States of total pure radiation in general relativity (*)

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Summary - *In this paper we investigate and exhibit space-times which admit states of pure radiation in the sense of Lichnerowicz. In § 1 the notion of special total pure radiation is introduced, and in § 2 we derive the canonical line element for this type of radiation. An additional type of spacetime admitting radiation is considered in § 3. A class of singular integrable electromagnetic fields for the space-times of § 2 are constructed in § 4. The final section is concerned with the radiation condition proposed by Zakharov.*

Introduction

LICHNEROWICZ, in his study of gravitational and electromagnetic radiation in general relativity, has introduced the notion of total pure radiation at a point of space-time. In this paper we investigate this notion and exhibit space -times and classes of space-times for which the LICHNEROWICZ radiation conditions are satisfied. The concept of total pure radiation will be reviewed at the end of this introduction. In § 1 we consider a special subcase of total pure radiation in a conformally flat space-time, and in § 2 we exhibit all of the canonical forms for the line element. Some of these results were announced by us in a recent note [11] in Comptes Rendus Académie des Sciences, Paris. In § 3 we consider a class of space-times, which are not conformally flat, that admit states of total pure radiation.

Classes of integrable singular electromagnetic fields for the space-times of the first two sections are constructed in § 4. In the final section the radiation condition proposed by ZAKHAROV is shown to be consistent with the LICHNEROWICZ conditions for the state of special total pure radiation considered in § 1 and § 2.

Let V_4 be a four-dimensional differentiable manifold with a RIEMANNIAN metric $g_{\alpha\beta}(x^\lambda)$ of hyperbolic normal signature. For brevity we will call such a V_4 a space-time. LICHNEROWICZ, [6], has defined a state of total pure radiation at a point $x \in V_4$ if the RIEMANN-CHRISTOFFEL curvature tensor $R_{\alpha\beta,\lambda\mu}$ and the RICCI tensor $R_{\alpha\beta} \overset{\text{def}}{=} R^{\gamma}_{\alpha,\gamma\beta}$ satisfy the following three conditions for a real non-zero null vector $\vec{e}$:

(1) $$e^\alpha R_{\alpha\beta,\lambda\mu} = 0,$$

(2) $$e_\alpha R_{\beta\gamma,\lambda\mu} + e_\beta R_{\gamma\alpha,\lambda\mu} + e_\gamma R_{\alpha\beta,\lambda\mu} = 0,$$

(3) $$R_{\alpha\beta} = \tau\, e_\alpha e_\beta,$$

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where τ is a non-zero scalar function of the local coordinates x^α, (τ may be constant). The case $\tau = 0$, which will not be considered in this paper, is called pure gravitational radiation by LICHNEROWICZ [6]

Throughout this paper $\vec{e}$ will always denote a real non-zero null vector with components e^α or e_α, and is called a fundamental vector of $R_{\alpha\beta,\lambda\mu}$. The notation of LICHNEROWICZ [5], [6], and our previous paper [11] will be employed in this paper.

1. - Special total pure radiation.

In this section we will show that the LICHNEROWICZ radiation conditions (1)-(3) are closely related to the existence of a null parallel vector field in a conformally flat V_4. A conformally flat V_4 will be denoted by C_4. It will be recalled that the necessary and sufficient condition that V_4 be a C_4 is that the WEYL conformal curvature tensor vanish identically, i.e.

$$(4) \qquad C^\alpha_{\beta,\lambda\mu} \overset{\text{def}}{=} R^\alpha_{\beta,\lambda\mu} - \frac{1}{2}\left(\delta^\alpha_\lambda R_{\beta\mu} - \delta^\alpha_\mu R_{\beta\lambda} + g_{\beta\mu} R^\alpha_\lambda - g_{\beta\lambda} R^\alpha_\mu\right) - \frac{R}{6}\left(\delta^\alpha_\mu g_{\beta\lambda} - \delta^\alpha_\lambda g_{\beta\mu}\right) = 0,$$

where $R^\alpha_\alpha \overset{\text{def}}{=} R$.

THEOREM 1. - The LICHNEROWICZ radiation conditions are satisfied by any C_4 which admits a parallel null vector field.

PROOF. - Let V_4 admit the parallel null vector field $\vec{e}$, i.e.

$$(5) \qquad \nabla_\lambda e^\alpha = 0.$$

It is clear that the integrability conditions of (5) yield (1). If V_4 is a C_4 then (1) can be re-written in the form, LEVINE [2],

$$(6) \qquad e_\lambda T_{\beta\mu} - e_\mu T_{\beta\lambda} = 0,$$

where

$$(7) \qquad T_{\alpha\beta} \overset{\text{def}}{=} R_{\alpha\beta} - \frac{R}{3} g_{\alpha\beta},$$

which has a consequence

$$(8) \qquad T_{\alpha\beta} = \tau\, e_\alpha e_\beta,$$

where we may assume τ is a non-zero scalar factor of proportionality ($\tau = 0$ gives a flat space for which (1) (3) are trivially satisfied). On forming $g^{\alpha\beta} T_{\alpha\beta}$ and using (7) and (8) we find that $R = 0$, whence $R_{\alpha\beta} = \tau\, e_\alpha e_\beta$, and (3) is satisfied. It now follows from (4), (7), (8) that

$$\nabla_\gamma R_{\alpha\beta,\lambda\mu} = t_\gamma R_{\alpha\beta,\lambda\mu}, \tag{9}$$

where

$$t_\gamma = \frac{1}{\tau}\partial_\gamma \tau, \left(\partial_\gamma \equiv \frac{\partial}{\partial x^\gamma}\right). \tag{10}$$

