On Permutation Groups.

OLAF TAMASCHKE

Riassunto - *Diese Arbeit versucht, die Theorie der Permutationsgruppen vom « kategoriellen » Standpunkt aus zu betrachten. Es werden in naheliegender Weise der Begriff der Permutationsstruktur und der Begriff des Homomorphismus einer Permutationsstruktur eingeführt (Abschnitt 1). In der so entstehenden Kategorie aller Permutationsstrukturen werden die Homomorphismen klassifiziert gemäß ihrem Verhalten gegenüber den Wielandtschen G-Relationen und den Bahnen der Stabilisatoren von endlich vielen Ziffern (Abschnitt 3). Jedem Homomorphismus wird sein Grad zugeordnet (Abschnitt 4). Diese Begriffsbildungen stehen in engem Zusammenhang mit einer Verallgemeinerung des Normalteilerbegriffes der Gruppentheorie. Zu jeder natürlichen Zahl n und zu jeder Untergruppe H einer Gruppe G wird der Begriff der n-fach G/H-normalen Untergruppe eingeführt (Abschnitt 7). Für homogene Räume gilt ein Homomorphiesatz (Theorem 8.5) analog zum Homomorphiesatz der Gruppentheorie; er besagt unter anderem, daß der Kern eines n-fachen Homomorphismus eines homogenen Raumes eine (n—1)-fach G/G_α-normale Untergruppe von G ist (wobei G_α der Stabilisator einer Ziffer α ist). Es gilt auch die Umkehrung dieses Satzes in dem Sinne, daß jede (n-1)-fach G/H-normale Untergruppe K einer Gruppe G Kern eines n-fachen Homomorphismus des durch die Nebenklassen Hg, g ∈ G, definierten homogenen Raumes auf den durch die Nebenklassen Kg, g ∈ G, definierten homogenen Raum ist (Theorem 8.6). Ferner gelten ein Erster und ein Zweiter Isomorphiesatz für homogene Räume (Theoreme 10.1 und 10.2).*

Auf den Begriff der n-fach G/H-normalen Untergruppe gründet sich der Begriff der n-fach G/H-subnormalen Untergruppe (Definition 11.1). Es läßt sich unter anderem ein Jordan-Hölder-Satz für 1-fache G/H-Kompositionsketten beweisen, wobei die Kompositionsfaktoren homogene Räume sind (Theorem 11.10).

Im letzten Abschnitt 12 wird gezeigt, daß die von den zweiseitigen Nebenklassen $G_\alpha g G_\alpha$, g ∈ G, erzeugte Halbgruppe G/G_α für den zugehörigen homogenen Raum die Bedeutung einer Art von «Endomorphismenring» besitzt.

Die Klasse $[G/G_\alpha]$ aller zu G/G_α isomorphen zweiseitigen Nebenklassen Halbgruppen wird der Typ des homogenen Raumes genannt (Definition 12.17). Dieser Begriff liefert eine Klassifizierung der homogenen Räume, bei der die 2-fachen Homomorphismen eine Rolle spielen (Theoreme 12.19, 12.20, 12.22).

The way of thinking in terms of categories and of homological algebra, which is invading now almost every branch of mathematics, does not seem to have gained much ground so far upon the theory of permutation groups. We are trying in this paper to set out in that direction.

Our investigations are based on *permutation structures* which are triples $(\Omega, G, \cdot)$ with Ω a set, G a group, and an external algebraic composition (for which the dot stands in the third place) which gives the acting of G on Ω in the usual way (Definition 1.1). A *homomorphism* of a permutation structure $(\Omega, G, \cdot)$ into a permutation structure $(\Omega', G', \cdot)$ is a pair (φ, ψ) where φ is a mapping of Ω into Ω' and ψ is a homomorphism of G into G' such that both mappings are compatible with the acting of G on Ω and the acting of G' on Ω' (Definition 1.2). These two notions yield the category $\mathfrak{P}$ of all permutation structures.

This obvious approach itself does not lead very far. To make it more useful we classify the homomorphisms of the permutation structures according to their behaviour with respect to two permutation concepts.

The first of these two concepts is WIELANDT's idea of n-nary G-relations on Ω ([7], § 10). For a positive integer n we say that a homomorphism (φ,ψ) satisfies $(\mathfrak{R}_n)$ if it maps every n-nary G-relation on Ω *onto* an n-nary G'-relation on Ω'. If ψ is an epimorphism, then $(\mathfrak{R}_n)$ trivially holds for every n. This fact shows that the conditions $(\mathfrak{R}_n)$ are not of prime importance for our classification.

More significant for the classification of homomorphisms are the orbits of the stabilizers of n letters. We say that a homomorphism (φ,ψ) satisfies $(\mathfrak{S}_n)$ if φ maps every orbit of the stabilizer in G of any n (not necessarily distinct) letters from Ω *onto* an orbit of the stabilizer in G' of the image letters.

There exist homomorphisms which satisfy $(\mathfrak{R}_n)$ for every positive integer n, but do not satisfy $(\mathfrak{S}_n)$ for any integer $n > 1$ (Lemma 3.5). There also exist homomorphisms which satisfy $(\mathfrak{S}_n)$ for every integer $n > 1$, but do not satisfy $(\mathfrak{R}_n)$ for every positive integer n (Lemma 3.6).

(φ,ψ) is called an *n-fold* homomorphism (n a positive integer) if $(\mathfrak{R}_n)$ *and* $(\mathfrak{S}_n)$ hold, and a *0-fold* homomorphism is simply a homomorphism in the original sense. Every n-fold homomorphism is also an m-fold homomorphism for every non-negative integer $m < n$. Therefore we can assign to every homomorphism (φ,ψ) as its *degree* the largest integer n such that (φ,ψ) is an n-fold but not an $(n+1)$-fold homomorphism if such an integer exists, and ∞ if it does not exist (Definition 4.5). Every isomorphism has degree ∞, but the converse does not hold (Lemma 4.7).

For every non-negative integer n the class of all permutation structures together with their n-fold homomorphisms form a subcategory $\mathfrak{P}_n$ of the category $\mathfrak{P}$ of all permutation structures. $\mathfrak{P}$ coincides with $\mathfrak{P}_0$.

In his Introduction to [7] WIELANDT has remarked that the theory of permutation groups can be characterized as the theory of conjugate subgroups and their intersections. Our approach to the theory of permutation groups proceeds in that direction as well. For homogeneous spaces, i.e. when G acts transitively on Ω, the condition $(\mathfrak{S}_n)$ is equivalent to the following generalization of normal subgroups.

Let n be a non-negative integer, let G be a group, and let H be a subgroup of G. A subgroup K of G is called *n-fold G/H-normal* if

$$H \leqq K \text{ and } Kg(H^{x_1} \cap \ldots \cap H^{x_n}) = Kg(K^{x_1} \cap \ldots \cap K^{x_n})$$

for all $g, x_1, \ldots, x_n \in G$. Thus a 0-fold G/H-normal subgroup is simply a subgroup of G containing H. A 1-fold G/H-normal subgroup is a G/H-normal subgroup in the sense of [3], Definition 1.9. To every subgroup K of G containing H we assign as its *G/H-degree* the largest non-negative integer n such that K is n-fold, but not $(n+1)$-fold G/H-normal, if such an integer exists, and ∞ if it does not exist (Definition 7.6). Every normal subgroup of G containing H has G/H-degree ∞. As an illustration we look at the alternating group A_5 which provides an example of a subgroup of G/H-degree 1 (7.13).

For every homomorphism (φ,ψ) of a homogeneous space $(\Omega, G, \cdot)$ we define the *kernel* $\mathrm{Ker}_\alpha(\varphi,\psi)$ of (φ,ψ) with respect to $\alpha \in \Omega$ as the subgroup of all those elements of G which ψ maps into the stabilizer of $\alpha\varphi$ in G'. The *Homomorphism Theorem* 8.5 shows that $\mathrm{Ker}_\alpha(\varphi,\psi)$ of an n-fold homomorphism (φ,ψ) is an $(n-1)$-fold G/G_α-normal subgroup of G where G_α denotes the stabilizer of α in G. If, in addition, ψ is an isomorphism of G, then the image space $(\Omega\varphi, G\psi, \cdot)$ is isomorphic to the homogeneous space $(\mathrm{Ker}_\alpha(\varphi,\psi){:}\, G, G, \cdot)$ which is given by the multiplication of the cosets $\mathrm{Ker}_\alpha(\varphi,\psi)g$, $g \in G$, by the elements of G. Conversely the *Canonic Epimorphism Theorem* 8.6 shows that every $(n-1)$-fold G/H-normal subgroup K of G defines an n-fold homomorphism (φ_K, i_G) of the homogeneous space $(H{:}G, G, \cdot)$ defined by the cosets Hg, $g \in G$, onto the homogeneous space $(K{:}G, G, \cdot)$ defined by the cosets Kg, $g \in G$, where $\varphi_K\colon Hg \to Kg$ and i_G is the identity mapping of G.

These two theorems show that the $(n-1)$-fold G/H-normal subgroups of G are exactly the kernels of the n-fold homomorphisms of the homogeneous space $(H{:}G, G, \cdot)$. Therefore G is called $\frac{1}{n}$-fold G/H-simple (n a non-negative integer) if $H < G$ and if there exists no n-fold G/H-normal subgroup properly between G and H. Hence G is $\frac{1}{0} = \infty$-fold G/H-simple if and only if H is a maximal subgroup of G. Furthermore G is called *G/H-simple of degree* $\frac{1}{n}$ if G is $\frac{1}{n}$-fold but not $\frac{1}{n-1}$-fold G/H-simple, if such a non-negative integer n exists, and G is called *G/H-simple of degree* 0 if such an integer does not exist. For example, if G has a normal subgroup properly between G and H, then G is G/H-simple of degree 0.

This notion of simplicity can be carried over to homogeneous spaces such that a homogeneous space $(\Omega, G, \cdot)$ is *$\frac{1}{n}$-fold simple* (Definition 9.8) if G is $\frac{1}{n-1}$-fold G/G_α-simple for $\alpha \in \Omega$ (Proposition 9.9). Also $(\Omega, G, \cdot)$ is *simple of degree* $\frac{1}{n}$ or 0 if G is G/G_α-simple of degree $\frac{1}{n-1}$ or 0 respectively. For instance $(\Omega, G, \cdot)$ is simple of (the highest possible) degree 1 if G acts as a primitive permutation group on Ω.

