left(\frac{9}{8}\frac{1}{n} + \frac{2}{\lambda}\right)^{-1}, \quad \zeta = 0 \tag{42}$$

and therefore that *the correspondence between* O_2 *and the plane of the section is projective.*

Especially when O_2 coincides with $\bar{P}_2$, the other intersection of t_n and the osculating conic of Γ at P

$$\eta = \frac{3}{8}\xi^2, \tag{43}$$

we have $\lambda = -\dfrac{8}{3}n$ and therefore the corresponding plane

$$\zeta - \frac{8}{3}n(\eta - n\xi) = 0. \tag{44}$$

The latter can, however, be constructed in the following manner: Denote by $\bar{P}_1$ the pole of t_n with regard to the osculating conic (43) and construct the corresponding plane $P\bar{P}_1\bar{P}_2$ of $\bar{P}_1$ with respect to $\mathfrak{B}$; then the BOMPIANI line l_5 of the plane section of T made by this plane must lie in the plane (44). In other words:

The plane (44) *is one face of the fundamental tetrahedron* $\mathrm{T}(\bar{\mathrm{P}}_1)$.

In fact, the coordinates of $\bar{P}_1$ are

$$x_0 = \frac{4}{3} n, \; 0, \; 0. \tag{45}$$

Therefore the corresponding point P_3 of the coordinates ([11])

$$\xi = 3x_0, \quad \eta = \frac{9}{2} x_0^2, \quad \zeta = \frac{9}{3} x_0^3$$

satisfy (44), which was to be proved.

10. Let us attend to the principal point O_1. The coordinates of this point must satisfy (38) and (41) so that

$$\left\{ \begin{aligned} \xi &= \frac{\frac{4}{3}\frac{1}{\lambda} + \frac{1}{n}}{\frac{1}{n\lambda} + \frac{1}{2}\frac{1}{n^2} - \frac{4}{5} n\theta_3}, \\ \eta &= \frac{1}{\frac{1}{n\lambda} + \frac{1}{2}\frac{1}{n^2} - \frac{4}{5} n\theta_3}, \\ \zeta &= \frac{\frac{4}{3} n}{\frac{1}{n\lambda} + \frac{1}{2}\frac{1}{n^2} - \frac{4}{5} n\theta_3}. \end{aligned} \right. \tag{46}$$

From this we infer that *when the plane revolves about the line* t_n *the corresponding principal point* O_1 *describes a range of points and the correspondence is projective.*

There is no difficulty in finding the base of the range. Elimination of λ from (46) gives the required:

$$\left\{ \begin{aligned} \xi &= \frac{3}{4} n\eta. \\ \xi &= \left(\frac{1}{3n} + \frac{16}{15}\theta_3 n^2\right)\eta + \frac{4}{3} n, \end{aligned} \right. \tag{47}$$

Here we will give a construction for this line. We notice first that it intersects the tangent t of Γ at $\bar{P}_1$, namely, the pole of t_n with respect to the osculating conic. In order to determine another point on (47) it is convenient to consider the line PP_3, i. e., the BOMPIANI line l_5. On the latter we

([11]) Cfr. B. SU, *Note on . . .* , loc. cit., p. 115.

have geometrically defined ∞^2 points Q_λ depending on a parameter λ ([12]), whose coordinates are

$$(48)\qquad \begin{cases} \xi = \dfrac{3x_0}{1 + \dfrac{81\lambda}{10(1-\lambda)}\theta_3 x_0^3}, \\ \eta = \dfrac{\frac{9}{2}x_0^2}{1 + \dfrac{81\lambda}{10(1-\lambda)}\theta_3 x_0^3}, \\ \zeta = \dfrac{\frac{9}{2}x_0^3}{1 + \dfrac{81\lambda}{10(1-\lambda)}\theta_3 x_0^3}. \end{cases}$$

Substituting (48) in (47) and remembering that

$$x_0 = \frac{4}{3} n,$$

we have that $Q_{-\frac{1}{2}}$ lies on the line in consideration. Thus follows the result:

For a fixed line t_n *the line described by the principal point* O_1 *is the join of* P_1 *and* $Q_{-\frac{1}{2}}$.

