Uniform approximation of continuous functions by rational functions (*).

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A Bruno Finzi nel suo 70mo compleanno.

Summary. - *Viene data la condizione necessaria e sufficiente perchè le funzioni razionali di una variabile, aventi poli di ordine prefissato in assegnati punti del piano complesso, costituiscano un sistema completo iu* C^0 (0, 1).

Let us denote, as usual, by C^0 (0, 1) the BANACH space of all complex-valued continuous functions g on the closed interval (0, 1) of the real axis, endowed with the norm $\|g\| = \max\limits_{(0,1)} |g(x)|$. Let $z_1, z_2, \ldots, z_k, \ldots$ be a sequence of points of the complex plane [1] such that, for every k, z_k does not lie on the closed interval (0, 1) of the real axis. Let $\nu_1, \nu_2, \ldots, \nu_k, \ldots$ be a sequence of positive integers. We assume that

i) the pair (z_h, ν_h) is distinct from the pair (z_k, ν_k) if $h \neq k$;

ii) if $z_k = z_{k_0}$ for some $k \neq k_0$, then there exist infinitely many k such that $z_k = z_{k_0}$.

Let us denote by R the set of all rational functions $r(z)$ of the complex variable z such that

i) each of them has its poles only in some points of the sequence $\{z_k\}$;

ii) if $r(z)$ has a pole in z_k, the order of the pole does not exceed ν_k.

We shall consider the following problem

P) Find the necessary and sufficient conditions to be satisfied by the sequences $\{z_k\}$ *and* $\{\nu_k\}$ *in order that the set* R *be dense in the space* $C^0(0, 1)$.

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(1) Throughout this paper, when we consider the « complex plane », we suppose that the point $z = \infty$ is a point of this plane.

A sufficient condition is classically known. In fact, if for every k we have $z_k = \infty$, the density of R in $C^0(0, 1)$ is insured by the celebrated theorem of WEIERSTRASS on the uniform approximation of a continuous function by polynomials [1].

If $\nu_k = 1$ and z_k is, for every k, real and negative, SZEGÖ [2] has shown that R is dense in $C^0(0, 1)$ if $\lim\limits_{k\to\infty} z_k = -\infty$.

SZÁSZ [3], assuming the same hypotheses on the z_k's, has extended SZEGÖ's result, proving the completeness of the sequence $\{(x - z_k)^m\}$ $(k = 1, 2, \dots)$ in the space $C^0(0, 1)$, if m is any number not a positive integer or 0.

More recently PORCELLI [4], [5], has considered the sequence

$$\{(1 + c_k x)^{-1}\} \qquad (k = 0, 1, 2, \dots), \tag{1}$$

where the c_k's are complex constants such that $c_k \neq 0$, $c_h \neq c_k$ $(h \neq k)$ and, if c_k is real, $c_k > -1$. He proves [4] that

i) the above sequence is complete in $C^0(0, 1)$ if and only if the series

$$\sum_{k=0}^{\infty} \left\{ 1 - \left| \frac{(1 + c_k)^{1/2} - 1}{(1 + c_k)^{1/2} + 1} \right| \right\} \text{ (}^2\text{)} \tag{2}$$

is divergent.

Assuming the following further hypotheses on the c_k's:

$$|\arg(1 + c_k)| \leq \theta_0 < \pi, \qquad |1 + c_k| \geq \delta > 0, \tag{3}$$

PORCELLI [4] proves that

ii) the sequence (1) is complete in $C^0(0, 1)$ if and only if the series $\sum\limits_{k=0}^{\infty} |c_k|^{-1/2}$ is divergent.

This result was already known to van HERK [6] in the particular case that the c_k's are real and such that $c_k < c_{k+1}$, $c_k \to +\infty$ as $k \to \infty$.

In this paper a complete answer to problem P) will be given and a general theorem obtained. From this result the theorems of WEIERSTRASS, SZEGÖ, van HERK and PORCELLI will be deduced as particular cases.

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(2) From the proof given by PORCELLI in his paper [4], one deduces that the branch to be chosen for the fractional power $(1+z)^{1/2}$ is the one determined by the condition $-\pi < \arg(1+z) \leq \pi$.

