On invariant immersions.

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Summary. - *Theory of immersions satisfying the condition that tangent spaces to an immersed submanifold are invariant under the curvature transformation.*

§ 1. - Definitions and Preliminaries.

Let M and $\overline{M}$ be differentiable manifolds with Riemannian metrics g and $\overline{g}$ respectively. To simplify notations, we denote them by (M, g) and $(\overline{M}, \overline{g})$, Let Γ and $\overline{\Gamma}$ be the Riemannian connections associated with (M, g) and $(\overline{M}, \overline{g})$ respectively and let ∇ and $\overline{\nabla}$ be the covariant differentiations with respect to Γ and $\overline{\Gamma}$ respectively. Let $\mathfrak{X}(M)$ and $\mathfrak{X}(\overline{M})$ be the Lie algebras of differentiable vector fields on M and $\overline{M}$ respectively. The curvature tensor fields R and $\overline{R}$ of Γ and $\overline{\Gamma}$ are given by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z$$

for $X, Y, Z \in \mathfrak{X}(M)$ and

$$\overline{R}(\overline{X}, \overline{Y})\overline{Z} = \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z} - \overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z} - \overline{\nabla}_{[\overline{X}, \overline{Y}]}\overline{Z}$$

for $\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\overline{M})$. Then we have

(1.1) $\quad R(X, Y) + R(Y, X) = 0,$

(1.2) $\quad R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ (BIANCHI'S 1st identity),

(1.3) $\quad (\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0$ (BIANCHI'S 2nd identity),

(1.4) $\quad g(R(X, Y)Z, U) = g(R(Z, U)X, Y),$

(1.5) $\quad g(R(X, Y)Z, U) + g(R(X, Y)U, Z) = 0$

for $X, Y, Z, U \in \mathfrak{X}(M)$ and

(1.1)' $\quad \overline{R}(\overline{X}, \overline{Y}) + \overline{R}(\overline{Y}, \overline{X}) = 0,$

(1.2)' $\quad \overline{R}(\overline{X}, \overline{Y})\overline{Z} + \overline{R}(\overline{Y}, \overline{Z})\overline{X} + \overline{R}(\overline{Z}, \overline{X})\overline{Y} = 0,$

$$(1.3)' \qquad (\bar\nabla_{\bar X}\bar R)(\bar Y, \bar Z) + (\bar\nabla_{\bar Y}\bar R)(\bar Z, \bar X) + (\bar\nabla_{\bar Z}\bar R)(\bar X, \bar Y) = 0,$$

$$(1.4)' \qquad \bar g(\bar R(\bar X, \bar Y)\bar Z, \bar U) = \bar g(\bar R(\bar Z, \bar U)\bar X, \bar Y),$$

$$(1.5)' \qquad \bar g(\bar R(\bar X, \bar Y)\bar Z, \bar U) + \bar g(\bar R(\bar X, \bar Y)\bar U, \bar Z) = 0$$

for $\bar X, \bar Y, \bar Z, \bar U \in \mathfrak{X}(\bar M)$.

Let $f : (M, g) \to (\bar M, \bar g)$ be an isometric immersion. The second fundamental form of $f : (M, g) \to (\bar M, \bar g)$ is, by definition, a mapping $II : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(\bar M)$ given by

$$(1.6) \qquad II(X, Y) = \bar\nabla_{fX} fY - f\nabla_X Y,$$

where f denotes the differential map of f.

Let $\mathfrak{N}(f(M))$ denote the set of all vector fields normal to $f(M)$.

PROPOSITION 1.1. - *$II(X, Y) \in \mathfrak{N}(f(M))$ for any $X, Y \in \mathfrak{X}(M)$.*

PROOF. - From the definition of Riemannian connections,

$$\begin{aligned} 2\bar g(\bar\nabla_{fX} fY, fZ) &= fX\cdot\bar g(fY, fZ) + fY\cdot\bar g(fX, fZ) - fZ\cdot\bar g(fX, fY) + \bar g([fX, fY], fZ) \\ &\qquad + \bar g([fZ, fX], fY) - \bar g([fY, fZ], fX) \\ &= X\cdot g(Y, Z) + Y\cdot g(X, Z) - Z\cdot g(X, Y) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) \\ &= 2g(\nabla_X Y, Z) \\ &= 2\bar g(f\nabla_X Y, fZ). \end{aligned}$$

Hence we have

$$\bar g(\bar\nabla_{fX} fY \quad f\nabla_X Y, fZ) = 0$$

for any $X, Y, Z \in \mathfrak{X}(M)$. This implies that $II(X, Y)$ is normal to $f(M)$.