The remaining radiation condition (2) is an obvious consequence of the second BIANCHI identity. End of proof.

The type of total pure radiation described in Theorem 1, i,e. $\vec{e}$ is an integrable parallel null vector field, and V_4 is a C_4, is called a state of special total pure radiation. It is worth noting that (9) is precisely the requirement that C_4 be one of a class of recurrent space-times. Recurrent space-times will be considered in § 3.

2. - Canonical line elements for special total pure radiation.

We now consider the general problem of the determination of canonical line elements of C_n (arbitrary signature) which admit a parallel null vector field. In the case that $n = 4$ and signature $+ \ldots$, by virtue of Theorem 1, this will give states of total pure radiation.

In [2] LEVINE has shown a C_n (not of constant curvature) can admit at most one linearly independent parallel vector field. If this vector field is null then from § 1 above, we may write

$$R_{\alpha\beta} = \tau\, e_\alpha e_\beta,\ R = 0, \tag{11}$$

where $e_\alpha = \partial_\alpha \lambda$ is the null field, λ is a (non-constant) scalar and $\tau \neq 0$ is a scalar. From (11) we may write

$$\nabla_\gamma(\rho R_{\alpha\beta}) = 0, \tag{12}$$

where ρ is a non-zero scalar. Spaces C_n (not of constant curvature) which satisfy (12) have been studied in detail by LEVINE and KATZIN, [3], [4], and will be denoted by C_n^*. The C_n^* consist of two types:

Type I. : $\rho =$ non-zero constant,

Type II. : $\rho \neq$ constant.

We now examine the canonical forms of the line elements of these two types.

TYPE I. - It is shown in [3], [4] that the line element for the Type I cases of C_n^* spaces can be reduced to one of the two canonical forms;

$$\Phi_1 = \frac{\sum_\alpha \varepsilon_\alpha (dx^\alpha)^2}{Q_1 + \varepsilon}, \quad \Phi_2 = \frac{\sum_\alpha \varepsilon_\alpha (dx^\alpha)^2}{Q_2 + M} \tag{13}$$

where Σ denotes the summation from 0 to $n-1$ on the indicated index, $\varepsilon_\alpha = \pm 1$, $\varepsilon = \pm 1$. In addition we have

$$\begin{cases} Q_1 \stackrel{\text{def}}{=} \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha x^\beta, \ (a_{\alpha\beta} = a_{\beta\alpha} = \text{const.}), \\ \sum_\alpha \varepsilon_\alpha a_{\alpha\beta} a_{\alpha\gamma} = 0, \ (\text{matrix } [a_{\alpha\beta}] \neq 0); \end{cases} \tag{14}$$

$$\begin{cases} Q_2 \stackrel{\text{def}}{=} \sum_{\alpha,\beta} \hat{a}_{\alpha\beta} x^\alpha x^\beta, \ (\hat{a}_{\alpha\beta} = \hat{a}_{\alpha\beta} = \text{const.}, \ \hat{a}_{\alpha 0} = \hat{a}_{\alpha 1} = 0), \\ \sum_\alpha \varepsilon_\alpha \hat{a}_{\alpha\beta} \hat{a}_{\alpha\gamma} = 0, \ (\text{matrix } [\hat{a}_{\alpha\beta}] \neq 0, \text{ for } \alpha, \beta = 2, \ldots, n-1), \\ M \stackrel{\text{def}}{=} x^0 + x^1; \ \varepsilon_0 = 1, \ \varepsilon_1 = -1 \ (\text{other } \varepsilon' s \text{ arbitrary sign}). \end{cases} \tag{14'}$$

For both canonical forms $R = 0$, and

$$R_{\alpha\beta} = \frac{n-2}{u} \partial_{\alpha\beta} u, \tag{15}$$

where $u^2 \stackrel{\text{def}}{=} Q_1 + \varepsilon$ for Φ_1, and $u^2 \stackrel{\text{def}}{=} Q_2 + M$ for Φ_2.