1. - Preliminary lemmas.

Let us consider the function

$$z=\frac{1}{2}+\frac{1}{4}\left(w+\frac{1}{w}\right), \tag{4}$$

which maps conformally the unit disk D, $|w|<1$, of the complex plane of the variable $w=w_1+iw_2$ onto an open set A, of the complex plane of the variable $z=x+iy$. A consists of the whole complex z-plane, slit along the segment S: $y=0$, $0\leq x\leq 1$.

If z is any point of the complex z-plane, we denote by $d(z)$ its distance from S. Let D_R be the disk $|w|<R$ $(0<R<1)$.

LEMMA 1. - *Let w be any point of $D-D_R$ and z its corresponding point in A. Two positive constants m_R and M_R exist such that*

$$m_R\frac{d(z)}{|z(z-1)|^{1/2}}\leq 1-|w|\leq M_R\frac{d(z)}{|z(z-1)|^{1/2}}. \tag{5}$$

Set $w'=|w|^{-1}w$ and $z'=2^{-1}+4^{-1}(w'+w'^{-1})$. We have

$$z-z'=\frac{1}{4}(w-w')\left[1-\frac{1}{ww'}\right]. \tag{6}$$

On the other hand, if we set $w=w_1+iw_2$, $r^2=(w_1-1)^2+w_2^2$, we have

$$\left|\frac{w^2-|w|}{w^2-1}\right|^2=\frac{(w_1-1)^2+4w_2^2+0(r^3)}{4r^2+0(r^3)}. \tag{7}$$

Let r_0 be such that $0<r_0<1-R$ and $|w^2-1|^2>r^2$ for $0<r<r_0$.

From (7) we deduce that in the disk $|w-1|<r_0$ we have

$$\frac{|w^2-|w||}{|w^2-1|}\leq c, \tag{8}$$

where c is a positive constant. This inequality holds also in the disk $|w+1|<r_0$. It follows, by eventually increasing c, that inequality (8) holds in the whole disk D. Then we have in $D-D_R$

$$\frac{|w^2-|w||}{|w|^2}\leq\frac{c}{R}\frac{|w^2-1|}{|w|}.$$

From (4) we easily deduce

$$\frac{|w^2-1|}{|w|}=4\,|z(z-1)|^{1/2}.$$

Then from (6) we get

$$d(z)\leq|z-z'|=\frac{1}{4}(1-|w|)\frac{|w^2-|w||}{|w|^2}\leq\frac{c}{R}(1-|w|)\,|z(z-1)|^{1/2},$$

which proves the first of inequalities (5).

Let E_R be the bounded domain, whose boundary is the ellipse:

$$x=2^{-1}+4^{-1}(R+R^{-1})\cos\varphi,\qquad y=4^{-1}(R^{-1}-R)\sin\varphi\qquad(-\pi\leq\varphi\leq\pi).$$

Let $E_R^{(1)}$, $E_R^{(2)}$, $E_R^{(3)}$ be the subsets of E_R defined by the conditions

$$E_R^{(1)}:\quad z\in E_R-S,\qquad x\leq 0;$$

$$E_R^{(2)}:\quad z\in E_R,\qquad x>0,\qquad y>\frac{1}{2}\left(\frac{1}{R}-R\right)x;$$

$$E_R^{(3)}:\quad z\in E_R,\qquad 0<x\leq\frac{1}{2},\qquad 0<y\leq\frac{1}{2}\left(\frac{1}{R}-R\right)x.$$

It is sufficient to prove the second inequality (5), when z is a point belonging to any of the above defined subsets of E_R. By obvious symmetry arguments the proof is then extended to every $z\in E_R-S$.