Further elimination of n from (47) gives the ruled surface of order 3:

$$(49)\qquad \frac{4}{9}\eta^3 + \frac{3}{5}\theta_3\zeta^3 - \xi\eta\zeta + \zeta^2 = 0.$$

11. If λ be eliminated from (41), then we obtain the locus of the line O_1O_2:

$$(50)\qquad \zeta + \frac{3}{4n}\xi\zeta - \frac{15}{8}\frac{1}{n^2}\eta\zeta - 2\xi\eta + \frac{2}{n}\eta^2 + \left(\frac{3}{5}\theta_3 + \frac{15}{32}\frac{1}{n^3}\right)\zeta^2 = 0,$$

a quadric corresponding to the line t_n. This result follows also from the fact that O_1 and O_2 describe two projective ranges of points on the lines (47) and t_n. The principal points O_1 and O_2 are therefore the points of contact of the quadric with the planes (38) and (12) respectively.

When the line t_n varies the corresponding quadric (50) envelopes an algebraic surface of order 6, as it may easily be seen by eliminating n from (50) and its derived equation with respect to n.

([12]) Cfr. B. Su, *Note on . . .*; loc. cit., p. 118.

In the same way we can show that the locus of the osculating cusped cubic is a cubic surface when the plane of the section revolves about the line t_n and that this cubic surface envelopes an algebraic surface of order 10.

12. As an application of the above results we shall derive some covariant figures for a pair of intersecting space curves and, in particular, for the asymptotic curves passing through a generic point on a surface.

Let C and $\bar{C}$ be two space curves having O for their point of intersection with distinct tangents t, $\bar{t}$ but with the same osculating plane $(t, \bar{t})$. If t and $\bar{t}$ be taken as axes x and y, then the expansions of the two curves in the neighbourhood of O are of the form (1) for C and

$$(51) \qquad \begin{cases} x = \alpha y^2 + \beta y^3 + \gamma y^4 + \delta y^5 + (6), \\ z = \rho y^3 + \sigma y^4 + \tau y^5 + \varphi y^6 + (7) \end{cases}$$

for $\bar{C}$.

For every line t_n through O and in the plane $(t, \bar{t})$ there exist two planes, each of which passes through one of t and $\bar{t}$; and corresponds to the tangent surface of the curve in the manner quoted in § 6. Denoting t_n by the equations (5), we obtain these plane s

$$(52) \qquad \frac{r}{a}\eta + \left(\frac{1}{2}\frac{s}{r} - \frac{b}{a} - \frac{a}{2n}\right)\zeta = 0,$$

$$(53) \qquad \frac{\rho}{\alpha}\xi + \left(\frac{1}{2}\frac{\sigma}{\rho} - \frac{\beta}{\alpha} - \frac{\alpha}{2}n\right)\zeta = 0.$$

Especially when t_n coincides with $\bar{t}$ the plane (52) becomes

$$(54) \qquad \frac{r}{a}\eta + \left(\frac{1}{2}\frac{s}{r} - \frac{b}{a}\right)\zeta = 0.$$

Similarly, when t_n coincides with t there is obtained the plane

$$(55) \qquad \frac{\rho}{\alpha}\xi + \left(\frac{1}{2}\frac{\rho}{\sigma} - \frac{\beta}{\alpha}\right)\zeta = 0.$$

These planes intersect in the dual line of BOMPIANI ([13]). Thus we have a new definition of this covariant line.

In general, the planes (52) and (53) corresponding to the same line t_n intersect in a line through O, whose locus is a quadric cone of the equation

$$(56) \qquad \left\{\frac{\rho}{\alpha}\xi + \left(\frac{1}{2}\frac{\rho}{\sigma} - \frac{\beta}{\alpha}\right)\zeta\right\}\left\{\frac{r}{a}\eta + \left(\frac{1}{2}\frac{s}{r} - \frac{b}{a}\right)\zeta\right\} = \frac{1}{4}a\alpha\zeta^2.$$

([13]) Cfr. my paper: *Invariants* . . . , loc. cit., p. 24.

