On the boundedness of the solutions of the equation $\dddot{x} + a\ddot{x} + f(x)\dot{x} + g(x) = p(t)$.

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Summary. - *In this paper my previous result* [1] *on the boundedness of solutions of* (1.1.1) *is fackled by use of a suitably chosen Liapounov function. This fresh approach leads to a more direct proof of the boundedness Theorem and makes for substantial reduction in each of my previous conditions on f and g.*

1. - Introduction.

1.1. - We shall be concerned here with the boundedness of the solutions of the differential equation

$$\dddot{x} + a\ddot{x} + f(x)\dot{x} + g(x) = p(t) \tag{1.1.1}$$

in which a is a constant and the functions f, g and p, dependent only on the arguments shown, are such that $f(x)$, $g'(x)$ and $p(t)$ are continuous for all x, t.

It was shown in a previous paper [1] that if $a > 0$ and if further (I) $g(0) = 0$, $g(x)/x \geq \delta_0 > 0\,(x \neq 0)$, (II) there are constants $\delta_1 > 0$, $\delta_2 > 0$ such that $a\delta_1 - \delta_2 > 0$ and such that $f(x) \geq \delta_1$ and $g'(x) \leq \delta_2$ for all x, and (III) $P(t) \equiv \int_0^t p(\tau\, d(\tau)$ satisfies

$$| P(t) | \leq A_0 < \infty \quad \text{for all } t \text{ considered,} \tag{1.1.2}$$

then every solution $x(t)$ of (1.1.1) ultimately satisfies

$$| x(t) | \leq D_1, \qquad | \dot{x}(t) | \leq D_1, \qquad | \ddot{x}(t) | \leq D_1, \tag{1.1.3}$$

where D_1 is a constant whose magnitude depends only on δ_0, δ_1, δ_2, A, f and g. In a subsequent note [2] it was pointed out, following a private comunication from Professor Pachale, that the proof given in [1] contains an important flaw, but that the defect can be rectified, if, for example, p were subjected to a further condition, namely:

$$| p(t) | \leq A_1 < \infty \quad \text{for all } t \text{ considered.} \tag{1.1.4}$$

In view of the fact that, for the special case $f \equiv$ constant, the result (1.1.3) can be proved (see, for example, [3], [4]) subject to one condition only on p, namely (1.1.4), one is naturally led to ask whether the use of both condition (1.1.2) and (1.1.4), which is apparently quite basic to my methods in [1], is in fact essential for the validity of (1.1.3) in the case when f is not necessarily constant.

One reason why the answer to the above question has proved elusive so far has been the difficulty in constructing a suitable Liapounov function: the function $W = W(x, y, z)$ used in [1], for instance, gives only a $\dot{W}$ (see Lemma 2 of [1]) satisfying only

$$\dot{W} \leq -\delta^* < 0, \qquad \text{if} \quad x^2(t) + y^2(t) \quad \text{is sufficiently large,}$$

so that the well known Yoshizawa-type technique which would require a result of the form

$$\dot{W} \leq -\delta^* < 0 \qquad \text{if} \quad x^2(t) + y^2(t) + z^2(t) \quad \text{is sufficiently large,} \tag{1.1.5}$$

is inapplicable here. The main object of the present paper is to give details of a fresh approach to the problem, involving the use of a combination of W with some other suitably chosen function, which not only leads to a much more direct proof of (1.1.3) than that given in [1], but also allows for a considerable reduction in each of my previous conditions on f and g.

1.2. Statement of the result. - The main result to be proved is the following

THEOREM. - *Suppose that $a > 0$ and that*

(i) *$g(x)$ sgn $x \to +\infty$ as $|x| \to \infty$*

(ii) *there are constants $\delta_1 > 0$, $\delta_2 > 0$ such that*

$$a\delta_1 - \delta_2 > 0 \tag{1.2.1}$$

and such that

$$f(x) \geq \delta_1 \quad \text{and} \quad g'(x) \leq \delta_2 \quad \text{for} \quad |x| \geq \xi_0 > 0$$

(iii) *$p(t)$ satisfies one or other of* (1.1.2), (1.1.4).

Then every solution $x(t)$ of (1.1.1) *ultimately satisfies* (1 1.3).

The form of the hypothesis (iii) above shows, in answer to the question raised earlier, that, subject to the given conditions on f and g any one (but not necessarily both) of the conditions on p would be quite sufficient for the boundedness of solutions of (1.1.1). Actually we shall see in § 3.4 that, in the case of the condition (1.1.4) the present method can be extended to allow for bounded functions $p(t, x, \dot{x}, \ddot{x})$ dependent explicitly on x, $\dot{x}$, and $\ddot{x}$.

Observe that, in place of our previous condition: $g(0) = 0$ and $g(x)/x \geq \geq \delta_0 > 0$ $(x \neq 0)$, we now have merely that $g(x) \operatorname{sgn} x \to +\infty$ as $|x| \to \infty$. Also the bounds on f and g' are now assumed to hold only for sufficiently large $|x|$.