We can prove the First Isomorphism Theorem 10.1 for homogeneous spaces and n-fold homomorphisms. As for the Second Isomorphisms Theorem

10.2 we prove the following result. If K is an n-fold G/H-normal subgroup, and if L is any subgroup of G containing H, then $K \cap L$ is an n-fold L/H-normal subgroup of L. But in general the homogeneous spaces $(K \cap L{:}L,\ L, \cdot)$ and $(K{:}KL,\ KL, \cdot)$ will not be isomorphic, yet they are «almost» isomorphic. The bijective mapping $(K \cap L)x \to Kx$ together with the injection of L into KL is a homomorphism of $(K \cap L{:}L,\ L, \cdot)$ into $(K{:}KL,\ KL, \cdot)$ which satisfies $(\mathfrak{S}_{n+1})$. Because of this fact we introduce another concept.

A homomorphism (φ, ψ) of a permutation structure $(\Omega, G, \cdot)$ into a permutation structure $(\Omega', G', \cdot)$ is called an *n-fold pre-isomorphism* if φ is a bijective mapping, if ψ is a monomorphism, and if $(\mathfrak{S}_n)$ holds (Definition 5.1). Two permutation structures $(\Omega, G, \cdot)$ and $(\Omega', G', \cdot)$ are called *n-fold pre-isomorphic* if they can be joined by a finite chain of homogeneous spaces such that for any two successive homogeneous spaces of that chain there exists an n-fold pre-isomorphism from the predecessor to the successor, or from the successor to the pre-deccessor (Definition 5.3). This concept gives a proper decomposition of the class of all permutation structures into classes of n-fold pre-isomorphic permütation structures. Thus our Second Isomorphism Theorem really is the Second Pre-isomorphism Theorem with the homogeneous spaces $(K \cap L{:}L,\ L, \cdot)$ and $(K{:}KL,\ KL, \cdot)$ as being $(n+1)$-fold pre-isomorphic.

There are two further reasons for introducing pre-isomorphy. The first is shown in the following, the second will be mentioned later. The n-fold G/H-normality leads to a notion of *n-fold G/H-subnormal subgroups* (Definition 11.1). We have various possibilities to assign factors to an *n-fold G/H-subnormal chain* $G = L_0 \geqq L_1 \geqq \dots \geqq L_r = L$. They are discussed at the beginning of Section 11. If we take the homogeneous spaces $(L_i{:}\ L_{i-1},\ L_{i-1}, \cdot)$ for $i=1, \dots,$ r as its factors, and if we try to prove a Jordan-Hölder Theorem for *n-fold G/H-composition chains*, then such a theorem would not hold with the isomorphy relation of the composition factors, but it might hold with the n-fold pre-isomorphy relation. All that we can show in this paper is that the *Theorem of Jordan and Hölder* is true for 1-fold G/H-composition chains and homogeneous spaces as factors (Theorem 11.10). This is a generalization of the Theorem of Jordan and Hölder for G/H-composition chains ([4], Theorem 3.3) where the subgroups are the same but where the factors are the double coset semigroups L_{i-1}/L_i.

Finally we suggest a classification of homogeneous spaces by double coset semigroups. For any homogeneous space $(\Omega, G, \cdot)$ the class $[G/G_\alpha]$ of all double coset semigroups isomorphic to the double coset semigroup G/G_α, $\alpha \in \Omega$, is called the *type* of $(\Omega, G, \cdot)$. For instance, all homogeneous spaces such that G acts 2-fold transitively on Ω with $|\Omega| > 2$ are of the same type (Lemma 12.18). If two homogeneous spaces are of the same type then the orbits of the stabilizers of one fixed letter are in one-to-one correspondence, but being of the same type implies more than this fact. We show that the double coset semigroup G/G_α has the meaning of a sort of «endomorphism ring» of the homogeneous space (Theorem 12.14). Every homomorphism (φ,ψ) of a homogeneous space $(\Omega,G,\cdot)$ into a homogeneous space $(\Omega',G',\cdot)$ which satisfies $(\mathfrak{S}_2)$ induces a homomorphism of the «endomorphism ring» of $(\Omega,G,\cdot)$ into the «endomorphism ring» of $(\Omega',G',\cdot)$, and also a homomorphism of the double coset semigroup G/G_α into the double coset semigroup $G'/G'_{\alpha\varphi}$ (Theorem 12.19). If φ is a surjective mapping, then condition $(\mathfrak{S}_2)$ is also necessary for inducing a homomorphism of the «endomorphism ring» (Theorem 12.20). 2-fold pre-isomorphic homogeneous spaces are of the same type (Theorem 12.22 and Corollary 12.23). This is the other motive for introducing the concept of n-fold pre-isomorphy since it subdivides the classes of the homogeneous spaces of the same type.

Several concepts introduced in this paper can be extended to arbitrary ordinal numbers instead of non-negative integers, but we do not go beyond integers in this paper.

1. - The Category of Permutation Structures

In order to deal properly with permutation groups, a permutation group cannot be considered as a mere group, but as a mathematical structure which consists of a set of letters, a group, and an external algebraic composition of that set with that group as its operator domain ([1], § 7, n° 2).

DEFINITION 1.1. - *Assume that*

(1) Ω *is a non-empty set,*

(2) G *is a group (whose unit element will be denoted by* 1),

(3) $(\alpha,g)\to\alpha g$ *is a mapping of* $\Omega\times G$ *into* Ω *such that* $(\alpha g)h=\alpha(gh)$ *and* $\alpha 1=\alpha$ *for all* $\alpha\in\Omega$ *and all* $g,h\in G$.

Then $(\Omega,G,\cdot)$ *is called a* permutation structure.

The dot in the third place of $(\Omega,G,\cdot)$ stands for the external algebraic composition. For clarity's sake we do not omit the external composition in the notation of a permutation structure.

For every $\Delta\subseteq\Omega$ and every $X\subseteq G$ we write

$$\Delta X=\{\delta x\mid \delta\in\Delta \text{ and } x\in X\}.$$

For every $g\in G$ the mapping

$$g\pi_\Omega:\ \alpha\to\alpha g$$

is a permutation of Ω, and the mapping

$$\pi_\Omega:\ g\to g\pi_\Omega$$

is a homomorphism of the group G into the symmetric group S^{Ω} ([1], § 7, n° 2, Proposition 1).

The permutation structure $(\Omega, G, \cdot)$ is called *faithful*, if π_{Ω} is a monomorphism, that is if Ker $\pi_{\Omega} = 1$.

$(\Omega, G\pi_{\Omega}, \cdot)$ is a permutation structure with respect to the external composition

$$(\alpha, g\pi_{\Omega}) \to \alpha(g\pi_{\Omega}) = \alpha g.$$

We call $(\Omega, G\pi_{\Omega}, \cdot)$ the *canonic representation* of $(\Omega, G, \cdot)$.

To make the class of all permutation structures a category it is quite obvious how to define the morphisms ([1], § 7, n° 4).

DEFINITION 1.2. - *Let $(\Omega, G, \cdot)$ and $(\Omega', G', \cdot)$ be permutation structures. Assume that*

(1) *φ is a mapping of Ω into Ω',*

(2) *ψ is a homomorphism of G into G',*

(3) *$(\alpha g)\varphi = (\alpha\varphi)(g\psi)$ for all $\alpha \in \Omega$ and all $g \in G$.*

Then the pair (φ, ψ) is called a homomorphism *of the permutation structure $(\Omega, G, \cdot)$ into the permutation structure $(\Omega', G', \cdot)$. The permutation structure*

$$\text{Im } (\varphi, \psi) := (\Omega\varphi, G\psi, \cdot)$$

with the restriction of the external composition of $(\Omega', G', \cdot)$ to $\Omega\varphi \times G\psi$ as its external algebraic composition is called the image *of (φ, ψ).*

A homomorphism (φ, ψ) is called an epimorphism, *a* monomorphism, *or an* isomorphism, *if φ and ψ are both surjective, injective, or bijective mappings respectively.*

PROPOSITION 1.3. - *The class of all permutation structures together with their homomorphisms form a category $\mathfrak{S}$.*

Here the product of a homomorphism (φ, ψ) of $(\Omega, G, \cdot)$ into $(\Omega', G', \cdot)$ with a homomorphism (φ', ψ') of $(\Omega', G', \cdot)$ into $(\Omega'', G'', \cdot)$ is defined componentwise as

$$(\varphi, \psi)\ (\varphi', \psi') := (\varphi\varphi', \psi\psi').$$

The pair (i_{Ω}, i_G) of identity mappings is a homomorphism of the permutation structure $(\Omega, G, \cdot)$ into itself.

For any two permutation structures $(\Omega, G, \cdot)$ and $(\Omega', G', \cdot)$ and any $\alpha' \in \Omega'$ the pair $(O_{\alpha'}, O_{G'})$ of mappings

$$O_{\alpha'}: \Omega \to \Omega' \text{ and } O_{G'}: G \to G'$$

which are defined by

$$\xi O_{\alpha'} = \alpha' \text{ for all } \xi \in \Omega,$$

$$gO_{G'} = 1' \ (= \text{unit element of } G') \text{ for all } g \in G,$$

is a homomorphism of $(\Omega, G, \cdot)$ into $(\Omega', G', \cdot)$. Such a homomorphism is called a *zero-homomorphism*. With these remarks it is easy to work out a detailed proof of 1.3.

We emphasize that our definitions of epimorphism and monomorphism are naive ones. It is easy to see that our epimorphisms are epic, and our monomorphisms are monic in the category $\mathfrak{P}$. We do not investigate the problem of determining all those homomorphisms which are epic, respectively monic, in the category $\mathfrak{P}$.

Every homomorphism of a mathematical structure can be considered as an approximation of that structure by a – more or less – simpler one. For the homomorphisms of the permutation structures we are going to introduce a measure for the degree of approximation.

2. - *G*-relations and Orbits of Stabilizers

In order to introduce a concept of the degree of a homomorphism we recall briefly some basic definitions and facts from the theory of permutation groups.

Let $(\Omega, G, \cdot)$ be a permutation structure, and let n be a positive integer. We set

$$\Omega^n = \underbrace{\Omega \times \dots \times \Omega}_{n} = \{(\alpha_1, \dots, \alpha_n) \mid \alpha_i \in \Omega\}.$$

Then $(\Omega^n, G, \cdot)$ is a permutation structure with respect to componentwise composition

$$((\alpha_1, \dots, \alpha_n), g) \rightarrow (\alpha_1, \dots, \alpha_n)g := (\alpha_1 g, \dots, \alpha_n g).$$

DEFINITION 2.1. – (WIELANDT [7], 10.1). *A subset R of Ω^n is called an* n-nary G-relation *on Ω, if R is G-invariant, that is if*

$$(\alpha_1, \dots, \alpha_n)g \in R \text{ for all } (\alpha_1, \dots, \alpha_n) \in R \text{ and all } g \in G.$$

DEFINITION 2.2. – ([7], p. 39). *An n-nary G-relation R on Ω is called* minimal, *if $R \neq \emptyset$, and if for every n-nary G-relation S on Ω*

$$S \subset R \text{ implies } S = \emptyset .$$

LEMMA 2.3. – ([7], *Proof of* 10.3). *An n-nary G-relation R on Ω is minimal if and only if R is an orbit of G on Ω^n, that is*

$$R = (\alpha_1, \dots, \alpha_n)G \text{ for some } (\alpha_1, \dots, \alpha_n) \in \Omega^n .$$

LEMMA 2.4. - ([7], 10.3). *Every n-nary G-relation on Ω is the set theoretical union of minimal n-nary G-relations.*

For any n elements $\alpha_1, \ldots, \alpha_n \in \Omega$, not necessarily all distinct, we denote by

$$G_{\alpha_1, \ldots, \alpha_n} := \{g \in G \mid \alpha_i g = \alpha_i \text{ for all } i = 1, \ldots, n\}$$

their *stabilizer* in G, which also is the stabilizer of $(\alpha_1, \ldots, \alpha_n) \in \Omega^n$ in G.