(Q. E. D.).

The following Proposition is fundamental.

PROPOSITION 1.2. - (*The equation of Gauss-Codazzi*).

$$\begin{aligned} \bar R(fX, fY)fZ &= fR(X, Y)Z + II(X, \nabla_Y Z) \\ &\quad - II(Y, \nabla_X Z) - II([X, Y], Z) \\ &\quad + \bar\nabla_{fX}\cdot II(Y, Z) - \bar\nabla_{fY}\cdot II(X, Z) \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$.

PROOF. - From the equation (1.6) we have

$$\begin{aligned}\overline{\nabla}_{fX}\overline{\nabla}_{fY}fZ &= \overline{\nabla}_{fX}f\nabla_Y Z + \overline{\nabla}_{fX}\cdot II(Y, Z)\\ &= f\nabla_X\nabla_Y Z + II(X, \nabla_Y Z) + \overline{\nabla}_{fX}\cdot II(Y, Z).\end{aligned}$$

Similarly we obtain

$$\overline{\nabla}_{fY}\overline{\nabla}_{fX}fZ = f\nabla_Y\nabla_X Z + II(Y, \nabla_X Z) + \overline{\nabla}_{fY}\cdot II(X, Z)$$

and

$$\overline{\nabla}_{[fX, fY]}fZ = f\nabla_{[X, Y]}Z + II([X, Y], Z).$$

These, together with the definition of curvature tensor fields, prove our assertion. (Q. E. D.).

An isometric immersion $f:(M, g)\to(\overline{M}, \overline{g})$ is said to be *minimal* at $x\in M$ provided that for one (and hence every) orthonormal frame $X_1, \ldots, X_n$ at x we have $\sum_{i=1}^{n} II(X_i, X_i) = 0$. f is said to be minimal if it is minimal at every point of M.

An isometric immersion $f:(M, g)\to(\overline{M}, \overline{g})$ is said to be *umbilic* at $x\in M$ if $II(X, X)$ has the same value for every unit vector X at x. f is said to be umbilic if it is umbilic at every point of M.

An isometric immersion $f:(M, g)\to(\overline{M}, \overline{g})$ is said to be *totally geodesic* if $II=0$.

Let (M, g) be a hypersurface of $(\overline{M}, \overline{g})$ with an isometric immersion f. Let ξ be the unit normal field to $f(M)$. By Proposition 1.1 we can write

$$II(X, Y) = H(X, Y)\cdot\xi.$$

H is a tensor field of type (0, 2) on M. We call H the *second fundamental tensor* of f. An isometric immersion f is umbilic if and only if $H = cg$, where c is a function on M.

$R(X, Y)$ (resp. $\overline{R}(\overline{X}, \overline{Y})$) defines, at each point of M(resp. $\overline{M}$), a linear transformation of the tangent space to M(resp. $\overline{M}$) at the point. We call them the *curvature transformations* of Γ and $\overline{\Gamma}$.

An isometric immersion $f:(M, g)\to(\overline{M}, \overline{g})$ is called an *invariant immersion* if, for any $X, Y\in\mathfrak{X}(M)$, the curvature transformation $\overline{R}(fX, fY)$ leaves the tangent space to $f(M)$ at each point invariant. M is called an *invariant submanifold* of $\overline{M}$.

An isometric immersion $f:(M, g)\to(\overline{M}, \overline{g})$ is called a *strongly invariant immersion* if, for any $\overline{X}, \overline{Y}\in\mathfrak{X}(\overline{M})$, the curvature transformation $\overline{R}(\overline{X}, \overline{Y})$ leaves the tangent space to $f(M)$ at each point invariant. M is called a *strongly invariant submanifold* of $\overline{M}$.

It is clear that a strongly invariant immersion is an invariant immersion.