In case a Type I C_n^* space admits a parallel null vector field it follows from (11) and (15) that u must satisfy

$$(\partial_{\alpha\beta} u)(\partial_{\rho\sigma} u) - (\partial_{\alpha\sigma} u)(\partial_{\beta\rho} u) = 0. \tag{16}$$

For Φ_1, (16) requires that $a_{\alpha\beta} a_{\rho\sigma} - a_{\alpha\sigma} a_{\beta\rho} = 0$, which implies $a_{\alpha\beta}$ must be of the form $a_{\alpha\beta} = \tau_0 a_\alpha a_\beta$, where $\tau_0 \neq 0$ (a_α, τ_0 constants). Furthemore by (14) we must have $\sum_\alpha \varepsilon_\alpha (a_\alpha)^2 = 0$. Thus $Q_1 = \tau_0 L^2$ where $L \stackrel{\text{def}}{=} \sum_\alpha a_\alpha x^\alpha$; and the components e_α of the null parallel field will have the from

$$e_\alpha = \partial_\alpha \lambda = \frac{a_\alpha}{\tau_0 L^2 + \varepsilon}, \quad \lambda = \int \frac{dL}{\tau_0 L^2 + \varepsilon}. \tag{17}$$

It is easy to verify that by a coordinate transformation we can reduce L to the form $L = \hat{a}(x^0 + x^{n-1})$, whit $\varepsilon_0 = 1$, $\varepsilon_{n-1} = -1$, ($a = \text{const.}$).

For Φ_2 of (13), (16) implies that

$$\hat{a}_{\alpha\sigma} c_\beta c_\rho + \hat{a}_{\rho\beta} c_\alpha c_\sigma - \hat{a}_{\alpha\beta} c_\rho c_\sigma - \hat{a}_{\rho\sigma} c_\alpha c_\beta = 0, \tag{18}$$

where $c_0 = c_1 = 1$, $c_2 = ... = c_{n-1} = 0$. If we put $\alpha = \beta = 0$ in (18) we find that $\hat{a}_{\rho\sigma} = 0$, which is a contradiction to the matrix condition on $[\hat{a}_{\alpha\beta}]$ of (14)'. Hence the type Φ_2 does not admit a parallel null vector field. Hence the line element of the only Type I C_n^* which admits such a field can be given the canonical form

$$\Phi_1 = \frac{\Sigma\, \varepsilon_\alpha (dx^\alpha)^2}{\tau_0(x^0 + x^{n-1})^2 + \varepsilon}, \qquad \left(\begin{array}{l} \varepsilon_\alpha^2 = 1,\ \varepsilon^2 = 1 \\ \varepsilon_0 = 1,\ \varepsilon_{n-1} = -1 \\ \tau_0 \neq 0 \text{ is const.} \end{array} \right). \tag{13'}$$

(the constant $\hat{a}$ of L has been absorbed in the τ_0).

TYPE II. - Since the parallel null vector e_α must satisfy $\nabla_\alpha e_\beta = 0$ it follows that e_α satisfies KILLING'S equation for a motion in C_n^*. We choose our local coordinate system (x^α) in C_n^* such that

$$\Phi = \frac{\Sigma\, \varepsilon_\alpha (dx^\alpha)^2}{u^2},$$

with $e^{-2\sigma} \overset{\text{def}}{=} u^2$. From LEVINE [1] we have that the associated contravariant components e_α satisfy

$$\varepsilon_\alpha \frac{\partial e}{\partial x^\beta} + \varepsilon_\beta \frac{\partial e^\beta}{\partial x^\alpha} = 0, \ (\alpha \neq \beta;\ \text{no summing}), \tag{19}$$

$$\sum_\beta e^\beta \frac{\partial \sigma}{\partial x^\beta} + \frac{\partial e^\beta}{\partial x^\alpha} = 0, \ (\alpha \text{ not summed}). \tag{20}$$

From [1] we know that the e^α which satisfy (19) and (20) must be of the form

$$e^\alpha = b^\alpha + \mu x^\alpha + \sum_\beta b_\beta^\alpha x^\beta - \frac{1}{2} \varepsilon_\alpha a_\alpha U, \tag{21}$$

where

$$\mu \overset{\text{def}}{=} a + \sum_\alpha a_\alpha x^\alpha, \tag{21'}$$

$$\varepsilon_\alpha b_\beta^\alpha + \varepsilon_\beta b_\alpha^\beta = 0, \ (\text{no summation}), \tag{22}$$

$$U \overset{\text{def}}{=} \sum_\alpha \varepsilon_\alpha (x^\alpha)^2,$$

and a, a_α, b^α, b^α_β are constants. The e^α of (21) satisfy (19) identically; (20) can be re-written as

(23) $$\sum_\alpha e^\alpha \frac{\partial u}{\partial x^\alpha} = \mu u, \ \mu = \frac{\partial e^0}{\partial x^0} = \dots = \frac{\partial e^{n-1}}{\partial x^{n-1}} .$$

From $e_\alpha = \partial_\alpha \lambda$ we may write $e^\alpha = \varepsilon_\alpha u^2 \partial_\alpha \lambda$, which allows us to write