PROPOSITION 2.5. - *For any fixed* $(\alpha_1, \ldots, \alpha_{n-1}) \in \Omega^{n-1}$

$$\alpha_n G_{\alpha_1, \ldots, \alpha_{n-1}} \to (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) G$$

is a mapping of the set of all orbits on Ω of the stabilizer $G_{\alpha_1, \ldots, \alpha_{n-1}}$ of n-1 (not necessarily all distinct) letters $\alpha_1, \ldots, \alpha_{n-1}$ of Ω into the set of all minimal n-nary G-relations on Ω.

If G operates $(n-1)$-fold transitively on Ω, then for any fixed $(\alpha_1, \ldots, \alpha_{n-1})$ the mapping of 2.5 yields a one-to-one correspondence of the orbits of $G_{\alpha_1, \ldots, \alpha_{n-1}}$ to the minimal n-nary G-relations on Ω. For $n = 2$ that remark was already made in [7], 10.6.

What we try to point out by our rather trivial observation 2.5 is that the orbits of the stabilizers of n-1 letters also have structural significance beside the n-nary G-relations, and, in certain instances, even a finer one than the G-relations. In the next section we shall see that in general the G-relations and the orbits of the stabilizers have independent meanings for the homomorphisms of permutation structures. But for the homomorphisms of the homogeneous spaces the orbits of the stabilizers have the prior importance.

3. - Homomorphisms which preserve G-relations and Homomorphisms which preserve Orbits of Stabilizers.

Let n be a positive integer, and let (φ, ψ) be a homomorphism of the permutation structure $(\Omega, G, \cdot)$ into the permutation structure $(\Omega', G', \cdot)$. Then (φ, ψ) also defines a homomorphism (φ_n, ψ) of the permutation structure $(\Omega^n, G, \cdot)$ into the permutation structure $(\Omega'^n, G', \cdot)$ by the componentwise definition

$$(\alpha_1, \ldots, \alpha_n)\varphi_n = (\alpha_1\varphi, \ldots, \alpha_n\varphi) \text{ for all } (\alpha_1, \ldots, \alpha_n) \in \Omega^n$$

which yields

$$\begin{aligned}((\alpha_1, \ldots, \alpha_n)g)\varphi_n &= ((\alpha_1 g)\varphi, \ldots, (\alpha_n g)\varphi) \\ &= ((\alpha_1 \varphi)(g\psi), \ldots, (\alpha_n \varphi)(g\psi)) \\ &= ((\alpha_1, \ldots, \alpha_n)\varphi_n)(g\psi)\end{aligned}$$

for all $(\alpha_1, \ldots, \alpha_n) \in \Omega^n$ and all $g \in G$.

Every n-nary G-relation R on Ω is mapped by φ_n *into* an n-nary G'-relation R' on Ω', namely

$$R\varphi_n = \{(\alpha_1, \ldots, \alpha_n)\varphi_n \mid (\alpha_1, \ldots, \alpha_n) \in R\} \subseteqq R' = (R\varphi_n)G',$$

but not necessarily *onto* R'. Obviously those homomorphisms will have a special meaning which have the following property.

($\mathfrak{R}_n$) *For every n-nary G-relation R on Ω the image $R\varphi_n$ is an n-nary G'-relation on Ω'.*

LEMMA 3.1. - *($\mathfrak{R}_n$) holds if and only if*

$$((\alpha_1, \ldots, \alpha_n)G)\varphi_n = (\alpha_1\varphi, \ldots, \alpha_n\varphi)(G\psi) = (\alpha_1\varphi, \ldots, \alpha_n\varphi)G'$$
$$\textit{for all } (\alpha_1, \ldots, \alpha_n) \in \Omega^n.$$

This follows from Lemmas 2.3 and 2.4.

LEMMA 3.2. - *($\mathfrak{R}_n$) implies ($\mathfrak{R}_m$) for every positive integer $m < n$.*

PROOF. - Assume that the homomorphism (φ, ψ) satisfies ($\mathfrak{R}_n$) and $m < n$. Let S be a minimal m-nary G-relation on Ω, hence $S = (\alpha_1, \ldots, \alpha_m)G$ for some $(\alpha_1, \ldots, \alpha_m) \in \Omega^m$. We set

$$\alpha_i = \alpha_m \text{ for } i = m+1, \ldots, n.$$

Then $R = (\alpha_1, \ldots, \alpha_n)G$ is an n-nary G-relation on Ω, and

$$R\varphi_n = (\alpha_1\varphi, \ldots, \alpha_n\varphi)G'$$

is a minimal n-nary G'-relation on Ω' by Lemma 3.1. But then

$$S\varphi_m = (\alpha_1\varphi \ldots, \alpha_m\varphi)G'$$

is a minimal m-nary G'-relation on Ω', and therefore (φ, ψ) satisfies ($\mathfrak{R}_m$) by Lemma 3.1.

LEMMA 3.3. - *Let (φ, ψ) be a homomorphism such that ψ is an epimorphism. Then ($\mathfrak{R}_n$) holds for every positive integer n.*

The proof follows from Lemma 3.1.

Assume now that $n > 1$. Every stabilizer $G_{\alpha_1, \ldots, \alpha_{n-1}}$ of any $n-1$ letters from Ω (not necessarily all distinct) is mapped by ψ into the stabilizer of the images $\alpha_1\varphi, \ldots, \alpha_{n-1}\varphi$ in G', namely

$$G_{\alpha_1, \ldots, \alpha_{n-1}}\psi \leqq G'_{\alpha_1\varphi, \ldots, \alpha_{n-1}\varphi}$$

and therefore every orbit of $G_{\alpha_1, \dots, \alpha_{n-1}}$ is mapped *into* an orbit of $G'_{\alpha_1\varphi, \dots, \alpha_{n-1}\varphi}$, that is

$$(\alpha_n G_{\alpha_1, \dots, \alpha_{n-1}})\varphi = (\alpha_n \varphi)(G_{\alpha_1, \dots, \alpha_{n-1}} \psi) \subseteq (\alpha_n \varphi)\; G'_{\alpha_1\varphi, \dots, \alpha_{n-1}\varphi}$$

but usually not *onto* it. We shall look at those homomorphisms (φ,ψ) which have the following property.

$(\mathfrak{S}_n)$ *For every element* $(\alpha_1, \dots, \alpha_n) \in \Omega^n$ *the orbit* $\alpha_n G_{\alpha_1, \dots, \alpha_{n-1}}$ *of the stabilizer of* $(\alpha_1, \dots, \alpha_{n-1})$ *in* G *is mapped onto an orbit of the stabilizer of* $(\alpha_1\varphi, \dots, \alpha_{n-1}\varphi)$ *in* G', *that is*

$$(\alpha_n G_{\alpha_1, \dots, \alpha_{n-1}})\varphi = (\alpha_n \varphi)(G_{\alpha_1, \dots, \alpha_{n-1}}\psi) = (\alpha_n \varphi)\, G_{\alpha_1\varphi, \dots, \alpha_{n-1}\varphi}\,.$$

For $n = 1$ *we set* $(\mathfrak{S}_1) = (\mathfrak{R}_1)$.

LEMMA 3.4. – $(\mathfrak{S}_n)$ *implies* $(\mathfrak{S}_m)$ *for every integer* m *such that* $1 < m < n$.

PROOF. – For every $(\alpha_1, \dots, \alpha_m) \in \Omega^m$ we set

$$\alpha'_i = \begin{cases} \alpha_i & \text{for } i = 1, \dots, m-1, \\ \alpha_{m-1} & \text{for } i = m, \dots, n-1, \\ \alpha_m & \text{for } i = n. \end{cases}$$

Then

$$\begin{aligned}(\alpha_m G_{\alpha_1, \dots, \alpha_{m-1}})\varphi &= (\alpha'_n G_{\alpha'_1, \dots, \alpha'_{n-1}})\varphi \\ &= (\alpha'_n \varphi)\, G'_{\alpha'_1\varphi, \dots, \alpha'_{n-1}\varphi} \\ &= (\alpha_m \varphi) G'_{\alpha_1\varphi, \dots, \alpha_{m-1}\varphi}\end{aligned}$$

proves $(\mathfrak{S}_m)$.

LEMMA 3.5. – *There exists a homomorphism* (φ,ψ) *such that* $(\mathfrak{R}_n)$ *holds for every positive integer* n, *but* $(\mathfrak{S}_n)$ *does not hold for any integer* $n > 1$.

PROOF. – Take any doubly transitive permutation group G' on Ω' such that $|\Omega'| > 2$. Set

$$\Omega = \{(\alpha,\beta) \mid \alpha,\beta \in \Omega' \;\; and \;\; \alpha \neq \beta\},$$
$$G = G',$$
$$\varphi : (\alpha,\beta) \to \alpha,$$
$$\psi = i_G \text{ the identity mapping of } G.$$

Then

$$((\alpha,\beta)g)\varphi = (\alpha g, \beta g)\varphi = \alpha g = ((\alpha,\beta)\varphi)g = ((\alpha,\beta)\varphi)(g\psi)$$

for all letters $(\alpha,\beta) \in \Omega$ and all $g \in G$, and therefore (φ,ψ) is a homomorphism of $(\Omega, G, \cdot)$ into $(\Omega', G', \cdot)$. There exist letters (α_1,α_2), $(\alpha_2,\alpha_3) \in \Omega$ because of $|\Omega'| > 2$.

$G_{(\alpha_1,\alpha_2)}$ is the stabilizer of the single letter $(\alpha_1,\alpha_2) \in \Omega$ in G.

$G_{(\alpha_1,\alpha_2)}\,\psi = G'_{\alpha_1,\alpha_2}$ is the stabilizer of the letters $\alpha_1,\alpha_2 \in \Omega'$ in G'.

$G'_{(\alpha_1,\alpha_2)\varphi} = G'_{\alpha_1}$ is the stabilizer of the single letter $\alpha_1 \in \Omega'$ in G'.