1.3. - Note that hypothesis (i) of the theorem implies the existence of a constant $\xi_1 > 0$ such that $g(x) \operatorname{sgn} x > 0$ for $|x| \geq \xi_1$, so that if we set $\Delta = \max(\xi_1, \xi_0)$, ξ_0 being the constant in hypothesis (ii) of the theorem, then we have, all at once, that

$$(1.3.1) \qquad g(x) \operatorname{sgn} x > 0, \qquad f(x) \geq \delta_1 \text{ and } g'(x) \leq \delta_2, \qquad \text{for } |x| \geq \Delta.$$

Throughout what follows D_1, D_2 stand for the constants defined by

$$(1.3.2) \qquad D_2 = \delta_2 \Delta + \max_{|x| \leq \Delta} |g(x)|, \qquad D_3 = \delta_1 \Delta + \max_{|x| \leq \Delta} |F(x)|,$$

where

$$F(x) = \int_0^x f(\lambda) d\lambda.$$

In fact, following the notation in [1], we shall generally use D's (with or without suffixes) for positive constants whose magnitudes depend only on δ_1, δ_2, Δ, A, f, and g. Any D which appears with an argument beside it stands for a positive constant whose magnitude depends on δ_1, δ_2, Δ, A, f, as well as the specific argument shown; thus, as an example, $D(d_1)$ denotes a constant whose magnitude depends only on δ_1, δ_2, Δ, A, f, g and d_1. The D's are not necessarily the same in each place unless they are numbered, but the D's: D_1, D_2, D_3, ... with suffixes attached retain their identities throughout.

The small letters d, d_1, d_2, ... without arguments, occuring in the text are positive constants which retain the same magnitudes through until they are fixed later, in § 2.4 and 3.3, as D's.

2. - The case $|P(t)| \leq A_0$ for all t.

2.1. Some useful preliminaires. - In order to be able to utilize the condition: $|P| \leq A_0$ it is convenient to consider the system

$$(2.1.1) \qquad \dot{x} = y, \qquad \dot{y} = z - ay + P(t), \qquad \dot{z} = -yf(x) - g(x)$$

which one obtains from (1.1.1) by setting $y = \dot{x}$, $z = \ddot{x} + a\dot{x} - P(t)$. It will be shown that subject to our conditions on f and g every solutions (x, y, z) of (2.1.1) ultimately satisfies

$$|x(t)| \leq D_4, \qquad |y(t)| \leq D_4, \qquad |z(t)| \leq D_4. \tag{2.1.2}$$

In view of the boundedness, assumed here, of $P(t)$, the boundedness of y and z in (2.1.2) also implies that

$$|\ddot{x}(t)| \leq (a+1)D_4 + A_0$$

so that the required conclusion of the theorem would follow once (2.1.2) is established.

In what follows let D_5, D_6, D_7 be constants defined by

$$D_5 = \max_{|x| \leq \Delta} |g'(x)|, \qquad D_6 = \frac{16 D_5 \Delta \sqrt{2}}{\pi}, \qquad D_7 = \frac{5 D_5 \Delta \sqrt{2}}{\pi}, \tag{2.1.3}$$

Also, for any given constant $d > 0$ let $\psi_d = \psi_d(\xi)$ be the function, defined for all real ξ by

$$\psi_d = \begin{cases} \operatorname{sgn} \xi, & |\xi| \geq d \\ \sin \dfrac{\pi \xi}{2d}, & |\xi| \leq d \end{cases} \tag{2.1.4}$$

Observe that this function $\psi_d(\xi)$ satisfies

$$|\psi_d(\xi)| \leq 1 \quad \text{for all} \quad \xi, \ d; \tag{2.1.5}$$

also that $\psi_d(\xi)$ is differentiable for all ξ: in fact

$$\psi_d'(\xi) = \frac{\pi}{2d} \cos\left(\frac{\pi\xi}{2d}\right) (|\xi| \leq d), \qquad \psi_d'(\xi) = 0 \ |\xi| \geq d \tag{2.1.6}$$

so thet, in particular

$$0 \leq \psi_d'(\xi) \leq \frac{\pi}{2d} \quad \text{for all} \ \xi; \ \text{and} \quad \psi_d'(\xi) \geq \frac{\pi}{2d\sqrt{2}} \left(|\xi| \leq \frac{d}{2}\right). \tag{2.1.7}$$

Our main tool in the proof of (2.1.2) is the function $V = V(x, y, z)$ defined by

$$V = W - U, \tag{2.1.8}$$

where $W = W(x, y, z)$ is the function of the previous paper [1]: that is

$$2W = 2a\int_0^x g(\lambda)d\lambda + F^2(x) - 2\delta x F(x) + 2\delta\int_0^x F(\lambda)d\lambda + \delta y^2 + z^2 + 2yg(x) + 2[F(x) - \delta x]z, \tag{2.1.9}$$

δ being a constant fixed (as is possible in view of (1.2.1) so that

$$\delta_1 > \delta > \delta_2/a; \tag{2.1.10}$$

and U is the function defined by

$$U = D_6 y\psi_d(z + F(x)) + D_7 y\psi_{2\Delta}(x), \tag{2.1.11}$$

the function $\psi_d(z+F)$ being interpreted hare as the function $\psi_d(\xi)$ (see (2.1.4)) with $\xi = z + F(x)$, and Δ the constant appearing in (1.3.1). Observe that the function V as defined above depends explicitly on the (so far) arbitrary constant $d > 0$. The actual proof of (2.1.2) will revolve around the fact established in §§ 2.2, 2.3 and 2.4, that, if d is sufficiently large, the function V so defined constitutes a Liapounov function for the system (2.1.1), satisfying a result analagous to (1.1.5).