(24) $$\theta_\alpha \overset{\text{def}}{=} \varepsilon_\alpha e^\alpha = u^2 \partial_\alpha \lambda = \varepsilon_\alpha x^\alpha \mu + \sum_\beta b_{\alpha\beta} x^\beta + b_\alpha - \frac{1}{2} a_\alpha U,$$

where $b_{\alpha\beta} = -b_{\beta\alpha} \overset{\text{def}}{=} \varepsilon_\alpha b^\alpha_\beta$; $b_\alpha \overset{\text{def}}{=} \varepsilon_\alpha b^\alpha$. From (24) we write

(25) $$\partial_\alpha \lambda = u^{-2} \theta_\alpha ,$$

so that

(26) $$\partial_{\alpha\beta} \lambda = u^{-2} (\partial_\beta \theta_\alpha) - 2u^{-3} \theta_\alpha (\partial_\beta u).$$

By (19) we find

(27) $$\partial_{\alpha\beta} \lambda = -u^{-1} ((\partial_\alpha u)(\partial_\beta \lambda) + (\partial_\alpha \lambda)(\partial_\beta u)), \ (\alpha \neq \beta),$$

which together with (26) implies

(28) $$\partial_\beta \theta_\alpha = u^{-1} (\theta_\alpha \partial_\beta u - \theta_\beta \partial_\alpha u), \ (\alpha \neq \beta).$$

This last equation shows that

(29) $$\theta_\alpha \partial_\gamma \theta_\beta + \theta_\beta \partial_\alpha \theta_\gamma + \theta_\gamma \partial_\beta \theta_\alpha = 0,$$
$$\partial_\beta \theta_\alpha = -\partial_\alpha \theta_\beta , \qquad (\alpha, \ \beta, \ \gamma \neq).$$

From (25) we define $X_{\alpha\beta}(\lambda) \overset{\text{def}}{=} \theta_\beta \partial_\alpha \lambda - \theta_\alpha \partial_\beta \lambda = 0$. One may directly verify that the integrability conditions $(X_{\alpha\beta}, X_{\gamma\delta})\lambda = 0$ are satisfied by use of (29).

If we now subsitute the expression θ_α given in (24) into (29) and require that the resulting equations are to be identically satisfied in the x^{α}'s, we obtain the following conditions on the constants a, a_α, b_α, $b_{\alpha\beta}$:

(30) $$b_\alpha b_{\beta\gamma} + b_\beta b_{\gamma\alpha} + b_\gamma b_{\alpha\beta} = 0,$$

(31) $$b_{\alpha\rho} b_{\beta\sigma} + b_{\beta\rho} b_{\sigma\alpha} + b_{\sigma\rho} b_{\alpha\beta} = 0,$$

(32) $$a\, b_{\alpha\beta} + a_\alpha b_\beta - a_\beta b_\alpha = 0,$$

(33) $$a_\alpha b_{\beta\gamma} + a_\beta b_{\gamma\alpha} + a_\gamma b_{\alpha\beta} = 0.$$

A detailed analysis of (30)-(33) based on the consideration of the two subcases $\mu \neq$ const., and $\mu =$ const. (see (21)′), results in the following four types of solutions:

(A)
$$b_\alpha = a_\alpha b - a B_\alpha ,$$
$$b_{\alpha\beta} = a_\alpha B_\beta - a_\beta B_\alpha ,$$

where a, b, a_α, B_α are arbitrary constants such that $a_1 \neq 0$, $B_1 = 0$.

(B)
$$a_\alpha = 0, \; b_{\alpha\beta} = 0,$$

and a, b_α are arbitrary, with $a \neq 0$.

(C)
$$a = a_\alpha = 0, \; b_{\alpha\beta} = b_\alpha \widehat{B}_\beta - b_\beta \widehat{B}_\alpha ,$$

where b_α, $\widehat{B}_\alpha$ are arbitrary, with $b_0 \neq 0$, $\widehat{B}_0 = 0$.

(D)
$$a = a_\alpha = b_\alpha = 0, \; b_{\alpha\beta} = B_\alpha \widehat{B}_\beta - B_\beta \widehat{B}_\alpha ,$$

where b_α, $\widehat{B}_\alpha$ are arbitrary, with $B_0 = \widehat{B}_1 = 0$, $B_1 = -\widehat{B}_0 \neq 0$.

The corresponding θ_α for these cases are given by

(A)
$$\theta_\alpha = \mu(\varepsilon_\alpha x^\alpha - B_\alpha) + a_\alpha \left(b - \frac{1}{2} U + \sum_\rho B_\rho x^\rho \right).$$

(B)
$$\theta_\alpha = a(\varepsilon_\alpha x^\alpha) + b_\alpha .$$

(C)
$$\theta_\alpha = b_\alpha + \sum_\rho (b_\alpha \widehat{B}_\rho - b_\rho \widehat{B}_\alpha) x^\rho .$$

(D)
$$\theta_\alpha = B_\alpha (\sum_\rho \widehat{B}_\rho x^\rho) - \widehat{B}_\alpha (\sum_\rho B_\alpha x^\rho).$$

The requirement that e^α is a null vector implies that $\sum_\rho \varepsilon_\alpha \theta_\alpha^2 = 0$.