Therefore we obtain

$$((\alpha_2,\alpha_3)G_{(\alpha_1,\alpha_2)})\varphi = ((\alpha_2,\alpha_3)\varphi)(G_{(\alpha_1,\alpha_2)}\psi)$$
$$= \alpha_2 G'_{\alpha_1,\alpha_2} = \{\alpha_2\} \subset \alpha_2 G'_{\alpha_1} = \Omega' - \{\alpha_1\}$$

because of the double transitivity of G' and of $|\Omega'| > 2$. This fact violates $(\mathfrak{S}_2)$ and hence also $(\mathfrak{S}_n)$ for every $n > 1$ by Lemma 3.4. But $(\mathfrak{R}_n)$ holds for every positive integer n because of Lemma 3.3.

LEMMA 3.6. - *There exists a homomorphism (φ,ψ) such that $(\mathfrak{S}_n)$ holds for every integer $n > 1$, but $(\mathfrak{R}_n)$ does not hold for any positive integer n.*

PROOF. - Let G' be a transitive permutation group on Ω', and take any subgroup G of G' such that $G'_\alpha \leqq G < G'$ for $\alpha \in \Omega'$. Set $\Omega = \alpha G$ and let φ and ψ be the injections of Ω into Ω' and of G into G' respectively. Then (φ,ψ) is a homomorphism of $(\Omega, G, \cdot)$ into $(\Omega', G', \cdot)$. For every $n > 1$ and every $(\alpha_1, \ldots, \alpha_{n-1}) \in \Omega^{n-1}$ we have

$$G_{\alpha_1, \ldots, \alpha_{n-1}} = G'_{\alpha_1, \ldots, \alpha_{n-1}} = G'_{\alpha_1} \cap \ldots \cap G'_{\alpha_{n-1}} \leqq G$$

since all G'_{α_i} are conjugate to G'_α in G. Therefore

$$(\alpha_n G_{\alpha_1, \ldots, \alpha_{n-1}})\varphi = (\alpha_n\varphi)(G_{\alpha_1, \ldots, \alpha_{n-1}}\psi)$$
$$= (\alpha_n\varphi)G'_{\alpha_1\varphi, \ldots, \alpha_{n-1}\varphi}$$

which means that $(\mathfrak{S}_n)$ holds.

On the other hand

$$((\alpha_1, \ldots, \alpha_n)G)\varphi = (\alpha_1, \ldots, \alpha_n)G \neq (\alpha_1, \ldots, \alpha_n)G' = (\alpha_1\varphi, \ldots, \alpha_n\varphi)G'$$

for every $(\alpha_1, \ldots, \alpha_n) \in \Omega^n$ and every positive integer n. Therefore $(\mathfrak{R}_n)$ does not hold for any positive integer n.

4. - The Degree of a Homomorphism

Our investigations of the foregoing section give the background for the following notion.

DEFINITION 4.1. - *Let n be a positive integer. A homomorphism* (φ,ψ) *of the permutation structure* $(\Omega, G, \cdot)$ *into the permutation structure* $(\Omega', G', \cdot)$ *is called an n-*fold homomorphism *if both* $(\mathfrak{R}_n)$ *and* $(\mathfrak{S}_n)$ *hold. Every homomorphism according to Definition* 1.2 *is called a* 0-fold homomorphism.

(φ,ψ) is a 1-fold homomorphism if and only if every $G\psi$-orbit on $\Omega\varphi$ is also a G'-orbit. Remember that $(\mathfrak{R}_1) = (\mathfrak{S}_1)$ by definition.

LEMMA 4.2. - *Let n be a positive integer, let* (φ,ψ) *be a homomorphism of the permutation structure* $(\Omega, G, \cdot)$, *into the permutation structure* $(\Omega', G', \cdot)$, *and* (φ',ψ') *be a homomorphism of the permutation structure* $(\Omega', G', \cdot)$ *into the permutation structure* $(\Omega'', G'' \cdot)$. *Then for the homomorphism* $(\varphi\varphi', \psi\psi')$ *of* $(\Omega, G, \cdot)$ *into* $(\Omega'', G'', \cdot)$ *the following hold.*

(1) *If* (φ,ψ) *and* (φ',ψ') *satisfy* $(\mathfrak{R}_n)$, *then* $(\varphi\varphi',\psi\psi')$ *satisfies* $(\mathfrak{R}_n)$.

(2) *If* (φ,ψ) *and* (φ',ψ') *satisfy* $(\mathfrak{S}_n)$, *then* $(\varphi\varphi',\psi\psi')$ *satisfies* $(\mathfrak{S}_n)$.

(3) *If* (φ,ψ) *and* (φ',ψ') *are n-fold homomorphisms, then* $(\varphi\varphi',\psi\psi')$ *is an n-fold homomorphism.*

PROOF. - I. Let (φ,ψ) and (φ',ψ') satisfy $(\mathfrak{R}_n)$. We have $(\varphi\varphi')_n = \varphi_n\varphi'_n$ and for all $(\alpha_1, \ldots, \alpha_n) \in \Omega^n$

$$\begin{aligned}((\alpha_1, \ldots, \alpha_n)G)(\varphi_n\varphi'_n) &= (((\alpha_1, \ldots, \alpha_n)G)\varphi_n)\varphi'_n \\ &= ((\alpha_1\varphi, \ldots, \alpha_n\varphi)G')\varphi'_n \\ &= (\alpha_1\varphi\varphi', \ldots, \alpha_n\varphi\varphi')G''.\end{aligned}$$

Therefore $(\mathfrak{R}_n)$ holds for $(\varphi\varphi',\psi\psi')$.

II. Let (φ,ψ) and (φ',ψ') satisfy $(\mathfrak{S}_n)$. Then for all $(\alpha_1, \ldots, \alpha_n) \in \Omega^n$

$$\begin{aligned}(\alpha_n G_{\alpha_1, \ldots, \alpha_{n-1}})\varphi\varphi' &= ((\alpha_n\varphi)G'_{\alpha_1\varphi, \ldots, \alpha_{n-1}\varphi})\varphi' \\ &= (\alpha_n\varphi\varphi')G''_{\alpha_1\varphi\varphi', \ldots, \alpha_{n-1}\varphi\varphi'}.\end{aligned}$$

Therefore $(\mathfrak{S}_n)$ holds for $(\varphi\varphi',\psi\psi')$.

III. (3) clearly follows from (1) and (2).

From 4.2(3) and from the fact that for every non-negative integer n the pair (i_Ω, i_G) of identity mappings is an n-fold homomorphism of the permutation structure $(\Omega, G, \cdot)$ into itself, we conclude the following

PROPOSITION 4.3. - *For every non-negative integer n the class of all permutation structures together with their n-fold homomorphisms form a category* $\mathfrak{F}_n$.

Note that $\mathfrak{F}_0 = \mathfrak{F}$.

LEMMA 4.4. - *Every n-fold homomorphism is also an m-fold homomorphism for all non-negative integers* $m \leqq n$.

This statement follows from the Lemmas 3.2 and 3.4, and from 4.1. Hence we have $\mathfrak{S}_m \subseteq \mathfrak{S}_n$ for all non-negative integers $m \leqq n$. Lemma 4.4 is also the basis for the

DEFINITION 4.5. - *Let* (φ,ψ) *be a homomorphism of the permutation structure* $(\Omega, G, \cdot)$ *into the permutation structure* $(\Omega', G', \cdot)$. *We define the* degree *of* (φ,ψ) *as*

$$\deg(\varphi,\psi) = \begin{cases} n & \text{if there exists a non-negative integer } n \text{ such that } (\varphi,\psi) \text{ is} \\ & \text{an } n\text{-fold but not an } (n+1)\text{-fold homomorphism,} \\ \infty & \text{if } (\varphi,\psi) \text{ is an } n\text{-fold homomorphism for all non-negative} \\ & \text{integers } n. \end{cases}$$

LEMMA 4.6. - *Let* (φ,ψ) *be a homomorphism of the permutation structure* $(\Omega, G, \cdot)$ *into the permutation structure* $(\Omega', G', \cdot)$ *such that* φ *is an injective mapping and* ψ *is an epimorphism. Then*

(1) $G_{\alpha_1, \dots, \alpha_n}\psi = G'_{\alpha_1\varphi, \dots, \alpha_n\varphi}$ *for all* $(\alpha_1, \dots, \alpha_n) \in \Omega^n$ *and all positive integers* n.

(2) $\deg(\varphi,\psi) = \infty$.

Especially, every isomorphism of a permutation structure has degree ∞; the more general Lemma 4.6 was pointed out to the author by DIETRICH HELMER.

PROOF. I. - $G_{\alpha_1, \dots, \alpha_n}\psi \leqq G'_{\alpha_1\varphi, \dots, \alpha_n\varphi}$ holds by Definition 1.2 for any homomorphism (φ,ψ) without further assumptions. By the additional hypotheses of our lemma the opposite inclusion holds as well. For if $g' \in G'_{\alpha_1\varphi, \dots, \alpha_n\varphi}$, then there exists a $g \in G$ such that $g' = g\psi$ since ψ is an epimorphism, and hence

$$(\alpha_i g)\varphi = (\alpha_i\varphi)(g\psi) = (\alpha_i\varphi)g' = \alpha_i\varphi \text{ for } i = 1, \dots, n.$$

It follows that

$$\alpha_i g = \alpha_i \text{ for } i = 1, \dots, n,$$

since φ is injective, and hence $g \in G_{\alpha_1, \dots, \alpha_n}$ and $g' = g\psi \in G_{\alpha_1, \dots, \alpha_n}\psi$. Thus we have proved (1).

II. For any positive integer n $(\mathfrak{R}_n)$ holds by 3.3, and $(\mathfrak{S}_n)$ holds by (1). Therefore $\deg(\varphi,\psi) = \infty$.

LEMMA 4.7. - *There exist homomorphisms of degree* ∞ *which are not isomorphisms.*

PROOF. - Let $(\Omega', G', \cdot)$ be a permutation structure such that G' has at least 2 orbits on Ω'. Let Ω be an orbit of G' on Ω'. We set $G = G'$, φ = injection of Ω into Ω', $\psi = i_G$. Then (φ,ψ) is a homomorphism of $(\Omega, G, \cdot)$ into $(\Omega', G', \cdot)$ such that $\deg(\varphi,\psi) = \infty$, but (φ,ψ) is no isomorphism.

5. - n-fold Pre-isomorphic Permutation Structures

Before we concentrate on the main subject of this paper, namely on homogeneous spaces, we introduce another concept that we shall need later for the Second Isomorphism Theorem and for a generalization of the Theorem of Jordan and Hölder.

In the following let n always denote a positive integer, and let $(\Omega, G, \cdot)$ and $(\Omega', G', \cdot)$ be permutation structures.