The following consequences of (1.3.1) and (1.3.2) will be useful in the verification of this property of V:

$$|g(x)| \leq \delta_2|x| + D_2 \tag{2.1.12}$$

$$F(x)\operatorname{sgn} x \geq \delta_1|x| - D_3. \tag{2.1.13}$$

2.2. **The unboundedness of V for arbitrarily large $x^2 + y^2 + z^2$.** - We prove here that

LEMMA 1. - *Subject to the conditions in hypothesis* (i) *and* (ii) *of the theorem, and for arbitrary $d > 0$, the function V satisfies*

$$V(x, y, z) \to +\infty \quad as \quad x^2 + y^2 + z^2 \to \infty. \tag{2.2.1}$$

PROOF. - We make use of the following re-arrangement of W, given in § 2.1 of [1]:

$$2W = \delta(y + \delta^{-1}g)^2 + (z + F - \delta x)^2 - \delta^2 x^2 + 2\delta\int_0^x F(\lambda)d\lambda + 2a\int_0^x g(\lambda)d\lambda - \delta^{-1}g^2 \tag{2.2.2}$$

and focus attention on the terms

$$W_1 \equiv 2\delta \int_0^x F(\lambda)d\lambda - \delta^2 x^2, \qquad W_2 \equiv 2a \int_0^x g(\lambda)d\lambda - \delta^{-1}g^2$$

arising in (2.2.2).

To obtain a lower bound for W_1 consider the function

$$\varphi_1(x) \equiv 2 \int_0^x F(\lambda)d\lambda - \delta_1 x^2 + 2D_3 \,|\, x \,|$$

where D_3 is given in (1.3.2). By considering the cases $x \geq 0$, $x < 0$ separately, and then making use of the definition of (2.1.13) it can be verified that $\varphi_1'(x) \operatorname{sgn} x \geq 0$, so that since $\varphi_1(0) = 0$, we have that $\varphi_1(x) \geq 0$, that is

$$2 \int_0^x F(\lambda)d\lambda \geq \delta_1 x^2 - 2D_3 \,|\, x \,|,$$

for all x. From this we have that

(2.2.3) $$W_1 \geq \delta(\delta_1 - \delta)x^2 - 2D_4 \,|\, x \,| \quad \text{for all} \quad x.$$

We turn now to the function W_2. Here set

$$\varphi_2(x) = W_2(x) + D_8$$

where $D_8 = \max_{|x| \leq \Delta} |\, W_2(x) \,|$. Then, clearly, $\varphi_2(x) \geq 0$ for $|\, x \,| \leq \Delta$. If $|\, x \,| \geq \Delta$ then, since

$$\varphi_2' = 2(a - \delta^{-1}g'(x))g(x)$$

it is clear from (1.3.1), and since $a - \delta^{-1}\delta_2 > 0$ (in view of (2.1.10)) we have that

$$\varphi_2'(x) \operatorname{sgn} x \geq 0 \quad \text{for} \quad |\, x \,| \geq \Delta.$$

But $\varphi_2(\pm \Delta) \geq 0$. Hence $\varphi_2(x) \geq 0$ for $|\, x \,| \geq \Delta$. Thus $\varphi_2(x) \geq 0$, that is,

$$W_2(x) \geq - D_8,$$

for all x

These estimates of W_1 and W_2 combine with (2.2.2) to give that

$$\begin{aligned}2W &\geq \delta(y+\delta^{-1}g)^2+(z+F-\delta x)^2+\delta(\delta_1-\delta)x^2-2D_3|x|-D_8\\ &\equiv W_0(x,\,y,\,z)\end{aligned}$$

say, for all x, y, z.

For the term U we have, in view of (2.1.5) that

$$|U|\leq(D_6+D_7)|y|,$$

(2.2.4)
$$\begin{aligned}2V &\geq W_0(x,\,y,\,z)-2(D_6+D_7)|y|,\\ &\equiv V_1(x,\,y,\,z),\end{aligned}$$

say. It remains only to show that

(2.2.5)
$$V_1\to\infty \quad \text{as} \quad x^2+y^2+z^2\to\infty$$

and the required result will then follow.

Since the coefficient, $\delta(\delta_1-\delta)$, of x^2 in W_0 is positive (in view of (2.1.10)) it is evident that (2.2.5) is true if $|x|$ is bounded but $y^2+z^2\to\infty$. Therefore the general result (2.2.5) will be proved as soon as it is established that

(2.2.6)
$$V_1\to\infty \quad \text{as} \quad |x|\to\infty.$$

Now let $|x|\geq D_2/\delta_2$. Then, by (2.1.12), $g(x)|\leq 2\delta_2|x|$, so that

$$\begin{aligned}&\delta(y+\delta^{-1}g)^2+\frac{1}{2}\delta(\delta_1-\delta)x^2\\ &\geq\delta(y+\delta^{-1}g)^2+\frac{1}{8}\delta(\delta_1-\delta)\delta_2^{-2}g^2\\ &\geq\delta_3(y^2+g^2)\end{aligned}$$

for some sufficiently small constant $\delta_3=\delta_3(\delta,\,\delta_1,\,\delta_2)>0$. Hence, when $|x|\geq D_2/\delta_2$,

$$V_1\geq\delta_3(y^2+g^2)+\frac{1}{2}\delta(\delta_1-\delta)x^2-2D_3|x|-(D_6+D_7)|y|-D_8,$$

so that, since

$$\delta_3y^2-(D_6+D_7)|y|\geq-(D_6+D_7)^2/4\delta_3\equiv-D_9,$$

for all y, we have that

$$V_1\geq\frac{1}{2}\delta(\delta_1-\delta)x^2-2D_3|x|-D_8-D_9.$$

The expression on the right-hand side here tends to $+\infty$ as $|x| \to \infty$, and thus (2.2.6) holds. This completes the verification of the Lemma.