Let w be a point of $D-D_R$. Suppose that the corresponding point z lies in $E_R^{(1)}$. We have from (4)

$$1-|w|\leq|w+1|\leq 2\,|z|^{1/2}(|z|^{1/2}+|z-1|^{1/2}).$$

There exists a constant c_R such that for $z\in E_R^{(1)}$ we have

$$2(|z|^{1/2}+|z-1|^{1/2})\leq\frac{c_R}{|z-1|^{1/2}}.\tag{9}$$

Then

$$1-|w|\leq\frac{c_R\,|z|}{|z(z-1)|^{1/2}},$$

which, for $z\in E_R^{(1)}$, is the inequality to be proved. Let us suppose that c_R be large enough that (9) holds also for $z\in E_R^{(2)}$. Then, for $z\in E_R^{(2)}$, we have

$$1 - |w| \leq c_R \frac{|z|}{|z(z-1)|^{1/2}} = \frac{c_R[(R^{-1}-R)^2+4]^{1/2}}{R^{-1}-R} \frac{|y|}{|z(z-1)|^{1/2}},$$

which proves the inequality for $z \in E_R^{(2)}$. From (4) we have

$$w = 2z - 1 - 2[z(z-1)]^{1/2}, \tag{4'}$$

where the branch of the fractional power is obtained by analytic continuation in A from the one which is positive when z is real and greater than 1. We have for $z \in E_R^{(3)}$

$$1 - |w| \leq |2iy - 2[z(z-1)]^{1/2} + 2[x(x-1)]^{1/2}|$$

$$= \left| 2iy - i\int_0^y \frac{2(x+i\eta)-1}{[(x+i\eta)(x+i\eta-1)]^{1/2}} d\eta \right| \leq c_1 |y| \left(1 + \frac{1}{|x(x-1)|^{1/2}}\right).$$

Since for $z \in E_R^{(3)}$, there exists a positive constant a_R such that $|x| \geq a_R |z|$, $|x-1| \geq a_R |z-1|$, from the last inequality we deduce, denoting by b_R a proper positive constant,

$$1 - |w| \leq b_R \frac{|y|}{|z(z-1)|^{1/2}}.$$

This completes our proof.

LEMMA 2. - *Let α be a complex-valued measure defined on the σ-ring $\{B\}_S$ of the Borel sets contained in the segment S. Suppose that for every $z \in A$ we have*

$$\int_S \frac{d\alpha_\xi}{\xi - z} = 0; \ (^3) \tag{10}$$

then α is identically zero.

Let $\varphi(x)$ be any complex-valued continuous function defined on the real axis and with a compact support. For $y > 0$, we get from (10)

$$0 = \int_{-\infty}^{+\infty} \varphi(x)dx \int_S \frac{d\alpha_\xi}{\xi - x - iy} - \int_{-\infty}^{+\infty} \varphi(x)dx \int_S \frac{d\alpha_\xi}{\xi - x + iy} =$$

$$= \int_S d\alpha_\xi \int_{-\infty}^{+\infty} \varphi(x) \frac{2iy}{(x-\xi)^2 + y^2} dx.$$

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(3) By writing $d\alpha_\xi$, we mean that the STIELTJES integration, with the measure α as integrator, is performed with respect to the variable ξ.

Since

$$\lim_{y \to 0^+} \int_{-\infty}^{+\infty} \varphi(x) \frac{2iy}{(x-\xi)^2 + y^2} dx = 2\pi i \varphi(\xi)$$

uniformly with respect to ξ, we have

$$\int_S \varphi(\xi) d\alpha_\xi = 0.$$

For the arbitrariness of φ, the proof follows.

LEMMA 3. - *Let β be a complex-valued measure defined on the σ-ring $\{B\}_\Gamma$ of the Borel sets contained in the boundary Γ of the unit disk D, $|w| < 1$. The function*

$$\psi(w) = \int_\Gamma \frac{d\beta_t}{t - w}$$

belongs to the class N of R. Nevanlinna (4).

For any B of $\{B\}_\Gamma$, set

$$\gamma(B) = \int_B \left(\frac{\partial t}{\partial s}\right)^{-1} d\beta,$$

where s denotes a curvilinear abscissa on Γ, increasing counterclockwise. We have

$$\frac{1}{t-w} \frac{\partial t}{\partial s} = \frac{\partial}{\partial s_t} \log |t - w| - i \frac{\partial}{\partial n_t} \log |t - w|,$$

where $\dfrac{\partial}{\partial s_t}$ and $\dfrac{\partial}{\partial n_t}$ denote, respectively, the tangential and the inner normal differentiation on Γ, with respect to the point t of Γ. Set

$$u(w) = \int_\Gamma \frac{\partial}{\partial n_t} \log |t - w| \, d\gamma_t, \qquad v(w) = \int_\Gamma \frac{\partial}{\partial s_t} \log |t - w| \, d\gamma_t,$$