DEFINITION 5.1. - *A homomorphism* (φ, ψ) *of* $(\Omega, G, \cdot)$ *into* $(\Omega', G', \cdot)$ *is called an* n-fold pre-isomorphism *if* φ *is a bijective mapping, if* ψ *is a monomorphism, and if* $(\mathcal{S}_n)$ *holds.*

LEMMA 5.2. - *Let* (φ, ψ) *be an n-fold pre-isomorphism of* $(\Omega, G, \cdot)$ *into* $(\Omega', G', \cdot)$. *Then for all* $(\alpha'_1, \ldots, \alpha'_n) \in \Omega'^n$

$$(\alpha'_n G'_{\alpha'_1, \ldots, \alpha'_{n-1}})\varphi^{-1} = (\alpha'_n \varphi^{-1}) G_{\alpha'_1 \varphi^{-1}, \ldots, \alpha'_{n-1}\varphi^{-1}}$$

PROOF. - Definition of $(\mathcal{S}_n)$.

DEFINITION 5.3. - $(\Omega, G, \cdot)$ *and* $(\Omega', G', \cdot)$ *are called* n-fold pre-isomorphic *if there exists a finite number of permutation structures*

$$(\Omega_i, G_i, \cdot) \qquad (i = 1, \ldots, m)$$

with the following properties.

(1) $(\Omega, G, \cdot) = (\Omega_1, G_1 \cdot)$ *and* $(\Omega_m, G_m, \cdot) = (\Omega', G', \cdot)$,

(2) *For every* $i = 1, \ldots, m-1$ *there exists an n-fold pre-isomorphism* (φ_i, ψ_i) *of* $(\Omega_i, G_i, \cdot)$ *into* $(\Omega_{i+1}, G_{i+1}, \cdot)$ *or of* $(\Omega_{i+1}, G_{i+1}, \cdot)$ *into* $(\Omega_i, G_i, \cdot)$.

PROPOSITION 5.4. - *Let* $(\Omega, G, \cdot)$ *and* $(\Omega', G', \cdot)$ *be n-fold pre-isomorphic permutation structures. Then there exists a bijective mapping* $\varphi\colon \Omega \to \Omega'$ *such that for all* $(\alpha_1, \ldots, \alpha_n) \in \Omega^n$

$$(\alpha_n G_{\alpha_1, \ldots, \alpha_{n-1}})\varphi = (\alpha_n \varphi) G'_{\alpha_1\varphi, \ldots, \alpha_{n-1}\varphi} \cdot$$

PROOF. - Let (φ_i, ψ_i) and $(\Omega_i, G_i, \cdot)$ for $i = 1, \ldots, m$ be n-fold pre-isomorphisms and permutation structures as in Definition 5.3. Set

$$\varepsilon_i = 1 \quad \text{if} \quad (\varphi_i, \psi_i) : (\Omega_i, G_i, \cdot) \to (\Omega_{i+1}, G_{i+1}, \cdot),$$

$$\varepsilon_i = -1 \quad \text{if} \quad (\varphi_i, \psi_i) : (\Omega_{i+1}, G_{i+1}, \cdot) \to (\Omega_i, G_i, \cdot), \ \varphi = \varphi_1^{\varepsilon_1} \cdots \varphi_{m-1}^{\varepsilon_{m-1}} \cdot$$

Our statement now follows from the definition of $(\mathcal{S}_n)$, and from Lemma 5.2.

LEMMA 5.5. - *Every isomorphism of any permutation structure is an n-fold pre-isomorphism for every non negative integer n.*

6. - Homogeneous Spaces

From now on we confine our investigations to those permutation structures $(\Omega, G, \cdot)$ where G acts transitively on Ω.

DEFINITION 6.1. - ([1], § 7, n° 6). *A permutation structure* $(\Omega, G, \cdot)$ *is called a* homogeneous space, *if for any fixed* $\alpha \in \Omega$ *the mapping* $(\alpha, g) \rightarrow \alpha g$ *is a surjective mapping of* Ω *onto itself, that is if*

$$\Omega = \alpha G \text{ for } \alpha \in \Omega.$$

A permutation structure $(\Omega, G, \cdot)$ is a homogeneous space if and only if $G\pi_\Omega$ is a transitive permutation group on Ω.

PROPOSITION 6.2. - *Every homomorphic image of a homogeneous space is a homogeneous space.*

PROOF. - Let (φ, ψ) be a homomorphism of the homogeneous space $(\Omega, G, \cdot)$ into a permutation structure $(\Omega', G', \cdot)$. Then $\Omega = \alpha G$ for every $\alpha \in \Omega$. Hence $\Omega\varphi = (\alpha\varphi)(G\psi)$, and therefore $\mathrm{Im}(\varphi, \psi) = (\Omega\varphi, G\psi, \cdot)$ is a homogeneous space.

LEMMA 6.3. - *Let* (φ, ψ) *be a homomorphism of the homogeneous space* $(\Omega, G, \cdot)$ *into the homogeneous space* $(\Omega', G', \cdot)$. *Then* (φ, ψ) *is a* 1*-fold homomorphism if and only if* φ *is a surjective mapping of* Ω *onto* Ω'.

PROOF. - $\mathrm{Im}(\varphi, \psi) = (\Omega\varphi, G\psi, \cdot)$ and $(\Omega', G', \cdot)$ are homogeneous spaces. It follows that

$$\Omega\varphi = (\alpha\varphi)(G\psi) \quad \text{and} \quad \Omega' = (\alpha\varphi)G' \text{ for every } \alpha \in \Omega.$$

Therefore (φ, ψ) is a 1-fold homomorphism if and only if the equality $\Omega\varphi = \Omega'$ holds.

PROPOSITION 6.4. - *For every non-negative integer n the homogeneous spaces together with their n-fold homomorphisms form a subcategory* $\mathcal{H}_n$ *of the category* $\mathcal{S}_n$.

We set $\mathcal{H} = \mathcal{H}_0$. Then $\mathcal{H}_n = \mathcal{H} \cap \mathcal{S}_n$.

LEMMA 6.5. - *Let G be a group, and let H be a subgroup of G. We set*

$$H : G = \{ Hg \mid g \in G \}.$$

Then $(H : G, G, \cdot)$ *is a homogeneous space with respect to the external composition*

$$(Hx, g) \rightarrow Hxg.$$

It is well known that every homogeneous space $(\Omega, G, \cdot)$ is isomorphic to the homogeneous space $(G_\alpha : G, G, \cdot)$ for any $\alpha \in \Omega$. We shall refine this statement. We shall see that the n-fold homomorphisms of the homogeneous spaces are linked with a special class of subgroups which we are now going to introduce.

7. - n-fold G/H-normal Subgroups

Let G be a group, H be a subgroup of G, and n a non-negative integer.

DEFINITION 7.1. - *A subgroup K of G is called* n-fold G/H-normal *if*

(1) $H \leqq K$,

(2) $Kg(H^{x_1} \cap \dots \cap H^{x_n}) = Kg(K^{x_1} \cap \dots \cap K^{x_n})$ *for all* $g, x_1, \dots, x_n \in G$.

LEMMA 7.2. - *Statement* 7.1(2) *is equivalent to*

(2′) $Kg(H \cap H^{x_1} \cap \dots \cap H^{x_{n-1}}) = Kg(K \cap K^{x_1} \cap \dots \cap K^{x_{n-1}})$ *for all* $g, x_1, \dots, x_{n-1} \in G$.

PROOF. - Obviously (2) implies (2′). If, conversely, (2′) holds, then

$$\begin{aligned} Kg(H^{x_1} \cap \dots \cap H^{x_n}) &= Kgx_1^{-1}(H \cap H^{x_2 x_1^{-1}} \cap \dots \cap H^{x_n x_1^{-1}})x_1 \\ &= Kgx_1^{-1}(K \cap K^{x_2 x_1^{-1}} \cap \dots \cap K^{x_n x_1^{-1}})x_1 \\ &= Kg(K^{x_1} \cap \dots \cap K^{x_n}) \end{aligned}$$

for all $g, x_1, \dots, x_n \in G$.

For $n = 0$ condition (2) can be considered as void, or the intersection of the empty set of subgroups can be considered as G. In either case we can state the following.

LEMMA 7.3. - *A subgroup K of G is 0-fold G/H-normal if and only if $H \leqq K$.*

For $n = 1$ condition (2′) reads as

$$(2'_1) \qquad KgH = KgK \text{ for all } g \in G.$$

Under the assumption of $H \leqq K$ it follows from $(2'_1)$ that

$$HgK \subseteqq KgK = KgH \text{ for all } g \in G,$$

and, by taking inverses,

$$KgH = (Hg^{-1}K)^{-1} \subseteqq (Kg^{-1}H)^{-1} = HgK$$

and therefore

$(2'_1)$ $$HgK = KgH \text{ for all } g \in G,$$

which means that K is G/H-normal in the sense of [3], Definition 1.9. Obviously $(2''_1)$ implies $(2'_1)$. Thus we have shown

LEMMA 7.4. – *A subgroup K of G is 1-fold G/H-normal if and only if K is G/H-normal.*

LEMMA 7.5. – *If K is an n-fold G/H-normal subgroup of G, then K is m-fold G/H-normal for every non-negative integer $m < n$.*

PROOF. – For every $g, x_1, \dots, x_m \in G$ we set $x_{m+1} = \dots = x_n = x_m$, and we obtain

$$\begin{aligned} Kg(H^{x_1} \cap \dots \cap H^{x_m}) &= Kg(H^{x_1} \cap \dots \cap H^{x_n}) \\ &= Kg(K^{x_1} \cap \dots \cap K^{x_n}) \\ &= Kg(K^{x_1} \cap \dots \cap K^{x_m}), \end{aligned}$$

and hence K is m-fold G/H-normal if it is n-fold G/H-normal and $m < n$.

DEFINITION 7.6. – *Let K be a subgroup of G containing H. We define the G/H-*degree *of K as*

$$\deg_{G/H} K = \begin{cases} n & \text{if there exists a non-negative integer } n \text{ such that } K \text{ is } n\text{-fold but not } (n+1)\text{-fold } G/H\text{-normal}, \\ \infty & \text{if } K \text{ is } n\text{-fold } G/H\text{-normal for all non-negative integers } n. \end{cases}$$

LEMMA 7.7. – *Let N be a normal subgroup of G containing H. Then the G/H-degree of N is ∞.*

Let n be a positive integer, and let K be an n-fold G/H-normal subgroup of G. Then by Lemma 7.4, Lemma 7.5 and [3], Section 1, the following hold.

7.8. $K = H(K \cap K^g)$ *for all* $g \in G$.

7.9. $H \leqq L \leqq G$ *implies* $KL = LK$.