It shall be noted that the polar of the plane $(t, \bar{t})$ with respect to (56) is also the dual line of BOMPIANI.

Especially when the curves C and $\bar{C}$ are the two asymptotic curves through a non-parabolic point O of a non-ruled non-degenerate surface S, then, using usual notations, we have the cone

$$\left(\xi + \frac{1}{4}\Psi\zeta\right)\left(\eta + \frac{1}{4}\Phi\zeta\right) = \frac{9}{16}\beta\gamma\zeta^2, \tag{57}$$

where β, γ are fundamental quantities of FUBINI and

$$\Phi = \frac{\partial \log \beta\gamma^2}{\partial u}, \quad \Psi = \frac{\partial \log \beta^2\gamma}{\partial v}. \tag{58}$$

This cone being covariant with respect to projective deformations was obtained by R. CALAPSO [14] and thus possesses a new definition. The above result gives further significances for the first edge of GREEN.

13. We proceed to establish certain further configurations projectively connected with the curves C and $\bar{C}$.

The projectivity between a plane $\bar{\omega}$ through t_n and the principal point O_2 of the section made by $\bar{\omega}$ suggests us to research what position on t_n the point O_2 should take in order that the corresponding planes with respect to the tangent surfaces of C and $\bar{C}$ always coincide.

For a plane through t_n there are in general two points O_2 and $\bar{O}_2$ on t_n, each of which stands for the second principal point of the plane section of the tangent surface of each curve made by $\bar{\omega}$, so that the correspondence between O_2 and $\bar{O}_2$ is projective. Hence the position we have to determine must be one of the double points of this projectivity. O is evidently the double point, since the corresponding planes coincide with $(t, \bar{t})$. It follows that there exists another point D on each line t_n and therefore one plane $\bar{\omega}$ through t_n satisfying the required condition.

For the purpose of calculating the coordinates of D we have merely to consider the principal point O_2 of the plane

$$\zeta + \lambda(\eta - n\xi) = 0 \tag{59}$$

with respect to the curve C, for the principal point $\bar{O}_2$ of the same plane

(14) R. CALAPSO, *Sugli enti proiettivi legati al generico punto di una superficie*, « Atti Acad. Gioenia », Catania, (5), **19** (1933), Mem. XIV, 1-6. I was not aware of the cone of Calapso until a paper of E. Bortolotti appeared. Cfr. ENEA BORTOLOTTI, *Quadriche di Moutard e fascio canonico*, « Rend. R. Accad. dei Lincei », (VI), **25** (1937), 158-165.

may easily be obtained by interchanging $a, b, c, d, \ldots$: $r, s, t, \ldots$; n and λ with $\alpha, \beta, \gamma, \delta, \ldots$; $\rho, \sigma, \tau, \ldots$; $\frac{1}{n}$ and $-\lambda n$ respectively.

From (19) and (20) we obtain

$$(60)\qquad \begin{cases} \mathfrak{A} = 2r, \quad \mathfrak{B} = 3\left(s - 2\dfrac{br}{a} - \dfrac{ar}{n}\right), \\ \mathfrak{C} = 21\dfrac{b^2r}{a^2} - 9\dfrac{cr}{a} + 15\dfrac{br}{n} + 3\dfrac{a^2r}{n^2} - 6\dfrac{r^2}{\lambda n} - 12\dfrac{bs}{a} - 8\dfrac{as}{n} + 4t. \end{cases}$$

Further substitution of these values in (23) gives

$$(61)\qquad u = -9\frac{c}{a} + 4\frac{t}{r} - \frac{21}{4}\frac{s^2}{r^2} + 9\frac{bs}{ar} + \left(\frac{5}{2}\frac{as}{r} - 6b\right)\frac{1}{n} - \frac{9}{4}\frac{a^2}{n^2} - 6\frac{r}{\lambda n}.$$