It is important to remark here, for future use, that since $V(x, y, z)$ is continuous in x, y and z the result (2.2.1) necessarily implies the existence of a constant D_{10} such that

$$V(x, y, z) \geq -D_{10} \quad \text{for all} \quad x, y, z. \tag{2.2.7}$$

2.3. **Estimates for $\dot{V}$.** - Let $V(t) \equiv V(x(t), y(t), z(t))$ where $(x(t), y(t), z(t))$ is any solution of (2.1.1). By an elementary calculation it can be verified from (2.1.8), (2.1.9) and (2.1.11) that

$$\dot{V} = -g(x)(F(x) - \delta x) + U(x)y^2 + V_1^* + V_2^*$$

where

$$U = a\delta - g'(x) + D_7\psi'_{2\Delta}(x), \quad \text{and}$$

$$\begin{aligned} V_1^* = &- D_6(z - ay)\psi_d(z + F(x)) - D_7(z - ay)\psi_{2\Delta}(x) \\ &+ D_6 yg(x)\psi'_d(z + F(x)), \end{aligned} \tag{2.3.1}$$

$$V_2^* = \{ \delta y + g(x) - D_6\psi_d(2 + F(x)) - D_7\psi_{2\Delta}(x) \} P(t). \tag{2.3.2}$$

About the coefficient $U(x)$ above it will be observed since $\psi'_{2\Delta}(x) \geq 0$ and since $g'(x) \leq \delta_2$ $(|x| \geq \Delta)$ that

$$\begin{aligned} U(x) &\geq a\delta - \delta_2 \qquad (|x| \geq \Delta) \\ &> 0 \end{aligned}$$

by (2.1.10). For $|x| \leq \Delta$ we have from (2.1.3) that $|g'(x)| \leq D_5$ and from (2.1.7) that $\psi'_{2\Delta}(x) \geq \pi/(4\Delta\sqrt{2})$, so that, here

$$U(x) \geq a\delta - D_5 + \pi D_7/(4\Delta\sqrt{2})$$

and by (2.1.7) this gives that

$$\begin{aligned} U(x) &\geq a\delta - D_5 + 5D_5/4 \\ &= a\delta + \frac{1}{4}D_5. \end{aligned}$$

Hence there exists a constant D_{11} such that $U(x) \geq D_{11}$ for all x; and thus

$$\dot{V} \leq -g(x)(F - \delta x) - D_{11}y^2 + V_1^* + V_2^*. \tag{2.3.3}$$

We are now in a position to embark on the estimates $\dot{V}$. Throughout what follows $(x, y, z) = (x(t), y(t), z(t))$ stands for an arbitrary solution of (2.1.1), and $d > 0$ the (so far arbitrary) constant in the definition (2.1.11) of U. Our first result is the following.

LEMMA 2. - *Corresponding to any d there exists a constants $D_{12}(d)$ such that if $|z + F(x)| \geq D_{12}(d)$ then*

$$\dot{V} \leq -1 \tag{2.3.4}$$

PROOF. - Assume to being with that $|z + F| \geq d$. Then by (2.1.6) the last term in (2.3.1) is zero so that

$$V_1^* = -D_6(z - ay)\psi_d(z + F) - D_7(z - ay)\psi_{2\Delta}(x).$$

This expression can be written thus

$$\begin{aligned} V_1^* &= -\{D_6(z + F)\psi_d(z + F) + D_7(z + F)\psi_{2\Delta}(x)\} \\ &\quad + \{D_6(ay + F)\psi_d(z + F) + D_7(ay + F)\psi_{2\Delta}(x)\} \\ &\equiv I_1 + I_2, \end{aligned}$$

say. Because $|z + F| \geq d$ it is evident from the definition (2.1.4) that

$$(z + F)\psi_d(z + F) = |z + F|. \tag{2.3.5}$$

Also, since $|\psi_{2\Delta}(x)| \leq 1$ anyway,

$$|(z + F)\psi_{2\Delta}(x)| \leq |z + F|;$$

and hence

$$I_1 \leq -|z + F|(D_6 - D_7),$$

so that, since $D_6 - D_7 = 11D_5\Delta\sqrt{2}/\mu$ (by (2.1.3) we have that

$$I_1 \leq -D_{13}|z + F|$$

for some constant $D_{13} > 0$. Coming to I_2, we also have, since $|\psi_d| \leq 1$ and $|\psi_{2\Delta}| \leq 1$, that

$$|I_2| \leq D_{14}(|y| + |F|)$$

for some constant D_{14}. Hence

$$V_1^* \leq -D_{13}|z + F| + D_{14}(|y| + |F|).$$

For the function V_2^* (given by (2.3.2)) the boundedness of ψ_d, $\psi_{2\Delta}$ and P leads to the estimate

$$V_2^* \leq D_{15} + \delta A_0 |y| + A_0 |g| .$$

and thus, by (2.3.3),

$$\dot{V} \leq - D_{13} |z + F| - D_{11} y^2 - g(F - \delta x) + D_{14}(|y| + |F|) + D_{15} \tag{2.3.6}$$

we shall now prove that

$$D_{11} y^2 + g(F - \delta x) - D_{14}(|y| + |F|) \to + \infty \quad \text{as} \quad x^2 + y^2 \to \infty. \tag{2.3.7}$$

Since such a result would imply the existence of a constant, D_{16} say, such that

$$D_{11} y^3 + g(F - \delta x) - D_{14}(|y| + |F|) + D_{15} \geq - D_{16}$$

for all x, y, and this in turn would imply, in view of (2.3.6) that

$$\dot{V} \leq - D_{13} |z + F| + D_{16};$$

it follows then that if (2.3.7) can be proved then $\dot{V} \leq -1$ provided that $|z + F| \geq$ $\geq \max [d, (D_{16} + 1) D_1^{-1}]$ which is the required conclusion of the lemma.