7.10. $H \leqq L \leqq G$ *implies that* KL *is* 1-*fold* G/L-*normal.*

7.11. $H \leqq L \leqq G$ *implies* $\mathfrak{N}_G(L) \leqq \mathfrak{N}_G(KL)$ (*where* $\mathfrak{N}_G(X)$ *denotes the normalizer of* X *in* G).

7.12. *Assume that* $H \leqq N \leqq L \leqq G$, *and let* N *be a normal subgroup of* G. *Then* L *is* 1-*fold* G/H-*normal if and only if* L *is normal in* G.

Our remarks show that n-fold G/H-normality can be considered as an approximation of normality. If H is a normal subgroup of G then every non-normal subgroup K of G containing H has G/H-degree 0.

EXAMPLE 7.13. - *Let $G = A_5$ be the alternating group on 5 letters, and let H be a 5-Sylowgroup of G. Then $K = \mathfrak{N}_G(H)$, the normalizer of H in G, has G/H-degree* 1.

PROOF. - We take $h = (1\ 2\ 3\ 4\ 5)$ and set $H = \langle h \rangle$. Then $K = \langle h, k \rangle$ for $k = (2\ 5)\ (3\ 4)$. The double coset decompositions

$$G = K \cup K(1\ 2\ 3)H = K \cup K(1\ 2\ 3)K$$

show that K is 1-fold G/H-normal. $x = (2\ 3)\ (4\ 5)$ centralizes k, but does not normalize H. Therefore $H \cap H^x = 1$, but $K \cap K^x = \langle k \rangle$. With $g = h^x$ we obtain

$$Kg(H \cap H^x) = Kg \neq Kg \cup Kg^{-1} = Kg(K \cap K^x).$$

This inequality shows that K is not 2-fold G/H-normal, and our statement follows.

8. - The Homomorphism Theorem

In the following let (φ, ψ) be a homomorphism of the homogeneous space $(\Omega, G, \cdot)$ into the permutation structure $(\Omega', G', \cdot)$.

DEFINITION 8.1. - *For every $\alpha \in \Omega$*

$$\mathrm{Ker}_\alpha(\varphi, \psi) = \{ g \in G \mid (\alpha\varphi)(g\psi) = \alpha\varphi \}$$

is called the kernel *of (φ, ψ) with respect to α.*

LEMMA 8.2. - (1) $(\mathrm{Ker}_\alpha(\varphi, \psi))^g = \mathrm{Ker}_{\alpha g}(\varphi, \psi)$ *for all $\alpha \in \Omega$ and all $g \in G$.*

(2) $\{ \mathrm{Ker}_\alpha(\varphi, \psi) \mid \alpha \in \Omega \}$ *is a class of conjugate subgroups of G.*

(3) $G_\alpha \leqq \mathrm{Ker}_\alpha(\varphi, \psi)$ *for every $\alpha \in \Omega$.*

(4) $\mathrm{Ker}\ \psi \leqq \mathrm{Ker}_\alpha(\varphi, \psi)$ *for every $\alpha \in \Omega$.*

(5) $(\mathrm{Ker}_\alpha(\varphi, \psi))\psi \leqq G'_{\alpha\varphi}$ *for every $\alpha \in \Omega$.*

The proof is obvious.

We denote by $\mathrm{Ker}\ \varphi$ the equivalence relation on Ω which is defined by φ, that is for $\alpha, \beta \in \Omega$ we set

$$\alpha \equiv \beta \pmod{\mathrm{Ker}\ \varphi} \text{ if and only if } \alpha\varphi = \beta\varphi.$$

The equivalence class of $\alpha \in \Omega$ modulo $\mathrm{Ker}\varphi$ is denoted by $\widehat{\alpha}$, and $\widehat{\Omega} = \Omega / \mathrm{Ker}\ \varphi$ denotes the set of all equivalence classes of Ω modulo $\mathrm{Ker}\varphi$.

We denote by $g \to \widehat{g} = g\mathrm{Ker}\psi$ the canonic epimorphism of G onto $\widehat{G} = G/\mathrm{Ker}\psi$. For every subgroup H of G we write $\widehat{H} = H\mathrm{Ker}\psi/\mathrm{Ker}\psi$.

LEMMA 8.3. - $$(\widehat{\alpha}, \widehat{g}) \to \widehat{\alpha}\,\widehat{g} = \widehat{\alpha g}$$

is a mapping of $\widehat{\Omega} \times \widehat{G}$ *into* $\widehat{\Omega}$ *which makes*

$$(\widehat{\Omega}, \widehat{G}, \cdot) = (\Omega/\mathrm{Ker}\,\varphi, G/\mathrm{Ker}\,\psi, \cdot)$$

a homogeneous space.

PROOF. - If $\widehat{\alpha} = \widehat{\beta}$ and $\widehat{g} = \widehat{h}$ for $\alpha, \beta \in \Omega$ and $g, h \in G$, then

$$(\alpha g)\varphi = (\alpha\varphi)(g\psi) = (\beta\varphi)(h\psi) = (\beta h)\varphi$$

and hence $\widehat{\alpha g} = \widehat{\beta h}$. 1.1(3) is easily checked.

PROPOSITION 8.4. -

(1) $\widehat{\varphi} : \widehat{\alpha} \to \alpha\varphi$ *is a bijective mapping of* $\widehat{\Omega} = \Omega/\mathrm{Ker}\,\varphi$ *onto* $\Omega\varphi$.

(2) $\widehat{\psi} : \widehat{g} \to g\psi$ *is an isomorphism of* $\widehat{G} = G/\mathrm{Ker}\,\psi$ *onto* $G\psi$.

(3) $(\widehat{\varphi}, \widehat{\psi})$ *is an isomorphism of* $(\widehat{\Omega}, \widehat{G}, \cdot)$ *onto* $\mathrm{Im}(\varphi, \psi) = (\Omega\varphi, G\psi, \cdot)$.

PROOF. - (1) and (2) are obvious. For all $\widehat{\alpha} \in \widehat{\Omega}$ and all $\widehat{g} \in \widehat{G}$

$$(\widehat{\alpha}\widehat{g})\widehat{\varphi} = (\widehat{\alpha g})\widehat{\varphi} = (\alpha g)\varphi = (\alpha\varphi)(g\psi) = (\widehat{\alpha}\widehat{\varphi})(\widehat{g}\widehat{\psi})$$

holds. Therefore $(\widehat{\varphi}, \widehat{\psi})$ is an isomorphism of $(\widehat{\Omega}, \widehat{G}, \cdot)$ onto $\mathrm{Im}(\varphi, \psi) = (\Omega\varphi, G\psi, \cdot)$.

THE HOMOMORPHISM THEOREM 8.5. - *Let n be a positive integer and assume that (φ, ψ) is an n-fold homomorphism of the homogeneous space $(\Omega, G, \cdot)$ into a permutation structure $(\Omega', G', \cdot)$. Set $K_\alpha = \mathrm{Ker}_\alpha(\varphi, \psi)$. Then for every $\alpha \in \Omega$:*

(1) K_α *is an* $(n-1)$*-fold* G/G_α*-normal subgroup of* G.

(2) $\widehat{K}_\alpha$ *is an* $(n-1)$*-fold* $\widehat{G}/\widehat{G}_\alpha$*-normal subgroup of* $\widehat{G} = G/\mathrm{Ker}\,\psi$.

(3) $\mathrm{Im}\,(\varphi, \psi)$ *is isomorphic to the homogeneous space* $(\widehat{K}_\alpha : \widehat{G}, \widehat{G}, \cdot)$.

PROOF. - I. For $n = 1$ statement (1) holds by 7.3 and 8.2(3). Therefore we can assume that $n > 1$. Take any elements $g, x_2, \ldots, x_{n-1} \in G$. By 7.2 we have to show that

$$K_\alpha g(G_\alpha \cap G_\alpha^{x_2} \cap \ldots \cap G_\alpha^{x_{n-1}}) = K_\alpha g(K_\alpha \cap K_\alpha^{x_2} \cap \ldots \cap K_\alpha^{x_{n-1}}).$$

We set $x_1 = 1$ and

$$\alpha x_i = \alpha_i \text{ for } i = 1, \ldots, n-1, \text{ and } \alpha_n = \alpha g.$$

Take any $y \in K_\alpha g(K_\alpha \cap K_\alpha^{x_2} \cap \ldots \cap K_\alpha^{x_{n-1}})$. There exist elements

$$k, k_i \in K_\alpha \text{ and } l \in K_\alpha \cap K_\alpha^{x_2} \cap \ldots \cap K_\alpha^{x_{n-1}}$$

such that

$$y = kgl \text{ and } l = k_i^{x_i} \text{ for } i = 1, \ldots, n-1.$$

Therefore

$$(\alpha_i\varphi)(l\psi) = (\alpha\varphi)(x_i\psi)(x_i^{-1}\psi)(k_i\psi)(x_i\psi) = \alpha_i\varphi \quad (i = 1, \ldots, n-1)$$

$$(\alpha^n\varphi)(l\psi) = (\alpha\varphi)(k\psi)(g\psi)(l\psi) = (\alpha\varphi)(y\psi) = (\alpha y)\varphi$$

which means that $l\psi \in G'_{\alpha_1\varphi, \ldots, \alpha_{n-1}\varphi}$ and, by property $(\mathfrak{S}_n)$ of (φ, ψ),

$$(\alpha y)\varphi \in (\alpha_n\varphi)G'_{\alpha_1\varphi, \ldots, \alpha_{n-1}\varphi} = (\alpha_n\varphi)(G_{\alpha_1, \ldots, \alpha_{n-1}}\psi).$$

Hence there exists

$$h \in G_{\alpha_1, \ldots, \alpha_{n-1}} = G_\alpha \cap G_\alpha^{x_2} \cap \ldots \cap G_\alpha^{x_{n-1}}$$

such that

$$(\alpha y)\varphi = (\alpha_n\varphi)(h\psi),$$

which implies

$$(\alpha\varphi)(gl)\psi = (\alpha\varphi)(gh)\psi,$$

$$(\alpha\varphi)(glh^{-1}g^{-1})\psi = \alpha\varphi,$$

$$glh^{-1}g^{-1} \in K_\alpha,$$

$$gl \in K_\alpha gh,$$

$$y = kgl \in K_\alpha gh \subseteqq K_\alpha gG_{\alpha_1, \ldots, \alpha_{n-1}} = K_\alpha g(G_\alpha \cap G_\alpha^{x_2} \cap \ldots \cap G_\alpha^{x_{n-1}}).$$

Therefore $K_\alpha g(K_\alpha \cap K_\alpha^{x_2} \cap \ldots \cap K_\alpha^{x_{n-1}}) \subseteqq K_\alpha g(G_\alpha \cap G_\alpha^{x_2} \cap \ldots \cap G_\alpha^{x_{n-1}})$. The converse inclusion holds by 8.2(3), and we have proved that K_α is $(n-1)$-fold G/G_α-normal.