(4) Set $w = \rho e^{i\theta}$. The function $\varphi(w)$ belongs to N if and only if

$$\int_{-\pi}^{\pi} \log^+ |\psi(\rho e^{i\theta})| \, d\theta \qquad (0 \leq \rho < 1)$$

is a bounded function of ρ. The proof of lemma 3 is already contained in [7] and [8]. We repeat it here for the convenience of the reader.

we have

$$\psi(w) = - i[u(w) + iv(w)].$$

Suppose $w = \rho e^{i\theta}$, $t = e^{i\tau}$. Since

$$\frac{\partial}{\partial n_t} \log | t - w | = \frac{1}{2} \frac{\rho^2 - 1}{1 + \rho^2 - 2\rho \cos(\tau - \theta)} - \frac{1}{2},$$

we obtain

$$u(\rho e^{i\theta}) = - \frac{1}{2} \int_{-\pi}^{\pi} \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\tau - \theta)} d\gamma_t - \frac{1}{2} \gamma(\Gamma). \tag{11}$$

From a known property of the POISSON-STIELTJES integral we deduce from (11)

$$\lim_{\rho \to 1} \int_{-\pi}^{\pi} q(e^{i\theta}) u(\rho e^{i\theta}) d\theta = - \pi \int_{-\pi}^{\pi} q(e^{i\theta}) d\gamma_\theta - \frac{1}{2} \gamma(\Gamma) \int_{-\pi}^{\pi} q(e^{i\theta}) d\theta,$$

for any complex-valued function q continuous on T. It follows, by the BANACH-STEINHAUS *uniform boundedness principle* (see [9], p. 26) that

$$\int_{-\pi}^{\pi} | u(\rho e^{i\theta}) | d\theta \leq L, \tag{12}$$

L being a constant independent on ρ.

Since u and v are a pair of harmonic functions satisfying the CAUCHY-RIEMANN system, $u_{w_1} = v_{w_2}$, $u_{w_2} = - v_{w_1}$, then the KOLMOGOROV inequality (see [10])

$$\int_{-\pi}^{\pi} | v(\rho e^{i\theta}) - v(0) |^p d\theta \leq M_p \Big(\int_{-\pi}^{\pi} | u(\rho e^{i\theta}) | d\theta \Big)^p \qquad (0 \leq \rho < 1), \tag{13}$$

for any p such that $0 < p < 1$, holds. M_p is a constant only depending on p. From (12) and (13) it follows that

$$\int_{-\pi}^{\pi} | \psi(\rho e^{i\theta}) |^p d\theta \leq c_p,$$

c_p being a constant independent on ρ. Consequently

$$\int_{-\pi}^{\pi} \log^+ |\psi(\rho e^{i\theta})| \, d\theta \leq \frac{c_p}{p} \ (^5).$$

2. - Main theorem.

In the statement of the next theorem we need to give a meaning to the function

$$F(z) = \frac{d(z)}{|z(z-1)|^{1/2}}$$

even when $z = \infty$. Since $F(z)$ is well defined in A for every $z \neq \infty$, it is natural to define $F(\infty)$ by assuming

$$F(\infty) = \lim_{z \to \infty} \frac{d(z)}{|z(z-1)|^{1/2}} = 1.$$

THEOREM. - *The set R is dense in $C^0(0, 1)$ if and only if the sequences $\{z_k\}$ and $\{\nu_k\}$ satisfy the following condition*

$$\boxed{\sum_{k=1}^{\infty} \frac{\nu_k d(z_k)}{|z_k(z_k - 1)|^{1/2}} = +\infty.} \tag{14}$$