The coordinates of O_2 are consequently given by

$$(62)\qquad \xi = -\frac{a}{nu}, \quad \eta = -\frac{a}{u}, \quad \zeta = 0.$$

In the same way we have that the coordinates of $\bar{O}_2$ are

$$(63)\qquad \xi = -\frac{\alpha}{\bar{u}}, \quad \eta = -\frac{\alpha n}{\bar{u}}, \quad \zeta = 0,$$

where

$$(64)\qquad \bar{u} = -9\frac{\gamma}{\alpha} + 4\frac{\tau}{\rho} - \frac{21}{4}\frac{\sigma^2}{\rho^2} + 9\frac{\beta\sigma}{\alpha\rho} + \left(\frac{5}{2}\frac{\alpha\sigma}{\rho} - 6\beta\right)n - \frac{9}{4}\alpha^2n^2 + 6\frac{\rho}{\lambda}.$$

From (62) and (63) it follows that these points $\bar{O}_2$ and O_2 coincide each other when, and only when,

$$(65)\qquad \alpha n u = a\bar{u}.$$

This gives the value of λ:

$$(66)\qquad \begin{aligned} 6(a\rho + \alpha r)\frac{1}{\lambda} = {} & \frac{9}{4}a\alpha^2n^2 - \frac{9}{4}a^2\alpha\frac{1}{n} + \\ & + \left\{\alpha\left(-9\frac{c}{a} + 4\frac{t}{r} - \frac{21}{4}\frac{s^2}{r^2} + 9\frac{bs}{ar}\right) - a\left(\frac{5}{2}\frac{\alpha\sigma}{\rho} - 6\beta\right)\right\}n + \\ & + \left\{a\left(9\frac{\gamma}{\alpha} - 4\frac{\tau}{\rho} + \frac{31}{4}\frac{\sigma^2}{\rho^2} - 9\frac{\beta\sigma}{\alpha\rho}\right) + \alpha\left(\frac{5}{2}\frac{as}{r} - 6b\right)\right\}. \end{aligned}$$

Substituting this in (61) and (62) we have that the coordinates of $O_2 \equiv \bar{O}_2 \equiv D$

are given by the equations

$$\frac{a\rho+\alpha r}{\eta}=\frac{9}{4}a^2\rho\frac{1}{n^2}-\frac{9}{4}\alpha^2 rn$$
$$+\rho\left(9\frac{c}{a}-4\frac{t}{r}+\frac{21}{4}\frac{s^2}{r^2}-9\frac{bs}{ar}\right)+r\left(\frac{5}{2}\frac{\alpha\sigma}{\rho}-6\beta\right)$$
$$-\left\{r\left(9\frac{\gamma}{\alpha}-4\frac{\tau}{\rho}+\frac{21}{4}\frac{\sigma^2}{\rho^2}-9\frac{\beta\sigma}{\alpha\rho}\right)+\rho\left(\frac{5}{2}\frac{as}{r}-6b\right)\right\}\frac{1}{n},$$
$$\eta=n\xi,\quad \zeta=0.$$

Therefore the locus of D is a nodal cubic in the plane $(t,\ \bar{t})$:

$$(a\rho+\alpha r)\xi\eta-\frac{9}{4}a^2\rho\xi^3+\frac{9}{4}\alpha^2 r\eta^3 \tag{67}$$
$$+\left\{r\left(9\frac{\gamma}{\alpha}-4\frac{\tau}{\rho}+\frac{21}{4}\frac{\sigma^2}{\rho^2}-9\frac{\beta\sigma}{\alpha\rho}\right)+\rho\left(\frac{5}{2}\frac{as}{r}-6b\right)\right\}\xi^2\eta$$
$$-\left\{\rho\left(9\frac{c}{a}-4\frac{t}{r}+\frac{21}{4}\frac{s^2}{r^2}-9\frac{bs}{ar}\right)+r\left(\frac{5}{2}\frac{\alpha\sigma}{\rho}-6\beta\right)\right\}\xi\eta^2=0,$$