This condition imposes the following restrictions on our cases:

(A)
$$\mathrm{a} = -\sum_\alpha \varepsilon_\alpha a_\alpha B_\alpha , \; \beta = -\frac{1}{2} \sum_\alpha \varepsilon_\alpha B_\alpha^2 ,$$

$$\sum_\alpha \varepsilon_\alpha a_\alpha^2 = 0 \text{ with } a_0 \neq 0, \; B_0 = 0, \; \mu = \sum_\alpha a_\alpha (x_\alpha - \varepsilon_\alpha B_\alpha)$$

$$\theta_\alpha = \varepsilon_\alpha (x^\alpha - \varepsilon_\alpha B_\alpha) \mu - \frac{1}{2} a_\alpha \sum \varepsilon_\beta (x^\beta - \varepsilon_\beta B_\beta)^2 .$$

(B) is excluded since we must have $a = 0$.

(C) $$\sum_\alpha \varepsilon\, b_\alpha^2 = 0,\ \Sigma\, \varepsilon_\alpha \widehat{B}_\alpha^2 = 0,\ \sum_\alpha \varepsilon_\alpha b_\alpha \widehat{B}_\alpha = 0,\ (b_0 \neq 0,\ \widehat{B}_0 = 0),$$

$$\theta_\alpha = (M+1)b_\alpha - \widehat{B}_\alpha K;\ K \overset{\text{def}}{=} \sum_\alpha b_\alpha x^\alpha,\ M \overset{\text{def}}{=} \sum_\alpha \widehat{B}_\alpha x^\alpha.$$

(D) $$\sum_\alpha \varepsilon_\alpha B_\alpha^2 = 0,\ \sum_\alpha \varepsilon_\alpha \widehat{B}_\alpha^2 = 0,\ \sum_\alpha \varepsilon_\alpha B_\alpha \widehat{B}_\alpha = 0,\ \theta_\alpha = M B_\alpha - \widehat{K}\widehat{B}_\alpha,$$

$$\widehat{K} \overset{\text{def}}{=} \sum_\alpha B_\alpha x^\alpha, \text{ and } B_0 = B_1 = 0,\ B_1 = -\widehat{B}_0 \neq 0.$$

These results can be simplified by use of appropriate coordinate transformations.

For case (A) consider the coordinate transformation $y^\alpha = x^\alpha - \varepsilon_\alpha B_\alpha$. In the y-coordinates replace a_α by $2a_\alpha$. Then, after changing y^α to x^α, we find $\theta_\alpha = \mu\varepsilon_\alpha x^\alpha - a_\alpha U$, where $\mu \overset{\text{def}}{=} \sum_\alpha a_\alpha x^\alpha$. Hence we may write

$$\theta_\alpha = u^2 \partial_\alpha \lambda = \mu^2 \partial_\alpha V,\ V \overset{\text{def}}{=} \mu^{-1} U. \tag{34}$$

This implies $(\partial_\alpha \lambda)(\partial_\beta V) - (\partial_\beta \lambda)(\partial_\alpha V) = 0$, so that we may write $\lambda = F_1(V)$, and by (34),

$$\frac{1}{u^2} = \frac{1}{\mu^2}\frac{dF_1}{dV}. \tag{35}$$

Next we make the inversion $x^\alpha = y^\alpha / W$, where $W \overset{\text{def}}{=} \sum_\alpha \varepsilon_\alpha (y^\alpha)^2$. In the new coordinates (after changing y^α to x^α) we find that

$$\frac{1}{u} = \frac{1}{\mu^2}\frac{dF_1}{d(\mu^{-1})},\ (\mu = \Sigma\, a_\alpha x^\alpha). \tag{36}$$

In Case (C) we define $Y \overset{\text{def}}{=} K/(M+1)^2$ so that $\theta_\alpha = (M+1)^2 \partial_\alpha Y$.

As in case (A) we may express λ in the from $\lambda = F_2(Y)$, and

$$\frac{1}{u^2} = \frac{1}{(M+1)^2}\frac{dF_2}{dY}. \tag{37}$$

For Case (D), put $X \overset{\text{def}}{=} K/M$, so $\theta_\alpha = M^2 \partial_\alpha X$, giving $\lambda = F_3(X)$, $u^{-2} = M^{-2}(dF_3/dX)$.

In (C) we may have $M = 0$, but $K \neq 0$. In (D) we have $K \neq 0$, $M \neq 0$.