PROPOSITION 1.3. - *An isometric immersion* $f:(M, g)\to(\bar{M}, \bar{g})$ *is an invariant immersion if and only if*

$$\bar{g}(\bar{R}(fX, fY)fZ, \xi)=0 \tag{1.7}$$

for $X, Y, Z\in\mathfrak{X}(M)$ *and* $\xi\in\mathfrak{N}(f(M))$.

PROPOSITION 1.4. - *An isometric immersion* $f:(M, g)\to(\bar{M}, \bar{g})$ *is a strongly invariant immersion if and only if*

$$\bar{g}(\bar{R}(fX, fY)fZ, \xi)=0, \tag{1.7}$$

$$\bar{g}(\bar{R}(fX, \xi)fZ, \xi')=0 \tag{1.8}$$

and

$$\bar{g}(\bar{R}(\xi, \xi')fZ, \xi'')=0 \tag{1.9}$$

for $X, Y, Z\in\mathfrak{X}(M)$ *and* $\xi, \xi', \xi''\in\mathfrak{N}(f(M))$.

PROPOSITION 1.5. - *Let* (M, g) *be a hypersurface of* $(\bar{M}, \bar{g})$. *Then* (M, g) *is an invariant hypersurface if and only if*

$$(\nabla_X H)(Y, Z)-(\nabla_Y H)(X, Z)=0$$

for $Y, Y, Z\in\mathfrak{X}(M)$.

§ 2. - Invariant immersions.

THEOREM 2.1. - *Every isometric immersion of a manifold into a manifold with constant curvature is an invariant immersion.*

PROOF. - Let $(\bar{M}, \bar{g})$ be a manifold with constant curvature and let $f:(M, g)\to(\bar{M}, \bar{g})$ be an isometric immersion. The curvature tensor field $\bar{R}$ of $(\bar{M}, \bar{g})$ is given by

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z}=k\{\bar{g}(\bar{Y}, \bar{Z})\bar{X}-\bar{g}(\bar{X}, \bar{Z})\bar{Y}\}$$

for $\bar{X}, \bar{Y}, \bar{Z}\in\mathfrak{X}(\bar{M})$, where k is a constant.

If we set $\bar{X}=fX$, $\bar{Y}=fY$, $\bar{Z}=fZ$ for $X, Y, Z\in\mathfrak{X}(M)$, then

$$\bar{R}(fX, fY)fZ=k\{g(Y, Z)fX-g(X, Z)fY\}.$$

This implies that $\bar{R}(fX, fY)fZ$ is tangent to $f(M)$. (Q. E. D.).

Conversely we have the following

THEOREM 2.2. – *If every isometrically immersed submanifold of $(\bar{M}, \bar{g})$ is an invariant submanifold, then $(\bar{M}, \bar{g})$ is a space of constant curvature.*

PROOF. – Let $f:(M, g)\to(\bar{M}, \bar{g})$ be an invariant immersion. Then we have

$$\bar{g}(\bar{R}(fX, fY)fX, \xi)=0$$

for any vectors X, Y tangent to M and any vector ξ normal to $f(M)$.

Since (M, g) and f are arbitrary, we may think of fX, fY and ξ as an arbitrary orthogonal triplet of vectors of $\bar{M}$. Now our assertion follows from the following lemma.

LEMMA (A. FIALKOW [1]). – *If*

$$\bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{X}, \bar{Z})=0$$

for every orthogonal triplet $\bar{X}$, $\bar{Y}$, $\bar{Z}\in T_{\bar{x}}(\bar{M})$ and for every point $\bar{x}\in\bar{M}$, *then* $(\bar{M}, \bar{g})$ *is a space of constant curvature.*

PROOF OF LEMMA. – The sectional curvature $\bar{K}(\bar{X}, \bar{Y})$ for the plane spanned by $\bar{X}$ and $\bar{Y}$ is given by

$$\bar{K}(\bar{X}, \bar{Y})=\frac{\bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{Y}, \bar{X})}{\bar{g}(\bar{X}, \bar{X})\bar{g}(\bar{Y}, \bar{Y})-(\bar{g}(\bar{X}, \bar{Y}))^2}.$$