It is quite clear that if $|x|$ is bounded, but $|y| \to \infty$, then

$$D_{11} y^2 + g(F - \delta x) - D_{14}(|y| + |F|) \to \infty.$$

Thus, to complete the proof of (2.3.7) it suffices now to verify the result for the case $|x| \to \infty$. To handle this case set

$$D_{17} = \frac{1}{3}(1 - \delta\delta_1^{-1}),$$

noting that, since $\delta\delta_1^{-1} < 1$ (by (2.1.10)) the constant D_{17} is strictly positive. If $|x|$ is sufficiently large we have from (1.3.1) and (2.1.13) that $g(x) \operatorname{sgn} x > 0$ and $F(x) \operatorname{sgn} x > 0$ so that then

$$\begin{aligned} g(F - \delta x) - 2D_{17} g F &= |g| [(1 - 2D_{17}) |F| - \delta |x|] \\ &= \delta\delta_1^{-1} |g| [(1 + \delta_1 \delta^{-1} D_{17}) |F| - \delta_1 |x|]. \end{aligned}$$

But, by (2.1.13),

$$(1 + \delta_1 \delta^{-1} D_{17}) |F| - \delta_1 |x| \to + \infty, \qquad \text{as} \quad |x| \to \infty,$$

so that since $|g| \to \infty$ as $|x| \to \infty$, we have that

$$g(F - \delta x) - 2D_{17} gF \to \infty \quad \text{as} \quad |x| \to \infty.$$

The particular consequence of this which is vital to our proof here is the existence of a constant D_{18} such that

$$g(F - \delta x) \geq 2D_{17}|g||F| \quad \text{for} \quad |x| \geq D_{18}. \tag{2.3.8}$$

Indeed, since

$$D_{11} y^2 - D_{14}|y| \geq D$$

for some constant D, (2.3.8) helps to show that, if $|x| \geq D_{18}$, then

$$\begin{aligned} D_{11}y^2 + g(F - \delta x) - D_{14}(|y| + |F|) - D \\ \geq 2D_{17}|g||F| - D_{14}|F| - D \\ \equiv \varphi_0, \end{aligned}$$

say. By putting φ_0 in the form

$$\varphi_0 = D_{17}|g| \cdot |F| + |F|(D_{17}|g| - D_{14}) - D$$

and then recalling that $|g|$ and $|F|$ both tend to infinity as $|x| \to \infty$, we see at once that

$$D_{11}y^2 + g(F - \delta x) - D_{14}(|y| + |y|) - D \to +\infty \quad \text{as} \quad |x| \to \infty.$$

This completes the proof of (2.3.7) and the lemma now follows, as indicated earlier.

Observe that all our results so far have been proved valid without restriction on d. Our next result is the only one whose validity will depend explicity on d being sufficiently large.

LEMMA 3. - *Let* $|z + F(x)| \leq d_1$. *Then there exist constants* D_{19}, $D_{20}(d_1)$ *such that if* $d \geq D_{19}$ *then*

$$\dot{V} \leq -1 \quad \textit{provided that} \quad x^2 + y^2 \geq D_{20}^2(d_1). \tag{2.3.9}$$

PROOF. - We start again with (2.3.3), but this time, since $|z + F(x)|$ is not being assumed greater than d, the last term in V_1^* is not zero as in Lemma 2. However, by (2.1.4),

$$(z + F)\psi_d(z + F) \geq 0$$

so that by splitting the term $(z - ay)$ in V_1^* (see (2.3.1)) in the form

$$z - ay = z + F - (ay + F)$$

and then using the various properties of ψ_d, we have here (with the same D_{14} as before) that

$$\begin{aligned} V_1^* &\leq D_7 \, | z + F | + D_{14}(| y | + | F |) + D_6 \, | yg | \, \psi_d'(z + F) \\ &\leq D_7 d_1 + D_{14}(| y | + | F |) + \frac{\pi D_6}{2d} | yg | \end{aligned}$$

where in the last step we have used the conditions: $| z + F | \leq d_1$. $| \psi_d'(\zeta) | \leq \geq \frac{1}{2} \pi d^{-1}$. The previous estimate V_2^* is valid here, so that

$$\dot{V} \leq - g(F - \delta x) - D_{11} y^2 + \frac{\pi D_6}{2d} | yg | + D_{14}(| y | + | F | + D_7 d_1. \tag{2.3.10}$$

Since $| yg | \leq \frac{1}{2}(y^2 + g^2)$ it is clear from (2.3.8) that, if $| x | \geq D_{18}$,

$$\begin{aligned} g(F - \delta x) + D_{11} y^2 - \frac{\pi D_6}{2d} | yg | &\geq 2D_{17} \, | g | \, | F | + D_{11} y^2 - \frac{\pi D_6}{4d}(y^2 + g^2) \\ &= V_3^* + V_4^* + D_{17} \, | g | \, | F | \end{aligned}$$

where

$$V_3^* = \left(D_{11} - \frac{1}{4} \pi D_6 d^{-1}\right) y^2, \qquad V_4^* = | g | \left(D_{17} \, | F | - \frac{1}{4} \pi D_6 d^{-1} \, | g |\right),$$

Observe now that if d is fixed so large that $\frac{1}{4} \pi D_6 d^{-1} \leq \frac{1}{2} D_{11}$ then

$$V_3^* \geq \frac{1}{2} D_{11} y^2.$$

For the term V_4^*, it is useful to note from (2.1.12) and (2.1.13) that if $| x |$ is sufficiently large

$$\begin{aligned} & D_{17} \, F | - \frac{1}{4} \pi D_6 d^{-1} \, | g | \\ & \geq \left(D_7 \delta_1 - \frac{1}{4} \pi D_6 d^{-1} \delta_2\right) | x | - \frac{1}{4} \pi D_6 D_2 d^{-1} - D_{17} D_3 \end{aligned}$$

so that if now d is fixed so that $\frac{1}{4} \pi D_6 \delta_2 d^{-1} < D^7 \delta_1$ then

$$| g | \left(D_{17} \, | F | - \frac{1}{4} \pi D_6 d^{-1} \, | g |\right) > 0$$

provided that $|x|$ is sufficiently large, say $|x| \geq D_{21}$.