whence we obtain that three points of inflexion lie on the line of the equations $\zeta=0$ and

$$\left\{r\left(9\frac{\gamma}{\alpha}-4\frac{\tau}{\rho}+\frac{21}{4}\frac{\sigma^2}{\rho^2}-9\frac{\beta\sigma}{\alpha\rho}\right)+\rho\left(\frac{5}{2}\frac{as}{r}-6b\right)\right\}\xi \tag{68}$$
$$-\left\{\rho\left(9\frac{c}{a}-4\frac{t}{r}+\frac{21}{4}\frac{s^2}{r^2}-9\frac{bs}{ar}\right)+r\left(\frac{5}{2}\frac{\alpha\sigma}{\rho}-6\beta\right)\right\}\eta$$
$$+a\rho+\alpha r=0,$$

a covariant line determined by the neighbourhoods of the 5th order of both the curves C *and* $\overline{\mathrm{C}}$. The lines joining D and each inflexion are

$$a^2\rho\xi^3-\alpha^2 r\eta^3=0 \tag{69}$$

and therefore apolar to the tangents t and $\bar{t}$.

If the corresponding plane $\bar{\omega}$ of D be denoted by the equation

$$u_1\xi+u_2\eta+u_3\zeta=0. \tag{70}$$

then we have

$$\left\{\begin{aligned}
u_1&=-6(a\rho+\alpha r)n,\\
u_2&=6(a\rho+\alpha r),\\
u_3&=\frac{9}{4}a\alpha^2 n^2-\frac{9}{4}\alpha a^2\frac{1}{n}\\
&\quad+a\left(9\frac{\gamma}{\alpha}-4\frac{\tau}{\rho}+\frac{21}{4}\frac{\sigma^2}{\rho^2}-9\frac{\beta\sigma}{\alpha\rho}\right)+\alpha\left(\frac{5}{2}\frac{as}{r}-6b\right)\\
&\quad-\left\{\alpha\left(9\frac{c}{a}-4\frac{t}{r}+\frac{21}{4}\frac{s^2}{r^2}-9\frac{bs}{ar}\right)+a\left(\frac{5}{2}\frac{\alpha\sigma}{\rho}-6\beta\right)\right\}
\end{aligned}\right. \tag{71}$$

and therefore the envelope of $\tilde{\omega}$ is a cone of the 3rd class:

$$\frac{9}{4}a\alpha^2u_1^3 + \frac{9}{4}\alpha a^2u_2^3 - 6(a\rho + \alpha r)u_1u_2u_3 + \tag{72}$$
$$+\left\{\alpha\left(9\frac{c}{a} - 4\frac{t}{r} + \frac{21}{4}\frac{s^2}{r^2} - 9\frac{bs}{ar}\right) + a\left(\frac{5}{2}\frac{\alpha\sigma}{\rho} - 6\beta\right)\right\}u_1^2u_2 +$$
$$+\left\{a\left(9\frac{\gamma}{\alpha} - 4\frac{\tau}{\rho} + \frac{21}{4}\frac{\sigma^2}{\rho^2} - 9\frac{\beta\sigma}{\alpha\rho}\right) + \alpha\left(\frac{5}{2}\frac{as}{r} - 6b\right)\right\}u_1u_2^2 = 0.$$

This cone, being determined also by the neighbourhoods of the 5th order of C *and* $\bar{C}$, *has three cuspidal tangent planes, which are concurrent in the line*

$$\xi : \eta : \zeta = \tag{73}$$
$$= \left\{\alpha\left(9\frac{c}{a} - 4\frac{t}{r} + \frac{21}{4}\frac{s^2}{r^2} - 9\frac{bs}{ar}\right) + a\left(\frac{5}{2}\frac{\alpha\sigma}{\rho} - 6\beta\right)\right\}$$
$$: \left\{a\left(9\frac{\gamma}{\alpha} - 4\frac{\tau}{\rho} + \frac{21}{4}\frac{\sigma^2}{\rho^2} - 9\frac{\beta\sigma}{\alpha\rho}\right) + \alpha\left(\frac{5}{2}\frac{as}{r} - 6b\right)\right\}$$
$$: -6(a\rho + \alpha r).$$

14. Having thus obtained several remarkable configurations, we shall now apply them to the interesting case where C and $\bar{C}$ are asymptotic curves of a surface and then derive certain new elements projectively associated at a generic point O of the surface.