An inspection of cases (A), (C), (D) shows that (A) is a special case of (C) with $M = 0$. Furthemore, both case (C) and case (D) can be represented in the form $u=(M+\gamma)G(Z)$, where $Z \stackrel{\text{def}}{=} K/(M+\gamma)$, with $\gamma=0$ or $\gamma=1$; $K \stackrel{\text{def}}{=} \sum_\alpha b_\alpha x^\alpha$, $M \stackrel{\text{def}}{=} \sum_\alpha B_\alpha x^\alpha$, with $b_0 = B_1 = 0$, $b_1 = -B_0 \neq 0$. If $M=0$ we may write

$$u = G(K) = G(Z); \tag{38}$$

if $M \neq 0$, $\gamma = 1$, a translation will change u to the form

$$u = M\,G(K/M) = M\,G(Z). \tag{39}$$

In both cases we must have $\sum_\alpha \varepsilon_\alpha b_\alpha^2 = 0$, $\sum_\alpha \varepsilon_\alpha B_\alpha^2 = 0$, $\sum_\alpha \varepsilon_\alpha b_\alpha B_\alpha = 0$. In the first case, (38), a coordinate transformation can reduce Z to the form $Z=\gamma(x^0 + x^{n-1})$, with $\varepsilon_0 = 1$, $\varepsilon_{n-1} = -1$. Finally, by a dilation $y^\alpha = \gamma x^\alpha$ we obtain $u = \gamma\, G(Z)$, $Z = x^0 + x^{n-1}$.

Thus for C_n^* of Type II we have the two canonical forms

$$\Phi_3 = \frac{\sum_\alpha \varepsilon_\alpha (dx^\alpha)^2}{\gamma^2\, G^2(Z)}, \quad Z = x^0 + x^{n-1}, \ \varepsilon_0 = 1, \ \varepsilon_{n-1} = -1; \tag{40}$$

and

$$\Phi_4 = \frac{\sum_\alpha \varepsilon_\alpha (dx^\alpha)^2}{M^2\, G^2(Z)}, \tag{41}$$

with $Z = K/M$, where $K = \gamma x^1 + b_2 x^2 + \dots + b_{n-1} x^{n-1}$, $M = -\gamma x^2 + B_2 x^2 + \dots + B_{n-1} x^{n-1}$, $(\gamma \neq 0)$, and $\varepsilon_0 \gamma^2 + \sum_{\alpha=2}^{n-1} \varepsilon_\alpha b_\alpha^2 = 0$, $\varepsilon_1 \gamma^2 + \sum_{\alpha=2}^{n-1} \varepsilon_\alpha B_\alpha^2 = 0$, $\sum_\alpha \varepsilon_\alpha b_\alpha B_\alpha = 0$.

For both of these Type II forms, (40), (41), we may easily verify that

$$R_{\alpha\beta} = (n-2) G^3 \frac{d^2 G}{dZ^2} \frac{\partial\lambda}{\partial x^\alpha} \frac{\partial\lambda}{\partial x^\beta}, \tag{42}$$

where $\partial\lambda/\partial x^\alpha = G^{-2}\, \partial Z/\partial x^\alpha$ is a parallel null vector field, $(\lambda = \int G^{-2}\, dZ)$.

To avoid the Type I C_n^* spaces we must have

$$G^3 \frac{d^2 G}{dZ^2} \neq \text{const}. \tag{43}$$

If $n = 4$ and $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2\ \varepsilon_3 = -1$, one can show that Φ_4 is not possible but Φ_3 will exist as a canonical form for Type II C_n^* space-times.

By Theorem 1 we may summarize the results of this section as follows.

THEOREM 2. - The line element of a C_4^* admitting a state of special total pure radiation can be given the canonical form

$$\Phi_1 = \frac{(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2}{\tau_0(x^0 + x^3)^2 + \varepsilon}, \quad (C_4^* \text{ of Type I}),$$

$$\Phi_3 = \frac{(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2}{\gamma^2 G^2(x^0 + x^3)}, \quad (C_4^* \text{ of Type II}).$$

In Φ_1, τ_0 is a non-zero constant, $\varepsilon = \pm 1$. In Φ_3, γ is a non-zero constant, and G satisfies (43) with $Z = x^0 + x^3$. (Note that although Φ_1 is of the form Φ_3 the corresponding G does not satisfy (43)).

3. - Radiation in other space-times.

In addition to the states of radiation indicated in the first two sections there exist radiation solutions in V_4 which need not be conformally flat. An example is given in the theorem to follow.

THEOREM 3. - The LICHNEROWICZ radiation conditions are satisfied for a V_4 wich is a non-special, symmetric, recurrent space (with proper signature).

PROOF. - Denote a V_4 of the type stated in Theorem 3 by K_4^*. Then a K_4^* satisfies the following conditions [7; p. 152],

$$\nabla_\gamma R_{\alpha\beta,\lambda\mu} = 0 \tag{44}$$

$$k_\alpha R_{\beta\gamma,\lambda\mu} + k_\beta R_{\gamma\alpha,\lambda\mu} + k_\gamma R_{\alpha\beta,\lambda\mu} = 0, \tag{45}$$

where k_α is a non-zero vector. In addition, a K_4^* admits a null parallel vector field e_α such that $e_\alpha = \varphi k_\alpha$ for some non-zero scalar φ [7; p. 173]. Hence conditions (1) and (2) are satisfied.