The above calculations show that if

$$d > \frac{1}{4}\pi D_6 \max(\delta_2 D_7^{-1}\delta_1^{-1},\ \delta_{11}^{-1}) \tag{2.3.11}$$

then

$$\dot{V} \leq -D_{17}|g||F| - \frac{1}{2}D_{11}y^2 + D_{14}(|y| + |F|) + D_7 d_1 \tag{2.3.12}$$

provided that $|x| \geq D_{21}$. By using arguments similar to those employed in the last part of the proof of the preceding lemma if can be shown that

$$-D_{17}|g||F| - \frac{1}{2}D_{11}y^2 + D_{14}(|y| + |F|) \to -\infty \quad \text{as} \quad |x| \to \infty.$$

Thus, subject to (2.3.11) we have that there is a constant $D_{22}(d_1)$ such that

$$\dot{V} \leq -1 \quad \text{if} \quad |x(t)| \geq D_{22}(d_1). \tag{2.3.13}$$

It remains now to tackle the case $|x(t)| \leq D_{22}(d_1)$, assuming d fixed by (2.3.11). We have here, from (2.3.10) and this boundedness of $|x(t)|$ that

$$\dot{V} \leq -D_{11}y^2 + D(d_1)|y| + D(d_1).$$

As the right handside tends to $-\infty$ as $|y| \to \infty$ it follows here that there is a constant $D_{23}(d_1)$ then

$$\dot{V} \leq -1 \quad \text{proved that} \quad |y(t)| \geq D_{23}(d_1). \tag{2.3.14}$$

The two estimates (2.3.13) and (2.3.14) combine to show that if (2.3.11) holds then

$$\dot{V} \leq -1 \quad \text{provided that} \quad x^2 + y^2 \geq D_{22}^2 + D_{23}^2$$

and the lemma is therefore established

2.4. **A combination of Lemmas 2 and 3.** - Consider now the function V (see (2.1.8)) but with d fixed by $d = D_{19}$ is the constant in Lemma 3. Define two new constants D_{24}, D_{25} by

$$D_{24} = D_{12}(D_{19}), \qquad D_{25} = D_{20}(D_{24}),$$

where $D_{12}(d)$ and $D_{20}(d_1)$ are the constants defined, for given d and d_1, in Lemmas 2 and 3 respectively. We shall now prove that

LEMMA 4. – *If* (x, y, z) *is any solution of* (2.1.1) *then subject to our conditions on* f, g *and* P,

(2.4.11) $$\dot{V} \leq -1 \quad \text{if} \quad x^2 + y^2 + (z + F(x))^2 \geq D_{24}^2 + D_{25}^2 .$$

PROOF. – Assume that

(2.4.2) $$x^2 + y^2 + (z + F(x))^2 \geq D_{24}^2 + D_{25}^2 .$$

If $|z + F| \geq D_{24}$ then $\dot{V} \leq -1$ by Lemma 2. Otherwise, that is if $|z+F| < D_{24}$, then by (2.4.2) we must have that $x^2 + y^2 \geq D_{25}^2$. But in the latter case Lemma 3 gives that $\dot{V} \leq -1$, and this proves the lemma.

2.5. **Completion of the proof of the theorem for the case P bounded.** – Let (x, y, z) ae any solution of (2.1.1), and let $V(t) \equiv V(x, y, z)$ where V is the particular V discussed in § 2.4 (that is with $d = D_{19}$). Since

$$x^2 + y^2 + (z + F(x))^2 \to \infty \quad \text{as} \quad x^2 + y^2 + z^2 \to \infty$$

the result of Lemma 4 obviously implies the existence of a constant D_{26} such that

(2.5.1) $$\dot{V} \leq -1 \quad \text{if} \quad x^2 + y^2 + z^2 \geq D_{26}^2 ,$$

analogous to (1.1.5).

The remainder of the proof of (2.1.2) can now be obtained by the standard Yoshizawa type technique using (2.2.1), (2.2.7) and (2.5.1) as required. The main arguments are exactly as in § 3 of [3], and further details will therefore be omitted here.

3. – The case: $|p(t)| \leq A_1 < \infty$ for all t considered.

3.1. – This time it is convenient to work with the system

(3.1.1) $$\dot{x} = y, \qquad \dot{y} = z - ay, \qquad \dot{z} = -yf(x) - g(x) + p(t)$$

obtained from (1.1.1) ay setting $y = \dot{x}$, $z = \ddot{x} + a\dot{x}$. To prove the theorem it will suffice to show that any solution (x, y, z) of (3.1.1) satisfies

(3.1.2) $$|x(t)| \leq D, \qquad |y(t)| \leq D, \qquad |z(t)| \leq D$$

for all sufficiently large t.