Condition (14) is sufficient. Let α be a complex-valued measure defined on $\{B\}_S$ such that, for any $r(z) \in R$,

$$\int_S r(\xi) d\alpha_\xi = 0. \tag{15}$$

Let us introduce the function

$$f(z) = \int_S \frac{d\alpha_\xi}{\xi - z},$$

which is analytic in A. Conditions (15) mean that $f(z)$ has in z_k a zero of order not less than ν_k $[\nu_k + 1$ if $z_k = \infty]$. In order to prove the theorem we need to show, because of lemma 2, that $f(z)$ vanishes identically in A. For any w of the unit disk D, set

$$\Phi(w) = f\left[\frac{1}{2} + \frac{1}{4}\left(w + \frac{1}{w}\right)\right].$$

(5) It must be remarked that we have proved the stronger result $\psi \in H^p$ for any p such that $0 < p < 1$.

Let Γ^+ (let Γ^-) be the subset of the boundary Γ of D defined by the conditions $|w|=1$, $\Im w>0$ [$|w|=1$, $\Im w<0$]. We have $\Gamma=\Gamma^+\cup\Gamma^-\cup\{1\}\cup\{-1\}$. Denote by $\tilde{B}^+$ (by $\tilde{B}^-$) the subset of S corresponding, through the (4), to the BOREL set B contained in Γ^+ (contained in Γ^-). Let μ be the measure, defined on $\{B\}_\Gamma$, which is determined by the following conditions

$$\mu(B)\begin{cases} =\alpha(\tilde{B}^+) & \text{if} \quad B\subset\Gamma^+, \\ =\alpha(\tilde{B}^-) & \text{if} \quad B\subset\Gamma^-, \\ =2\alpha(\{1\}) & \text{if} \quad B=\{1\}, \\ =2\alpha(\{0\}) & \text{if} \quad B=\{-1\}. \end{cases}$$

After elementary computations we obtain

$$\Phi(w)=\frac{2w}{1-w^2}\left\{\int_\Gamma\frac{e^{i\varphi}d\mu_\varphi}{w-e^{i\varphi}}+w\int_\Gamma\frac{d\mu_\varphi}{w-e^{-i\varphi}}\right\}.$$

Let us set, using (4'),

$$w_k=2z_k-1-[z_k(z_k-1)]^{1/2}. \tag{16}$$

The function

$$\Phi_0(w)=\int_\Gamma\frac{e^{i\varphi}d\mu_\varphi}{w-e^{i\varphi}}+w\int_\Gamma\frac{d\mu_\varphi}{w-e^{-i\varphi}}$$

has in w_k a zero of order at least ν_k. Since $\Phi_0(w)$ — because of lemma 3 — belongs to N, the series

$$\sum_{k=1}^{\infty}\nu_k(1-|w_k|) \tag{17}$$

must be convergent [6], unless $\Phi_0(w)$ is identically zero in D. From lemma 1 and from (14), it follows that the series (17) diverges. Hence $\Phi_0(w)$ must vanish identically in D, i.e. $f(z)$ must vanish identically in A.

Condition (14) is necessary. We do not need to give any proof if for infinitely many k we have $z_k=\infty$. Let us suppose that $z_k\neq\infty$ for any k. Suppose, by absurd, that R is dense in $C^0(0,1)$ and the series (14) is convergent. From the sequence $\{w_k\}$, with w_k given by (16), let us deduce the sequence $\{w_s^*\}$ constructed by repeating ν_k times each w_k. Since the series

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[6] See [9] p. 273.

$$\sum_{s=1}^{\infty} (1 - |w_s^*|)$$

is convergent, the BLASCHKE product

$$B(w) = \prod_{s=1}^{\infty} \frac{w - w_s^*}{\overline{w}_s^* w - 1} \frac{\overline{w}_s^*}{|w_s^*|}$$

defines a bounded analytic function in the unit disk D. We set, using (4'),

$$f(z) = \frac{1}{[z(z-1)]^{1/2}} B\{2z - 1 - 2[z(z-1)]^{1/2}\},$$

The function $f(z)$ is analytic in A and $f(\infty) = 0$. Set for $0 \leq \xi \leq 1$

$$f^+(\xi) = \lim_{y \to 0^+} f(\xi + iy), \qquad f^-(\xi) = \lim_{y \to 0^-} f(\xi + iy).$$