II. The canonic epimorphism $g \to \widehat{g} = g\,\mathrm{Ker}\psi$ of G onto $\widehat{G} = G/\mathrm{Ker}\psi$ clearly maps K_α onto an $(n-1)$-fold $\widehat{G}/\widehat{G}_\alpha$-normal subgroup of $\widehat{G}$.

III. $\varepsilon: \widehat{K}_\alpha \widehat{x} \to \widehat{\alpha}\widehat{x} = \widehat{\alpha x}$ is a bijective mapping of $\widehat{K}_\alpha : \widehat{G}$ onto $\widehat{\Omega}$ such that

$$((\widehat{K}_\alpha \widehat{x})\widehat{g})\,\varepsilon = (\widehat{K}_\alpha \widehat{xg})\,\varepsilon = \widehat{\alpha}\,\widehat{xg} = (\widehat{\alpha}\,\widehat{x})\,\widehat{g} = ((\widehat{K}_\alpha\,\widehat{x})\,\varepsilon)\,\widehat{g}$$

for all x, $g \in G$. Therefore $(\varepsilon, i_{\widehat{G}})$ is an isomorphism of $(\widehat{K}_\alpha : \widehat{G}, \widehat{G}, \cdot)$ onto $(\widehat{\Omega}, \widehat{G}, \cdot)$. (3) now follows from Proposition 8.4(3).

THE CANONIC EPIMORPHISM THEOREM 8.6. - *Let n be a positive integer, let G be a group, H a subgroup of G, and K an $(n-1)$-fold G/H-normal subgroup of G. Denote by φ_K the surjective mapping $Hx \to Kx$ of $H : G$ onto $K : G$, and by i_G the identity mapping of G. Then*

(1) (φ_K, i_G) *is an n-fold epimorphism of the homogeneous space* $(H : G, G, \cdot)$ *onto the homogeneous space* $(K : G, G, \cdot)$.

(2) $\mathrm{Ker}_H(\varphi_K, i_G) = K$.

We call (φ_K, i_G) the *canonic epimorphism* or the *projection* of $(H : G, G, \cdot)$ onto $(K : G, G, \cdot)$, and we call $(K : G, G, \cdot)$ the *factor space* of $(H : G, G, \cdot)$ modulo K. We write $(H : G, G, \cdot)/(H : K, K, \cdot) := (K : G, G, \cdot)$.

PROOF. - (φ_K, i_G) clearly is an epimorphism of $(H : G, G, \cdot)$ which satisfies $(\mathfrak{R}_n)$ by Lemma 3.3. $(\mathfrak{S}_n)$ for (φ_K, i_G) is equivalent to the $(n-1)$-fold G/H-normality of K by Definition 7.1. This proves (1), and (2) is obvious from Definition 8.1.

9. - $\frac{1}{n}$-fold G/H-simple Groups and $\frac{1}{n}$-fold Simple Homogeneous Spaces

Theorems 8.5 and 8.6 show that the $(n-1)$-fold G/H-normal subgroups of G are exactly the kernels of the n-fold homomorphisms of the homogeneous space $(H : G, G, \cdot)$. This fact permits the following definition.

DEFINITION 9.1. - *Let n be a non-negative integer, let G be a group, and let H be a proper subgroup of G. Then G is called* $\frac{1}{n}$-fold G/H-simple *if there exists no n-fold G/H-normal subgroup properly between G and H.*

We set $\frac{1}{0} = \infty$, and hence G is ∞-fold G/H-simple if and only if H is a maximal subgroup of G. Also G is 1-fold G/H-simple if and only if G is G/H-simple in the sense of [3], Definition 1.14.

LEMMA 9.2. - *If G is $\frac{1}{n}$-fold G/H-simple, then G is also $\frac{1}{m}$-fold G/H-simple for every integer $m > n$.*

PROOF. - Lemma 7.5.

DEFINITION 9.3. - *A group G is called G/H*-simple of degree $\frac{1}{n}$ *if there exists a non-negative integer n such that G is $\frac{1}{n}$-fold but not $\frac{1}{n-1}$-fold G/H-simple, and it is called G/H*-simple of degree 0 *if it is not $\frac{1}{n}$-fold G/H-simple for any non-negative integer n.*

For instance, if G has a normal subgroup properly between G and H, then G is G/H-simple of degree 0 (Lemma 7.7). For $G = A_5$ and H a 5-Sylowgroup of A_5 Example 7.13 shows that A_5 is G/H-simple of degree $\frac{1}{2}$.

In order to introduce analogous notions for homogeneous spaces we have to sort out a class of epimorphisms which have an essential meaning for the notion of simplicity. For instance, the epimorphism (i_Ω, π_Ω) of a homogeneous space $(\Omega, G, \cdot)$ onto its canonic respresentation $(\Omega, G\pi_\Omega, \cdot)$ (cf. Section 1) is «almost» an isomorphism, and is therefore rather unessential for our present considerations.

In the following let (φ, ψ) be a homomorphism of the homogeneous space $(\Omega, G, \cdot)$ into the permutation structure $(\Omega', G', \cdot)$ such that φ is a surjective mapping. Then $(\Omega', G', \cdot)$ is a homogeneous space, and the following three statements hold.

LEMMA 9.4. - *For any* $\alpha \in \Omega$

$$\mathrm{Ker}_\alpha(\varphi,\psi)g \rightarrow (\alpha g)\varphi = (\alpha\varphi)(g\psi)$$

is a bijective mapping of $\mathrm{Ker}_\alpha(\varphi,\psi) : G$ *onto* Ω'.

The proof is obvious. Now we describe, under the above assumption, all those homomorphisms which are close to isomorphisms from the point of view of simplicity.

PROPOSITION 9.5. - *If, moreover,* $\mathrm{Ker}_\alpha(\varphi,\psi) = G_\alpha$ *for an* $\alpha \in \Omega$, *then*

(1) φ *is a bijective mapping of* Ω *onto* Ω'.

(2) $\mathrm{Ker}\ \psi \leqq \mathrm{Ker}\ \pi_\Omega$.

PROOF. - We set $K_\alpha = \mathrm{Ker}_\alpha(\varphi,\psi)$.

I. Since $(\Omega, G, \cdot)$ is a homogeneous space every element $\beta \in \Omega$ can be written as $\beta = \alpha g$ for some $g \in G$, and

$$\varphi_1 : \alpha g \rightarrow G_\alpha g$$

is a bijective mapping of Ω onto $G_\alpha : G$. Moreover

$$\varphi_2 : K_\alpha g \rightarrow (\alpha g)\varphi = (\alpha\varphi)(g\psi)$$

is a bijective mapping of $K_\alpha : G$ onto Ω' by Lemma 9.4. $K_\alpha = G_\alpha$ now implies that $\varphi = \varphi_1 \varphi_2$ is a bijective mapping of Ω onto Ω'.

II. Take any element $k \in \operatorname{Ker} \psi$. Then

$$((\alpha g) k)\varphi = ((\alpha g)\varphi)(k\psi) = (\alpha g)\varphi \text{ for all } g \in G,$$

and therefore

$$\beta k = \beta \text{ for all } \beta \in \Omega$$

since φ is a bijective mapping. It follows that $k\pi_\Omega = i_\Omega$, and hence $\operatorname{Ker}\psi \leqq \leqq \operatorname{Ker} \pi_\Omega$.

There is still the other extreme of homomorphisms we have to eliminate, namely the zero homomorphisms (cf. Section 1).

PROPOSITION 9.6. - (φ,ψ) *is a zero homomorphism if and only if*

$$\operatorname{Ker}_\alpha(\varphi,\psi) = G \textit{ for } \alpha \in \Omega.$$

The proof follows from Lemma 9.4.

DEFINITION 9.7. - *An epimorphism* (φ,ψ) *of the homogeneous space* $(\Omega,G,\cdot)$ *is called* essential *if* $G_\alpha < \operatorname{Ker}_\alpha(\varphi,\psi) < G$ *for* $\alpha \in \Omega$.

It is clear that Definition 9.7 is independent of $\alpha \in \Omega$.

DEFINITION 9.8. - *Let n be a positive integer, and let* $(\Omega,G,\cdot)$ *be a homogeneous space such that* $|\Omega| > 1$. *Then* $(\Omega,G,\cdot)$ *is called* $\frac{1}{n}$-fold simple *if every n-fold epimorphism is unessential.*

PROPOSITION 9.9. - *A homogeneous space* $(\Omega,G,\cdot)$ *is* $\frac{1}{n}$*-fold simple if and only if G is* $\frac{1}{n-1}$*-fold* G/G_α*-simple for any* $\alpha \in \Omega$.

PROOF. I. - Assume that $(\Omega,G,\cdot)$ is $\frac{1}{n}$-fold simple. Let $K < G$ be an $(n-1)$-fold G/G_α-normal subgroup of G. The canonic epimorphism (φ_K, i_G) of $(G_\alpha : G, G, \cdot)$ (Theorem 8.6) is unessential by our hypotheses because $(G_\alpha : G, G, \cdot)$ is isomorphic to $(\Omega,G,\cdot)$. Therefore $K = \operatorname{Ker}_H(\varphi_K, i_G) = G_\alpha$ by Definition 9.7, and hence G is $\frac{1}{n-1}$-fold G/G_α-simple.

II. Assume that G is $\frac{1}{n-1}$-fold G/G_α-simple. Let (φ,ψ) be an n-fold epimorphism of $(\Omega,G,\cdot)$. By the Homomorphism Theorem 8.5 $K_\alpha = \operatorname{Ker}_\alpha(\varphi,\psi)$ is an $(n-1)$-fold G/G_α-normal subgroup of G, and hence $K_\alpha = G_\alpha$ or $K_\alpha = G$

by Definition 9.1. Therefore (φ, ψ) is unessential, and $(\Omega, G, \cdot)$ is $\frac{1}{n}$-fold simple.

LEMMA 9.10. - *If the homogeneous space* $(\Omega, G, \cdot)$ *is* $\frac{1}{n}$*-fold simple, then it is also* $\frac{1}{m}$*-fold simple for every integer* $m > n$.

PROOF. - Proposition 9.9 and Lemma 9.2.

DEFINITION 9.11. - *A homogeneous space* $(\Omega, G, \cdot)$ *is called* simple of degree $\frac{1}{n}$ *if there exists a positive integer* n *such that* $(\Omega, G, \cdot)$ *is* $\frac{1}{n}$*-fold but not* $\frac{1}{n-1}$*-fold simple, and it is called* simple of degree 0 *if it is not* $\frac{1}{n}$*-fold simple for any positive integer* n.