Let $\bar{X}$, $\bar{Y}$, $\bar{Z}$ be an orthonormal triplet. Then $\bar{X}$, $\bar{Y}+\bar{Z}$, $\bar{Y}-\bar{Z}$ is an orthogonal triplet and hence we have

$$\bar{g}(\bar{R}(\bar{X}, \bar{Y}+\bar{Z})\bar{X}, \bar{Y}-\bar{Z})=0.$$

This implies

$$\bar{K}(\bar{X}, \bar{Y})=\bar{K}(\bar{X}, \bar{Z}).$$

Let $\bar{U}$ be any vector orthogonal to $\bar{X}$. Then $\bar{U}$ can be written as

$$\bar{U}=a\bar{Y}+b\bar{Z}$$

for some orthonormal triplet $\bar{X}$, $\bar{Y}$, $\bar{Z}$ and we have

$$\bar{K}(\bar{X}, \bar{U})=\bar{K}(\bar{X}, \bar{Y}).$$

This implies that the sectional curvature is the same for all sections which contain any given vector $\bar{X}$. Hence we have

$$\bar{K}(\bar{X},\ \bar{Y}) = \bar{K}(\bar{Y},\ \bar{Z}) = \bar{K}(\bar{Z},\ \bar{U})$$

for arbitrary $\bar{X}$, $\bar{Y}$, $\bar{Z}$ and $\bar{U}$. This, together with Theorem of F. SHUR, implies that $(\bar{M},\ \bar{g})$ is a space of constant curvature. (Q. E. D.).

PROPOSITION 2.3. - *Every totally geodesic submanifold is an invariant submanifold.*

PROOF. - This is clear from the equation of GAUSS-CODAZZI. (Q. E. D.).

PROPOSITION 2.4. - *Every connected invariant umbilic hypersurface has constant mean curvature.*

PROOF. - Let M be a connected manifold and $f:(M,\ g)\to(\bar{M},\ \bar{g})$ an invariant umbilic immersion. Let ξ be the unit normal vector field to $f(M)$.

If we set $H = \rho g$, then $\rho = h/n$, where h is the mean curvature and $n = \dim M$. From Proposition 1.5 we have

$$(X\rho)g(Y,\ Z) - (Y\rho)g(X,\ Z) = 0$$

for $X,\ Y,\ Z \in \mathfrak{X}(M)$. If we take $Y = Z$ orthogonal to X, then

$$X\rho = 0.$$

Since X is arbitrary and M is connected, ρ is a constant. Hence the mean curvature is constant. (Q. E. D.).

§ 3. - Strongly invariant immersions

PROPOSITION 3.1. - *If $f:(M,\ g)\to(\bar{M},\ \bar{g})$ is a strongly invariant immersion, then*

$$\bar{R}(fX,\ \xi)fZ = 0$$

for $X,\ Z \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{N}(f(M))$.

PROOF. - From the equation (1.4)′ and (1.7) we have

$$\bar{g}(\bar{R}(fX,\ \xi)fZ,\ fY) = 0$$

for $X,\ Y,\ Z \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{N}(f(M))$. This implies that $\bar{R}(fX,\ \xi)fZ$ is normal to $f(M)$. This, together with (1.8), implies that $\bar{R}(fX,\ \xi)fZ = 0$. (Q. E. D.).

PROPOSITION 3.2. – *If* $f:(M, g)\rightarrow(\bar{M}, \bar{g})$ *is a strongly invariant immersion, then*

$$\bar{R}(fX, \bar{Y})\xi = 0$$

for $X\in\mathfrak{X}(M)$, $\bar{Y}\in\mathfrak{X}(\bar{M})$ *and* $\xi\in\mathfrak{N}(f(M))$.

PROOF. – From (1.2)′ and Proposition 3.1, we have

$$\bar{R}(fX, fY)\xi = 0$$

for $X, Y\in\mathfrak{X}(M)$ and $\xi\in\mathfrak{N}(f(M))$. From (1.8) and (1.5)′ we have

$$\bar{g}(\bar{R}(fX, \xi)\xi', fY) = 0$$

and from (1.9) and (1.5)′ we have

$$\bar{g}(\bar{R}(fX, \xi)\xi', \xi'') = 0$$

for $X, Y\in\mathfrak{X}(M)$ and $\xi, \xi', \xi''\in\mathfrak{N}(f(M))$. This implies $\bar{R}(fX, \xi)\xi' = 0$. Hence we have $\bar{R}(fX, \bar{Y})\xi = 0$ for $X\in\mathfrak{X}(M)$, $\bar{Y}\in\mathfrak{X}(\bar{M})$ and $\xi\in\mathfrak{N}(f(M))$. (Q. E. D.).