Our main tool for the proof of (3.1.2) is the function $V = V(x, y, z)$ defined by

$$V = W - U \tag{3.1.3}$$

where W is the same as before (see (2.1.9)) but U is now given by

$$U = D_6^* y\psi_d(z + F(x) - \delta x) + D_7 y\psi_{2\Delta}(x), \tag{3.1.4}$$

where

$$D_6^* = D_6 + A_1 . \tag{3.1.5}$$

The only difference the between this and the previous U (2.1.1I) lies in the coefficient of y, which, in the previous case, was $D_6\psi_d(z + F)$ while now we have $D_9^*\psi_d(z + F - \delta x)$. However since $|\psi_d| \leq 1$ and $|\psi_{2\Delta}| \leq 1$ the same arguments as were used for Lemma 1 will also give here that the present V satisfies

$$V(x, y, z) \to +\infty \quad \text{as} \quad x^2 + y^2 + z^2 \to \infty. \tag{3.1.6}$$

3.2. - To complete the proof of the theorem it suffices now, in view of (3.1.6) and in view of the remarks in § 2.5, to show that if (x, y, z) is any solution of (3.1.1) then $\dot{V} = \dfrac{d}{dt} V(x, y, z)$ satisfies a result analogous to (2.5.1).

An elementary calculation from (3.1.1), and from our present definition of V, will show that

$$\dot{V} = -g(F - \delta x) - U(x)y^2 + V_5^* + V_6^*, \tag{3.2.1}$$

where $U(x)$ is the same as before (§ 2.3) and therefore satisfies

$$U(x) \geq D_{11} > 0 \quad \text{for all} \quad x, \tag{3.2.2}$$

and

$$\begin{aligned} V_5^* = &- D_6^*(z - ay)\psi_d(z + F - \delta x) - D_7(z - ay)\psi_{2\Delta}(x) \\ &+ \delta D_6^*\psi_d'(z + F - \delta x)y^2 + D_6^* yg\psi_d'(z + F - \delta x), \end{aligned} \tag{3.2.3}$$

$$V_6^* = \{ (z + F - \delta x) - D_6^* y\psi_d'(z + F - \delta x) \} p(t). \tag{3.2.4}$$

Our starting point for the estimate of $\dot{V}$ is the result

$$\dot{V} \leq -g(F - \delta x) - D_{11}y^2 + V_5^* + V_6^* \tag{3.2.5}$$

obtained from (3.2.1) and (3.2.2). The procedure is exactly the same as before except that now we consider the cases

$$|z+F-\delta x| \ \text{large}, \qquad |z+F-\delta x| \ \text{small}$$

as against the cases

$$|z+F| \ \text{large}, \qquad |z+F| \ \text{small}$$

which were dealt with in § 2.3.

Our first result, analagous to Lemma 2, is that

LEMMA 2'. - *Corresponding to any d there exists a constant $D_{27}(d)$ such that if $|y+F-\delta x| \geq D_{27}(d)$, then*

$$\dot{V} \leq -1 \tag{3.2.6}$$

PROOF. - Assume to begin with that $|z+F-\delta x| \geq d$. Then, by (2.1.6), $\psi_d'(z+F-\delta x)=0$, so that, from (3.2.3) and (3.2.4),

$$V_5^* = -D_6^*(z-ay)\psi_a(z+F-\delta x) - D_7(z-ay)\psi_{2\Delta}(x)$$

$$V_6^* = (z+F-\delta x)p(t).$$

Since $|p| \leq A_1$, it is clear that

$$|V_6^*| \leq A_1 |z+F-\delta x|.$$

To estimate the V_5^* (given above) use the re-arrangement

$$\begin{aligned} V_5^* = &-D_6^*(z+F-\delta x)\psi_d(z+F-\delta x) + D_6^*(ay+F-\delta x)\psi_d(z+F-\delta x) \\ &- D_7(z+F-\delta x)\psi_{2\Delta}(x) + D_7(ay+F-\delta x)\psi_{2\Delta}(x). \end{aligned}$$

Since $\xi\psi_d(\xi)=|\xi|$ (if $|\xi|\geq d$) and $|\psi_d(\xi)|\leq 1$ for all ξ, d it is clear from the re-arrangement that

$$V_5^* \leq -(D_6^* - D_7)|z+F-\delta x| + D_{28}(|y|+|F-\delta x|.$$

for some constant D_{28}. Thus

$$V_5^* + V_6^* \leq -(D_6^* - D_7 - A_1)|z+F-\delta x| + D_{28}(|y|+|F-\delta x|).$$

But, by (3.1.5) and (2.1.3)

$$D_6^* - D_7 - A_1 > 0$$

and so, by (3.2.1),

$$\dot{V} \leq - D_{29} \,|\, z + F - \delta x \,| - D_{11} y^2 - (F - \delta x) g + D_{28} (|\, y \,| + |\, F - \delta x \,|), \tag{3.2.7}$$

for some constant $D_{29} > 0$.