Setting, as usual, $\theta = \log \beta\gamma$ and

$$L = \theta_{uu} - \frac{1}{2}\theta_u{}^2 - 2p_{11} - \beta\theta_v - \beta_v,$$
$$M = \theta_{vv} - \frac{1}{2}\theta_v{}^2 - 2p_{22} - \gamma\theta_u - \gamma_u,$$

the nodal cubic (67) in case is given by

$$\frac{45}{4}\beta\gamma(\beta\xi^3 - \gamma\eta^3) - 20\beta\gamma\xi\eta \tag{74}$$
$$+ 3\gamma\left\{-2L + 5\theta_u(\log\gamma)_u + \frac{\beta_{uu}}{\beta} - \frac{5}{4}\Phi^2 + \frac{25}{6}\beta\Psi\right\}\xi\eta^2$$
$$- 3\beta\left\{-2M + 5\theta_v(\log\beta)_v + \frac{\gamma_{vv}}{\gamma} - \frac{5}{4}\Psi^2 + \frac{25}{6}\gamma\Phi\right\}\xi^2\eta = 0.$$

Therefore *the three points of inflexion are the intersections of the tangents of*

SEGRE *and the covariant line:*

$$(75)\quad \begin{cases} \zeta = 0, \\ 3\beta \left\{ -2M + 5\theta_v (\log\beta)_v + \dfrac{\gamma_{vv}}{\gamma} - \dfrac{5}{4}\Psi^2 + \dfrac{25}{6}\gamma\Phi \right\} \xi \\ -3\gamma \left\{ -2L + 5\theta_u (\log\gamma)_u + \dfrac{\beta_{uu}}{\beta} - \dfrac{5}{4}\Phi^2 + \dfrac{25}{6}\beta\Psi \right\} \eta + 20\beta\gamma = 0. \end{cases}$$

In the same way we obtain a cone of the 3rd class, namely,

$$(76)\quad \frac{45}{4}\beta\gamma(\gamma u_1^3 + \beta u_2^3) - 40\beta\gamma u_1 u_2 u_3$$
$$+ 3\beta \left\{ -2M + 5\theta_v (\log\beta)_v + \frac{\gamma_{vv}}{\gamma} - \frac{5}{4}\Psi^2 + \frac{25}{6}\gamma\Phi \right\} u_1 u_2^2$$
$$+ 3\gamma \left\{ -2L + 5\theta_u (\log\gamma)_u + \frac{\beta_{uu}}{\beta} - \frac{5}{4}\Phi^2 + \frac{25}{6}\beta\Psi \right\} u_1^2 u_2 = 0.$$

The three cuspidal tangent planes are concurrent in the covariant line

$$(77)\quad \xi : \eta : \zeta = 3\gamma \left\{ -2L + 5\theta_u (\log\gamma)_u + \frac{\beta_{uu}}{\beta} - \frac{5}{4}\Phi^2 + \frac{25}{6}\beta\Psi \right\}$$
$$: 3\beta \left\{ -2M + 5\theta_v (\log\beta)_v + \frac{\gamma_{vv}}{\gamma} - \frac{5}{4}\Psi^2 + \frac{25}{6}\gamma\Phi \right\}$$
$$: -40\beta\gamma$$

and each of them intersects the tangent plane of the surface in one tangent of SEGRE.