PROPOSITION 3.3. – *If* $f:(M, g)\rightarrow(\bar{M}, \bar{g})$ *is a strongly invariant immersion, then*

$$\bar{R}(fX, \xi) = 0$$

for $X\in\mathfrak{X}(M)$ *and* $\xi\in\mathfrak{N}(f(M))$.

PROOF. – From (1.5)′ and (1.8) we have

$$\bar{g}(\bar{R}(fX, \xi)\xi', fZ) = 0.$$

From (1.4)′ and (1.9) we have

$$\bar{g}(\bar{R}(fX, \xi)\xi', \xi'') = 0.$$

Hence we have $\bar{R}(fX, \xi)\xi' = 0$. This, thogether with Proposition 3.1, implies $\bar{R}(fX, \xi)\bar{Y} = 0$ for $X\in\mathfrak{X}(M)$, $\bar{Y}\in\mathfrak{X}(\bar{M})$ and $\xi\in\mathfrak{N}(f(M))$. (Q. E. D.).

THEOREM 3.4. – *There exists no strongly invariant immersion of a manifold into a manifold with non-zero constant curvature.*

PROOF. – Let $(\bar{M}, \bar{g})$ be a manifold with non-zero constant curvature and let $f:(M, g)\rightarrow(\bar{M}, \bar{g})$ be a strongly invariant immersion. From (1.8) we have

$$\bar{g}(\bar{R}(fX, \xi)fX, \xi) = 0.$$

This implies that the sectional curvature for the section defined by fX and ξ is zero. Since $(\bar{M}, \bar{g})$ is of constant curvature, $(\bar{M}, \bar{g})$ must be flat.

(Q. E. D.).

Since a strongly invariant immersion is necessarily invariant immersion, Theorems 2.2 and 3.4 imply

THEOREM 3.5. - *If every isometrically immersed submanifold of* $(\bar{M}, \bar{g})$ *is a strongly invariant submanifold, then* $(\bar{M}, \bar{g})$ *is flat.*

It is clear that *every submanifold of a flat manifold is a strongly invariant submanifold.*

THEOREM 3.6. - *Let* (M, g) *be a strongly invariant umbilic hypersurface of* $(\bar{M}, \bar{g})$. *Then the scalar curvature of* (M, g) *is not less than that of* $(\bar{M}, \bar{g})$. *In particular, they are the same if and only if* (M, g) *is a totally geodesic hypersurface of* $(\bar{M}, \bar{g})$.

PROOF. - Let ξ be the unit normal field to $f(M)$.

The *Ricci tensor field* S of (M, g) is a tensor field of type $(0, 2)$ on M defined as follows:

$$S(X, Y) = \text{trace}\{Z \to R(Z, X)Y\}.$$

If $Z_1, \ldots, Z_n$ are orthonormal, then

$$S(X, Y) = \sum_{i=1}^{n} g(R(Z_i, X)Y, Z_i).$$

The tensor field σ of type $(1, 1)$ defined by $S(X, Y) = g(\sigma(X), Y)$ is called the *Ricci transformation* of (M, g). The Ricci tensor field $\bar{S}$ and the Ricci transformation $\bar{\sigma}$ of $(\bar{M}, \bar{g})$ is defined similarly.

From the equation of GAUSS-CODAZZI (Proposition 1.2) we have

$$\begin{aligned}&\bar{S}(fX, fY) - \bar{g}(\bar{R}(\xi, fX)fY, \xi)\\ &= S(X, Y) - \frac{h^2}{n} g(X, Y) + \frac{h^2}{n^2} g(X, Y),\end{aligned}$$

where h is the mean curvature of the immersion and $n = \dim M$.

This, together with Proposition 1.4, implies

$$S(X, Y) = \frac{n-1}{n} h^2 g(X, Y) + \bar{S}(fX, fY).$$

Hence we have

$$r = \frac{n-1}{n} h^2 + \bar{r} - \bar{S}(\xi, \xi),$$

where r and $\bar{r}$ are the scalar curvatures of (M, g) and $(\bar{M}, \bar{g})$ respectively. On the other hand, from (1.8) we have $\bar{S}(\xi, \xi) = 0$ and hence

$$r = \frac{n-1}{n} h^2 + \bar{r}.$$

This implies $r \geqq \bar{r}$, and $r = \bar{r}$ if and only if $h = 0$. (Q. E. D.)