The result (3.2.7) is analogous to the result (2.3.6) arising in the course of Lemma 2 and the arguments given there can be adapted readily to our present case to show that

$$\dot{V} \leq -1 \quad \text{if} \quad |\, z + F - \delta x \,| \geq D(d).$$

Next we have, analagous to Lemma 3,

LEMMA 3'. - *Let* $|\, z + F - \delta x \,| \leq d_2$. *Then there exist constants* D_{29}, $D_{30}(d_2)$ *such that if* $d \geq D_{29}$, then

$$\dot{V} \leq -1 \quad \textit{provided that} \quad x^2 + y^2 \geq D_{30}^2(d_2). \tag{3.2.8}$$

PROOF. - We start once again from (3.2.5). We take V_5^* (see (3.2.3)) in the form

$$\begin{aligned} V_5^* = & - D_6^*(z + F - \delta x)\psi_d(z + F - \delta x) + D_6^*(ay + F - \delta x)\psi_d(z + F - \delta x) \\ & - D_7(z + F - \delta x)\psi_{2\Delta}(x) + D_7(ay + F - \delta x)\psi_{2\Delta}(x) \\ & + \delta D_6^* y^2 \psi_d'(z + F - \delta x) + D_6^* y g \psi_d'(z + F - \delta x). \end{aligned}$$

From this, by making use of the following:

$$|\, z + F - \delta x \,| \leq d_2, \qquad |\, \psi_d'(\xi) \,| \leq \frac{1}{2} \pi d^{-1}, \qquad |\, \psi_d(\xi) \,| \leq 1,$$

we see that

$$V_5^* \leq \frac{1}{2} \pi d^{-1} D_6^* (y^2 + |\, yg \,|) + D (|\, y \,| + |\, F - \delta x \,|) + D(d_2).$$

As for the function V_6^* it is clear that

$$V_6^* \leq \frac{1}{2} \pi d^{-1} D_6^* A_1 \,|\, y \,| + A_1 d_2$$

Hence, by (3.2.5),

$$\begin{aligned} \dot{V} \leq & - (F - \delta x) g - D_{11} y^2 + D (|\, y \,| + |\, F - \delta x \,| \\ & + \frac{1}{2} \pi d^{-1} (y^2 + |\, yg \,| + |\, y \,|) + D(d_2). \end{aligned} \tag{3.2.9}$$

By using the arguments immediately following (2.3.10) it can be shown that if d is sufficiently large, say $d \geq D_{31}$, then

$$(F - \delta x)g + D_{11}y^2 - \frac{1}{2}\pi d^{-1}(y^2 + |yg|)$$
$$\geq D_{17}|g||F| + \frac{1}{2}D_{11}y^2$$

provided that $|x|$ is large enough, say $|x| \geq D_{32}$. Thus, if

$$|x| \geq D_{32} \quad \text{and} \quad d \geq D_{31}$$

then

$$\dot{V} \leq - D_{17}|g||F| - \frac{1}{2}D_{11}y^2 + D(|y| + |F - \delta x|) + D(d_2). \tag{2.3.10}$$

Assume now that $d \geq D_{31}$. Then, as in Lemma 3, it can be shown from (3.2.10) that

$$\dot{V} \leq -1 \quad \text{provided that} \quad |x| \geq D_{33}(d_2),$$

Note that the coefficient

$$- D_{11} + \frac{1}{2}\pi d^{-1}$$

of $y^{\cdot}$ on the right hand side of (3.2.9) is strictly negative so long as d is large enough, say $d \geq D_{34}$. Thus if $|x| \leq D_{33}(d_2)$ as it stands $|x| \leq D_{33}(d_2)$ then, so long as $d \geq D_{34}$, the expression on the right hand side of (3.2.9) can be majorized by an expression of the form

$$- Dy^2 + (|y| + 1)D(d_2)$$

and as this is strictly negative if $|y|$ is sufficiently large, say for $|y| \geq D(d_2)$, the result (3.2.8) can be obtained from (3.2.9) in exactly the same way as in Lemma 3.

3.3. Completion of the proof of the theorem. - Let us now set $d = D_{29}$ in (3.1.4), and let D_{35}, D_{36} be defined by

$$D_{35} = D_{27}(U_{29}), \qquad D_{36} = D_{30}(D_{35})$$

where the $D_{23}(d)$, $D_{30}(d_2)$ are the constants defined, for given d, d_2, by Lemmas 2′, 3′ respectively

It is easy to see now, just as in § 2.4, that the results of Lemmas 2′ and 3′ imply that

$$\dot{V} \leq -1 \quad \text{if} \quad x + y^2 + (z + F - \delta x)^2 \geq D_{85}^2 + D_{86}^2 . \tag{3.3.1}$$

Since

$$x^2 + y^2 + (z + F - \delta x)^2 \to \infty \quad \text{as} \quad x^2 + y^2 + z^2 \to \infty,$$

the result (3.3.1) in turn implies the existence of a constant D_{87} such that

$$\dot{V} \leq -1 \quad \text{if} \quad x^2 + y^2 + z^2 \geq D_{87}^2 .$$

This proves the result (2.5.1), for our function V defined by (3.1.3), and the result (3.1.2) now follows.

This completes our verification of the theorem.

3.4. **A further generalization.** - It will have been observed from a review of the main steps in §§ 3.1, 3.2 and 3.3, that our arguments would have worked equally for a function $p = p(t, x, \dot{x}, \ddot{x})$ dependent explicitly on t, x, $\dot{x}$ and $\ddot{x}$, so long as $p(t, x, y, z)$ is uniformly bounded. Thus the boundedness result (1.1.3) holds also for the equation

$$\dddot{x} + a\ddot{x} + f(x)\dot{x} + g(x) = p(t, x, \dot{x}, \ddot{x})$$

subject to our usual condition on a, f and g, if there is a constant $A_1 < \infty$ such that

$$|p(t, x, y, z)| \leq A_1$$

uniformly in t, x, y and z.

REFERENCES

[1] J. O. C. EZEILO, *Proc. London Math, Soc.*, 13 (1963), pp. 99-124.

[2] — —, *Ibid* (1967) (in press).

[3] — —, *Journal London Math. Soc.*, 38 (1963), pp. 11-16.

[4] V. A. PLISS, *Soviet Mathematics*, (Doklay) 2 (1961), pp. 930-932.