A coordinate system can be chosen locally so that the metric of K_4^* has the form [7; pp. 176, 182]

$$ds^2 = \psi (dx^0)^2 + \sum_{\alpha,\beta=1}^{2} k_{\alpha\beta}\, dx^\alpha dx^\beta + 2dx^0 dx^3, \tag{46}$$

where $\psi = \sum_{\alpha,\beta=1}^{2} a_{\alpha\beta} x^\alpha x^\beta$, $|k_{\alpha\beta}| \neq 0$, $|a_{\alpha\beta}| \neq 0$, and $a_{\alpha\beta}, k_{\alpha\beta}$ are constants.

In this coordinate system we have $R_{00} = A \overset{\text{def}}{=} k^{\alpha\beta} a_{\alpha\beta}$, [8, p. 57; 7, p. 179], all other $R_{\alpha\beta} = 0$. In addition, in this coordinate system, $e^\alpha = \delta^\alpha_3$, so $e_\alpha = \delta_{\alpha 0}$ [7, p. 176]. It follows we may write $R_{\alpha\beta} = A\, e_\alpha e_\beta$ (in any coordinate system), and hence condition (3) is satisfied (case $A=0$ is exluded as this would give a flat space). It is easily shown the $a_{\alpha\beta}$ and $k_{\alpha\beta}$ of metric (46) can always be chosen to give the hyperbolic normal signature.

4. - Singular electromagnetic fields

For each state of total pure radiation we can construct a singular electromagnetic field by requiring that the source-free MAXWELL equations

$$\nabla_\alpha F^{\alpha\beta} = 0, \tag{47}$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \tag{48}$$

be satisfied in addition to the LICHNEROWICZ conditions (1), (3). It is well known, [6], that $F_{\alpha\beta}$ is singular with fundamental vector $\vec{k}$ if and only if

$$F_{\alpha\beta} = k_\alpha m_\beta - m_\alpha k_\beta \tag{49}$$

with

$$k_\alpha k^\alpha = k_\alpha m^\alpha = 0, \;\; m^\alpha m_\alpha = -1. \tag{50}$$

To obtain the states of total pure radiation in a C_4 we identify the fundamental vectors of $F_{\alpha\beta}$ and $R_{\alpha\beta,\lambda\mu}$, $\vec{k} = \vec{e}$, and write (47), (48) in a C_4 whose metric is written in the general form

$$\Phi = \frac{\eta_{\alpha\beta}\, dx^\alpha\, dx^\beta}{u^2}, \tag{51}$$

where $\eta_{\alpha\beta}$ is the MINKOWSKI metric, $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$. This form includes the canonical forms discusses in § 2.

In C_4, (47) simplifies to

$$\eta^{\rho\sigma}\, \partial_\rho F_{\alpha\beta} = 0, \tag{52}$$

and as in [11] we may choose the $\vec{e}$ and $\vec{m}$ to have the components

$$e_\alpha = e_0(\lambda)\, (\delta^0_\alpha + \varepsilon \delta^1_\alpha), \tag{53}$$

$$m_\beta = m_0(x^\alpha)\, (\delta^0_\beta + \varepsilon \delta^1_\beta) + m_2(x^\alpha)\, \delta^2_\beta + m_3\, (x^\alpha) \delta^3_\beta, \tag{54}$$

where $e_0(\lambda)$ is a function of $\lambda \overset{\text{def}}{=} x^0 + \varepsilon x^1$ of class $C^k (k \geq 1)$, $\varepsilon = \pm 1$, and m_0, m_2, and m_3 are arbitrary functions of x^α of class $C^k (k \geq 1)$ such that

$$(m_2)^2 + (m_3)^2 = \frac{1}{u^2}, \tag{55}$$

In a previous paper, [12], we have shown how to construct a class of non-integrable singular electromagnetic fields in a C_4 with a metric of the form (51). By omitting the requirement that the field be non-integrable (i.e. Lemma B of [12]) and identifying the fundamental vectors of $F_{\alpha\beta}$ and $R_{\alpha\beta,\lambda\mu}$ the results of [12] allow us to state the following:

THEOREM 4· - Let $\vec{e}$ be a fundamental vector of both $F_{\alpha\beta}$ and $R_{\alpha\beta,\lambda\mu}$. For a state of total pure radiation in C_4 with metric (51) a class of integrable singular electromagnetic fields is given by a holomorphic function $F(\mathfrak{z};\lambda)$, where $\mathfrak{z} \stackrel{\text{def}}{=} x^3 + ix^2$. Furthermore (53) and (54) may now be written

$$e_\alpha = \varepsilon u\,|F|\,(\delta^0 + \varepsilon\delta^1),$$

$$m_\beta = m_0(x^\alpha)\,(\delta^0_\beta + \varepsilon\delta^1_\beta) + \frac{\varepsilon}{u|F|}\,[(Re\;F)\,\delta^2_\beta + (Im\;F)\,\delta^3_\beta].$$