§ 4. - Invariant Kählerian immersions

Let M be an even dimensional differentiable manifold and J an almost complex structure on M, that is, a tensor field of type (1, 1) on M satisfying $J^2 = -I$, where I denotes the field of identity endomorphisms.

A Riemannian metric g on M is called a Hermitian metric if the almost complex structure J on M is an isometry with respect to g.

A triplet (M, J, g) is said to be KÄHLERIAN if it satisfies

$$\nabla J = 0,$$

where ∇ denotes the covariant differentiation with respect to the Riemannian connection determined by g.

Let (M, J, g) and $(\bar{M}, \bar{J}, \bar{g})$ be two Kählerian manifolds. An isometric immersion $f : (M, J, g) \to (\bar{M}, \bar{J}, \bar{g})$ is said to be Kählerian if it satisfies

$$\bar{J} \circ f = f \circ J.$$

THEOREM 4.1. - *Every Kählerian submanifold of a Kählerian manifold with constant holomorphic curvature is an invariant submanifold.*

PROOF. - Let $(\bar{M}, \bar{J}, \bar{g})$ be a Kählerian manifold with constant holomorphic curvature. Then the curvature tensor field $\bar{R}$ of $(\bar{M}, \bar{J}, \bar{g})$ is given by

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = k\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \bar{\Omega}(\bar{Y}, \bar{Z})\bar{J}\bar{X} - \bar{\Omega}(\bar{X}, \bar{Z})\bar{J}\bar{Y} - 2\bar{\Omega}(\bar{X}, \bar{Y})\bar{J}\bar{Z}\}$$

for $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\bar{M})$, where k is a constant and $\bar{\Omega}$ is the fundamental 2-form of $(\bar{M}, \bar{J}, \bar{g})$, that is, $\bar{\Omega}(\bar{X}, \bar{Y}) = \bar{g}(\bar{J}\bar{X}, \bar{Y})$.

Let $f : (M, J, g) \to (\bar{M}, \bar{J}, \bar{g})$ be an Kählerian immersion.

If we set $\bar{X} = fX$, $\bar{Y} = fY$, $\bar{Z} = fZ$ for $X, Y, Z \in \mathfrak{X}(M)$, then

$$\begin{aligned} \bar{R}(fX, fY)fZ = k\{ & g(Y, Z)fX - g(X, Z)fY + \\ & + \Omega(Y, Z)fJX - \Omega(X, Z)fJY - 2\Omega(X, Y)fJZ\}, \end{aligned}$$

where Ω is the fundamental 2-form of (M, J, g).

This implies that $\bar{R}(fX, fY)fZ$ is tangent to $f(M)$ for $X, Y, Z \in \mathfrak{X}(M)$, that is, f is an invariant immersion. (Q. E. D.).

COROLLARY. - *Every Kählerian submanifold of a complex projective space with Fubini-Study metric is an invariant submanifold.*

Conversely we have the following

THEOREM 4.2. - *If every Kählerian submanifold of a Kählerian manifold $(\bar{M}, \bar{J}, \bar{g})$ is an invariant submanifold, then $(\bar{M}, \bar{J}, \bar{g})$ is a Kählerian manifold with constant holomorphic curvature.*

PROOF. - Let $f:(M, J, g) \to (\bar{M}, \bar{J}, \bar{g})$ be an invariant Kählerian immersion. Then we have

$$\bar{g}(\bar{R}(fX, \bar{J}fX)fX, \xi) = 0$$

for any vector X tangent to M and any vector ξ normal to $f(M)$.

Since (M, J, g) and f are arbitrary, we may think of fX and ξ as an arbitrary orthogonal couple of vectors of $\bar{M}$. Now our assertion follows from the following lemma.

LEMMA. - *If*

$$\bar{g}(\bar{R}(\bar{X}, \bar{J}\bar{X})\bar{X}, \bar{Y}) = 0$$

for every orthogonal couple $\bar{X}, \bar{Y} \in T_{\bar{x}}(\bar{M})$ and for every point $\bar{x} \in \bar{M}$, then $(\bar{M}, \bar{J}, \bar{g})$ is a Kählerian manifold with constant holomorphic curvature.