It is known that the limits $f^+(\xi)$ and $f^-(\xi)$ exist almost everywhere in $(0, 1)$ and the functions $f^+(\xi)$ and $f^-(\xi)$ are LEBESGUE integrable functions in $(0, 1)$. Moreover, for any $z \in A$, we have

$$f(z) = \frac{1}{2\pi i} \int_0^1 \frac{f^+(\xi) - f^-(\xi)}{\xi - z} d\xi.$$

Since $f(z)$ has in z_k a zero of order not less than ν_k, then, for any $r(z) \in R$, we have

$$\int_0^1 [f^+(\xi) - f^-(\xi)] r(\xi) d\xi = 0,$$

which contradicts the assumption that R is dense in $C^0(0, 1)$. A similar argument can be used if $z_{k_0} = \infty$, $z_k \neq \infty$ for $k \neq k_0$.

3. - Particular cases.

If we assume $z_k = \infty$ and $\nu_k = k$, the series in the left hand side of (14) reduces to $\sum_{k=1}^{\infty} k$. Thus we obtain, as a particular case of our result, the classical WEIERSTRASS polynomial approximation theorem.

I we assume $\nu_k = 1$, $z_k = -\lambda_k$ with λ_k real and positive and $\lim\limits_{k\to\infty} \lambda_k = +\infty$, we have

$$\frac{d(z_k)}{|z_k(z_k - 1)|^{1/2}} = \left(\frac{\lambda_k}{1 + \lambda_k}\right)^{1/2}.$$

Since $\lim\limits_{k\to\infty} \lambda_k^{1/2}(1 + \lambda_k)^{-1/2} = 1$, the series in (14) is, in this case, divergent. This proves that the result of SZEGÖ, which we quoted in the introduction, is included in our theorem.

The problem considered by PORCELLI which we have mentioned in the introduction, is a particular case of problem P) when we assume $z_k = -c_k^{-1}$ and $\nu_k = 1$. The series (2) becomes

$$\sum_{k=0}^{\infty} \left\{1 - \left|\frac{(1 - z_k^{-1})^{1/2} - 1}{(1 - z_k^{-1})^{1/2} + 1}\right|\right\},$$

which, after some manipulations, can be written

$$\sum_{k=0}^{\infty} (1 - |w_k|),$$

where w_k is given by (16). It follows, because of lemma 1, that the series (2) is divergent if and only if the series in (14) — with $\nu_k = 1$ — is divergent. This proves that our theorem includes the theorem i) of PORCELLI which we quoted in the introduction.

Suppose that the points z_k lie in a subset A' of A such that, given R $(0 < R < 1)$, two positive constants p_R and P_R exist such that, for $z \in A' \cap E_R$,

$$p_R |z|^{1/2} \leq \frac{d(z)}{|z(z-1)|^{1/2}} \leq P_R |z|^{1/2}. \tag{18}$$

Then in the statement of our main theorem the series in (14) can be replaced by the series $\sum\limits_{k=1}^{\infty} |z_k|^{1/2}$.

Assume $z_k = -c_k^{-1}$ and suppose that conditions (3) are satisfied. Let us denote by 0 the origin of the z-plane, by U the point (1, 0) and by Z the point (x, y). Let C_1 be the arc of circle which lies on the half-plane $y \geq 0$ and is determined by the condition that, for $Z \in C_1$, the angle $0\widehat{Z}U$ is constant and equal to θ_0. Let C_2 be the symmetric of C_1 with respect to the x-axis. The first of conditions (3) imposes to z_k not to lie in the bounded domain B_1, whose boundary is $C_1 \cup C_2$. The second condition, assuming $0 < \delta < 1$, imposes to z_k not to belong to the disk B_2 of center $[(1 - \delta^2)^{-1}, 0]$ and radius $[(1 - \delta^2)^{-2} - (1 - \delta^2)^{-1}]^{1/2}$ Assuming $A' = (A - B_1) \cap (A - B_2)$ and R so small that $E_R \supset \overline{B_1 \cup B_2}$, it is easily seen that conditions (18) are satisfied. It follows that the theorem ii) of PORCELLI, quoted in the introduction, which includes the one of van HERK, is a particular case of our general result.

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