Using the well know expressions of $\vec{E}$, $\vec{H}$, the momentum tensor $\tau_{\alpha\beta}$, the energy density W, and the POYNTING vector $\vec{P}$, we find that

$$E^\alpha = -u^{-5}[(Re\;F)\delta^\alpha_2 + (\mathrm{Im}\;F)\delta^\alpha],$$

$$H^\alpha = u^{-5}[Im\;F)\delta^\alpha_2 - (Re\;F)\delta^\alpha_3],$$

$$\tau_{\alpha\beta} = u^2|F|^2(\delta^0_\alpha\delta^0_\beta + \delta^1_\alpha\delta^1_\beta + \varepsilon\delta^0_\alpha\delta^1_\beta),$$

$$W = u^4|F|^2,\quad P_\alpha = \varepsilon u\;|F|^2\,\delta^1_\alpha.$$

5. - Zakharov's radiation condition.

In a recent paper, [9], V. D. ZAKHAROV has proposed the equation

$$\Box R_{\alpha\beta,\lambda\mu} = 0, \tag{56}$$

where $\Box \stackrel{\text{def}}{=} g^{\rho\sigma}\nabla_\rho\nabla_\sigma$, serve as a general criterion for the existence of gravitational radiation. The explicit expression ($n = 4$)

$$\begin{aligned}\Box R_{\alpha\beta,\lambda\mu} = {} & \nabla\;\;\nabla_\mu R_{\beta\lambda} + \nabla_\beta\nabla_\lambda R_{\alpha\mu} - \nabla_\alpha\nabla_\lambda R_{\alpha\mu} - \nabla_\beta\nabla_\mu R_{\alpha\lambda} \\ & + R_{\alpha\sigma}R^\sigma_{\beta,\lambda\mu} - R_{\beta\sigma}R^\sigma_{\alpha,\lambda\mu} \\ & - 2(R^\rho_{\alpha,\sigma\mu}R^\sigma_{\lambda,\rho\beta} - R^\rho_{\alpha,\sigma\lambda}R^\sigma_{\mu,\rho\beta}) - R^\sigma_{\rho,\alpha\beta}R^\rho_{\sigma,\lambda\mu}\,,\end{aligned} \tag{57}$$

can be easily derived by using the RICCI and BIANCHI identities. It is clear that (57) is a complicated conditiou on $R_{\alpha\beta}$ and $R_{\alpha\beta,\lambda\mu}$. If V_4 is an EINSTEIN space (57) simplifies considerably, and this case was originally investigated by ZAKHAROV, [9], and more recently by ZUND and MAHER, [13], who used the VAN DER WAERDEN spinor formalism.

We show here that ZAKHAROV'S condition (56) is always satisfied for a state of special total pure radiation (see § 1).

Consider then a C_4 admitting a parallel null vector field e_α. Then (9) is satisfied, where we may assume $t_\gamma \neq 0$ ($t_\gamma = 0$ shows (56) satisfied trivially). Also, we have $R_{\alpha\beta} = \tau e_\alpha e_\beta$, and $R=0$. Hence the well known condition in a C_4,

$$R_{\alpha\beta\gamma} \overset{\text{def}}{=} \nabla_\gamma R_{\alpha\beta} - \nabla_\beta R_{\alpha\gamma} + \frac{1}{6}(g_{\alpha\gamma}\,\partial_\beta R - g_{\alpha\beta}\,\partial_\gamma R) = 0,$$

reduces to $\nabla_\gamma R_{\alpha\beta} = \nabla_\beta R_{\alpha\gamma}$. This implies

$$\partial_\alpha \tau = \rho\, e_\alpha, \tag{58}$$

where ρ is a scalar factor.

From (9), (10) we derive

$$\square\, R_{\alpha\beta,\lambda\mu} = g^{\gamma\delta}[\tau^{-1}\nabla_\gamma\nabla_\delta\tau - \tau^{-2}(\partial_\gamma\tau)(\partial_\delta\tau)]\,R_{\alpha\beta,\lambda\mu}. \tag{59}$$

By (58), $g^{\gamma}(\partial_\gamma\tau)(\partial_\delta\tau) = 0$ (since e_α is null). Since $\nabla_\gamma\nabla_\delta\tau = \nabla_\delta\nabla_\gamma\tau$, we find by use of (58), that $\partial_\alpha\rho = \mu e_\alpha$ for some scalar factor μ. Hence $g^{\gamma\delta}(\nabla_\gamma\nabla_\delta\tau) = \mu(g^{\gamma\delta}e_\gamma e_\delta) = 0$, and thus the right side of (59) is zero.

This gives us

THEOREM 5. - *For a state of special total pure radiation* ZAKHAROV'S *condition is always satisfied.*

In general it is not to be expected that a state of total pure radiation, in the sense of LICHNEROWICZ, will satisfy (56).

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