PROOF OF LEMMA. - Let $\bar{X}, \bar{Y}, \bar{J}\bar{X}, \bar{J}\bar{Y}$ be an orthonormal quadruplet of vectors at $\bar{x} \in \bar{M}$. Then $\bar{X} + \bar{Y}, \bar{J}\bar{X} - \bar{J}\bar{Y}$ is an orthogonal couple and hence we have

$$\bar{g}(\bar{R}(\bar{X} + \bar{Y}, \bar{J}\bar{X} + \bar{J}\bar{Y})\bar{X} + \bar{Y}, \bar{J}\bar{X} - \bar{J}\bar{Y}) = 0.$$

This implies

$$\bar{K}(\bar{X}, \bar{J}\bar{X}) = \bar{K}(\bar{Y}, \bar{J}\bar{Y}). \tag{4.1}$$

Let $\bar{U}, \bar{V}$ be an arbitrary couple of unit vectors at $\bar{x} \in \bar{M}$.

If $\{\{\bar{U}, \bar{V}\}\}$ (1) is a holomorphic section, that is, $\bar{J}\{\{\bar{U}, \bar{V}\}\} = \{\{\bar{U}, \bar{V}\}\}$, then it is clear that

$$\bar{K}(\bar{U}, \bar{J}\bar{U}) = \bar{K}(\bar{V}, \bar{J}\bar{V}).$$

If $\{\{\bar{U}, \bar{V}\}\}$ is not a holomorphic section, then we can take unit vectors $\bar{X} \in \{\{\bar{U}, \bar{J}\bar{U}\}\}^{\perp}$ (2) and $\bar{Y} \in \{\{\bar{V}, \bar{J}\bar{V}\}\}^{\perp}$ which determine a holomorphic section $\{\{\bar{X}, \bar{Y}\}\}$. Then (4.1) implies

$$\bar{K}(\bar{U}, \bar{J}\bar{U}) = \bar{K}(\bar{X}, \bar{J}\bar{X}) = \bar{K}(\bar{Y}, \bar{J}\bar{Y}) = \bar{K}(\bar{V}, \bar{J}\bar{V}).$$

(1) $\{\{\bar{U}, \bar{V}\}\}$ denotes the section determined by $\bar{U}$ and $\bar{V}$.

(2) $\perp$ denotes the orthogonal complement.

Hence we have proved that the holomorphic sectional curvature is the same for all holomorphic sections at $\bar{x} \in \bar{M}$. This, together with an analogy of Theorem of F. SHUR, implies that $(\bar{M}, \bar{J}, \bar{g})$ is a space of constant holomorphic curvature. (Q. E. D.).

THEOREM 4.2. - *There exists no strongly invariant Kählerian immersion of a Kählerian manifold with non-zero constant holomorphic curvature.*

PROOF. - Let $(\bar{M}, \bar{J}, \bar{g})$ be a Kählerian manifold with non-zero constant holomorphic curvature. Then the curvature tensor field $\bar{R}$ of $(\bar{M}, \bar{J}, \bar{g})$ is given by

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = k\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \bar{\Omega}(\bar{Y}, \bar{Z})\bar{J}\bar{X} - \bar{\Omega}(\bar{X}, \bar{Z})\bar{J}\bar{Y} - 2\bar{\Omega}(\bar{X}, \bar{Y})\bar{J}\bar{Z}\},$$

where k is a non-zero constant.

Let $f: (M, J, g) \to (\bar{M}, \bar{J}, \bar{g})$ be a strongly invariant Kählerian immersion. If we set $\bar{X} = fX$, $\bar{Y} = \xi$, $\bar{Z} = fZ$ for $X, Z \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{N}(f(M))$, then

$$\bar{R}(fX, \xi)fZ = k\{-g(X, Z)\xi + \bar{\Omega}(\xi, fZ)fJX - \Omega(X, Z)\bar{J}\xi - 2\bar{\Omega}(fX, \xi)fJZ\}.$$

$\bar{R}(fX, \xi)fZ$ is tangent to $f(M)$ if and only if $k = 0$. (Q. E. D.).

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