15. We shall conclude this paper with a remark on the elements determined by the neighbourhoods of the 6th order of both curves C and $\overline{C}$. As was shown before, there are two quadrics, each of them being constructed with respect to each curve and to the same line t_n in the plane $(t, \overline{t})$. These quadrics intersect in t_n and a twisted cubic so that there is such a twisted cubic corresponding to t_n. It is easily seen that this element may be found by using the value of u given by (61) and that of v derived from (19), (20) and (23):

$$(78)\quad v = 9\frac{bc}{a^2 r} - 6\frac{d}{ar} - 9\frac{b^2 s}{a^2 r^2} + \frac{9}{2}\frac{cs}{ar^2} + 2\frac{bt}{ar^2} + \frac{5}{2}\frac{u}{r^2} +$$
$$+ \frac{23}{8}\frac{s^3}{r^4} + \frac{3}{4}\frac{bs^2}{ar^3} - 6\frac{st}{r^3} + \frac{1}{n}\left(3\frac{c}{r} + 6\frac{b^2}{ar} - 9\frac{bs}{r^2} - \frac{3}{2}\frac{at}{r^2} + \frac{27}{8}\frac{as^2}{r^3}\right) +$$
$$+ \frac{1}{n^2}\left(\frac{15}{4}\frac{ab}{r} - \frac{15}{8}\frac{a^2 s}{r^2}\right) + \frac{5}{8}\frac{a^3}{rn^3} + \frac{1}{\lambda}\left\{\frac{3}{2}\frac{a}{n^2} - \left(21\frac{b}{a} + \frac{7}{2}\frac{s}{r}\right)\frac{1}{n}\right\}.$$

Without expressing the equations of these quadrics we merely note that one constructed by means of $\bar{C}$ and t is given by

$$\begin{aligned}(79)\qquad \zeta\Big\{&\Big(-9\frac{c}{a}+4\frac{t}{r}-\frac{21}{4}\frac{s^2}{r^2}+9\frac{bs}{ar}\Big)\eta\\&+\Big(9\frac{bc}{ar}-6\frac{d}{r}-9\frac{b^2s}{ar^2}+\frac{9}{2}\frac{cs}{r^2}+2\frac{bt}{r^2}+\frac{5}{2}\frac{au}{r^2}\\&+\frac{23}{8}\frac{s^3}{r^4}+\frac{3}{4}\frac{bs^2}{r^3}-6\frac{ast}{r^3}\Big)\zeta+a\Big\}\\&-\xi\Big\{6r\eta+\Big(21\frac{b}{a}+\frac{7}{2}\frac{s}{r}\Big)\zeta\Big\}=0.\end{aligned}$$

In the same way we obtain another quadric for $\bar{C}$ and t:

$$\begin{aligned}(80)\qquad \zeta\Big\{&\Big(-9\frac{\gamma}{\alpha}+4\frac{\tau}{\rho}-\frac{21}{4}\frac{\sigma^2}{\rho^2}+9\frac{\beta\sigma}{\alpha\rho}\Big)\xi\\&+\Big(9\frac{\beta\gamma}{\alpha\rho}-6\frac{\delta}{\rho}-9\frac{\beta^2\sigma}{\alpha\rho^2}+\frac{9}{2}\frac{\gamma\sigma}{\rho^2}+2\frac{\beta\tau}{\rho^2}+\frac{5}{2}\frac{\alpha\varphi}{\rho^2}\\&+\frac{23}{8}\frac{\sigma^3}{\rho^4}+\frac{3}{4}\frac{\beta\rho^2}{\rho^3}-6\frac{\alpha\sigma\tau}{\rho^3}\Big)\zeta+\alpha\Big\}\\&-\eta\Big\{6\rho\xi+\Big(21\frac{\beta}{\alpha}+\frac{7}{2}\frac{\sigma}{\rho}\Big)\zeta\Big\}=0.\end{aligned}$$

It follows that these quadrics intersect in t, $\bar{t}$ and a conic, whose plane passes through O when, and only when

$$(81)\qquad a\rho-\alpha r=0.$$

In the case of two asymptotic curves of a surface the last condition is evident, so that we are led to a conic through O projectively connected to the surface at O.