LEMMA 9.12. - *Let* $(\Omega, G, \cdot)$ *be a homogeneous space such that* $|\Omega| > 1$. *Then the following statements are equivalent.*

(1) $(\Omega, G, \cdot)$ *is* 1*-fold simple.*

(2) G_α *is a maximal subgroup of* G *for every* $\alpha \in \Omega$.

(3) $G\pi_\Omega$ *is a primitive permutation group on* Ω.

10. - The Isomorphism Theorems

In the following, G will always denote a group, H a subgroup of G, and n a positive integer.

THE FIRST ISOMORPHISM THEOREM 10.1. - *Let* K *be an* n*-fold* G/H*-normal subgroup of* G*, and let* L *be a subgroup of* G *containing* K. *Then*

(1) L *is* n*-fold* G/H*-normal if and only if* L *is* n*-fold* G/K*-normal.*

(2) *If* L *is* n*-fold* G/H*-normal, then*

$$((H:G,\ G,\cdot)/(H:K,\ K,\cdot))/(K:L,\ L,\cdot) = (H:G,\ G,\cdot)/(H:L,\ L,\cdot).$$

PROOF. - For all $g, x_1, \dots, x_n \in G$

$$Kg(H^{x_1} \cap \dots \cap H^{x_n}) = Kg(K^{x_1} \cap \dots \cap K^{x_n})$$

holds, and therefore

$$\begin{aligned} Lg\, H^{x_1} \cap \dots \cap H^{x_n}) &= LKg(H^{x_1} \cap \dots \cap H^{x_n}) \\ &= LKg(K^{x_1} \cap \dots \cap K^{x_n}) \\ &= Lg(K^{x_1} \cap \dots \cap K^{x_n}). \end{aligned}$$

This equation means that n-fold G/H-normality of L is equivalent to n-fold G/K-normality of L. (2) obviously follows from (1) and our notation of the factor space of a homogeneous space introduced after Theorem 8.6.

THE SECOND (PRE-) ISOMORPHISM THEOREM 10.2. - *Let K be an n-fold G/H-normal subgroup of G, and let L be a subgroup of G containing H. Then*

(1) *$K \cap L$ is an n-fold L/H-normal subgroup of L.*

(2) *K is an n-fold KL/H-normal subgroup of KL.*

(3) *The pair (φ, ψ), where φ is the bijective mapping*

$$(K \cap L)x \rightarrow Kx$$

of $K \cap L : L$ onto $K : KL$ and ψ is the injection of L into KL, is an $(n+1)$-fold pre-isomorphism of the homogeneous space

$$(K \cap L : L,\ L, \cdot) = (H : L,\ L,\ )/(H : K \cap L,\ K \cap L, \cdot)$$

onto the homogeneous space

$$(K : KL,\ KL, \cdot) = (H : KL,\ KL, \cdot)/(H : K,\ K, \cdot).$$

PROOF. I. - Take any elements $g, x_2, \ldots, x_n \in L$ and any element

$$y \in (K \cap L)g((K \cap L) \cap (K \cap L)^{x_2} \cap \ldots \cap (K \cap L)^{x_n}).$$

Then there exist elements

$$k \in K \cap L \text{ and } l \in (K \cap L) \cap (K \cap L)^{x_2} \cap \ldots \cap (K \cap L)^{x_n}$$

such that $y = kgl$. But

$$gl \in Kg(K \cap K^{x_2} \cap \ldots \cap K^{x_n}) = Kg(H \cap H^{x_2} \cap \ldots \cap H^{x_n})$$

since K is n-fold G/H-normal. Hence there exist elements

$$k' \in K \text{ and } h \in H \cap H^{x_2} \cap \ldots \cap H^{x_n}$$

such that $gl = k'gh$. From $H \leqq L$ it follows that

$$k' = glh^{-1}g^{-1} \in K \cap L,$$

and therefore

$$y = kk'gh \in (K \cap L)g(H \cap H^{x_2} \cap \ldots \cap H^{x_n}).$$

Thus we have proved

$$(K\cap L)\,g((K\cap L)\cap(K\cap L)^{x_2}\cap\ldots\cap(K\cap L)^{x_n})\subseteq(K\cap L)\,g(H\cap H^{x_2}\cap\ldots\cap H^{x_n}).$$

Since the converse inclusion trivially holds we obtain the equality

$$(K\cap L)\,g(H\cap H^{x_2}\cap\ldots\cap H^{x_n})=(K\cap L)\,g((K\cap L)\cap(K\cap L)^{x_2}\cap\ldots\cap(K\cap L)^{x_n}).$$

Therefore $K\cap L$ is an n-fold L/H-normal subgroup of L by 7.2.

II. (2) is obvious since it merely means the restriction of the equation 7.1 (2) to elements $g, x_1, \ldots, x_n \in KL$. Note that KL is a subgroup of G by 7.9.

III. φ: $(K\cap L)x \to Kx$ is a bijective mapping of $K\cap L:L$ onto $K:KL$. For every $g, x_1, \ldots, x_n \in L$

$$\begin{aligned} Kg(K^{x_1}\cap\ldots\cap K^{x_n}) &= Kg(H^{x_1}\cap\ldots\cap H^{x_n})\\ &\subseteq Kg((K\cap L)^{x_1}\cap\ldots\cap(K\cap L)^{x_n})\\ &\subseteq Kg(K^{x_1}\cap\ldots\cap K^{x_n}). \end{aligned}$$

Therefore equality holds instead of $\subseteq$, and we obtain

$$\begin{aligned} &((K\cap L)\,g((K\cap L)^{x_1}\cap\ldots\cap(K\cap L)^{x_n}))\varphi\\ &= Kg((K\cap L)^{x_1}\cap\ldots\cap(K\cap L)^{x_n})\\ &= Kg(K^{x_1}\cap\ldots\cap K^{x_n}) \end{aligned}$$

which means property $(\mathfrak{S}_{n+1})$ for (φ, ψ). Therefore (φ, ψ) is an $(n+1)$-fold pre-isomorphism of $(K\cap L:L, L,\cdot)$ onto $(K:KL, KL,.)$ according to Definition 5.1.

11. - n-fold G/H-subnormal Subgroups

For any subgroup H of a group G there are several possibilities to define a factor structure of G modulo H, for example:

1. The *double coset semigroup* G/H, that is the semigroup (with respect to the «complex» multiplication) generated by the double cosets HgH, $g\in G$. This factor structure has been investigated in a more general context in [2], [3], [4], and [5].

2. The *coset semigroup*, that is the semigroup (with respect to the «complex» multiplication) generated by the cosets Hg, $g\in G$. This possibility has been discussed by WIELANDT in [8].

3. The *homogeneous space* $(H:G, G,\cdot)$.

The third factor structure is the richest and can be considered as a refinement of the second, just as the second can be considered as a refinement of the first. We shall discuss their relationship to each other in the next section. Here we want to investigate a different problem.

We consider each type of factor structure as a mathematical structure in its own right. Each of them is, or can be (case 2), provided with its own notion of homomorphy so that we obtain a category. Thus we have

1. The category of all double coset semigroups ([2], [3]).
2. The category of all coset semigroups.
3. The category of all homogeneous spaces.

For the third category we have in fact the choice of infinitely many notions of homomorphisms, namely for each non-negative integer n we can take the n-fold homomorphisms as the morphisms of a category. Therefore the last category is subdivided into the categories $\mathcal{H}_n$ (Proposition 6.4).

Each of these categories gives rise to a notion of normality such that the subgroups which are normal relative to such a category are exactly the kernels of its homomorphisms. Each of these concepts of normality leads to a notion of subnormality.

To every chain

$$G = L_0 \geqq L_1 \geqq \dots \geqq L_r = L$$

of subgroups which are subnormal with respect to one of these categories we can assign factors in different ways. We can choose one type of factor structure and take as factors the factor structure of that type for each L_{i-1} modulo L_i. This procedure will be particularly fruitful if a Jordan-Hölder Theorem can be proved for a certain choice of subnormality and a certain choice of factor structure.

For subnormality and factor structure both taken with respect to the category of double coset semigroups this has been done in [4]. We investigate the same problem for the categories $\mathcal{H}_n$ in this section.

In the following let n be a positive integer, G a group, and H a subgroup of G.

DEFINITION 11.1. - *A subgroup L of G is called* n-fold G/H-subnormal, *if there exists a finite chain*

$$G = L_0 \geqq L_1 \geqq \dots \geqq L_r = L$$

of subgroups of G such that L_i is an n-fold L_{i-1}/H-normal subgroup of L_{i-1} for each $i = 1, \dots, r$. Such a chain is called an n-fold G/H-subnormal chain; *it is called an* n-fold G/H-composition chain *if $L_{i-1} > L_i$ and if there is no*

n-fold L_{i-1}/H-normal subgroup of L_{i-1} properly between L_{i-1} and L_i (which means that L_{i-1} is $\frac{1}{n}$-fold L_{i-1}/L_i-simple by Definition 9.1 *and the First Isomorphism Theorem* 10.1) *for every $i = 1, \ldots, r$.*

LEMMA 11.2. - *A subgroup L of G is* 1*-fold G/H-subnormal if and only if it is G/H-subnormal, in the sense of* [4], *Definition* 1.1.

PROOF. - Lemma 7.4.

LEMMA 11.3. - *If L is n-fold G/H-subnormal then L is m-fold G/H-subnormal for every positive integer $m < n$.*

PROOF. - Lemma 7.5.

LEMMA 11.4. - *If K is an n-fold G/H-subnormal subgroup of G, and L is an n-fold K/H-subnormal subgroup of K, then L is an n-fold G/H-subnormal subgroup of G.*

THEOREM 11.5. - *If K is an n-fold G/H-subnormal subgroup of G and $H \leqq L \leqq G$, then $K \cap L$ is an n-fold L/H-subnormal subgroup of L.*

PROOF. - Take an n-fold G/H-subnormal chain

$$G = K_0 \geqq K_1 \geqq \ldots \geqq K_r = K$$

from G to K as in Definition 11.1. $K_0 \cap L = G \cap L = L$ is an n-fold L/H-normal subgroup of L. Assume that $K_{i-1} \cap L$ is already proved to be n-fold L/H-subnormal. Since K_i is an n-fold K_{i-1}/H-normal subgroup of K_{i-1} by assumption, $K_i \cap L = K_i \cap (K_{i-1} \cap L)$ is an n-fold $K_{i-1} \cap L/H$-normal subgroup of $K_{i-1} \cap L$ by the Second Isomorphism Theorem 10.2. Therefore $K_i \cap L$ in n-fold L/H-subnormal in L by 11.4, and the theorem follows.

THEOREM 11.6. - *If K and L are n-fold G/H-subnormal subgroups of G, then $K \cap L$ is an n-fold G/H-subnormal subgroup of G.*

PROOF. - $K \cap L$ is n-fold L/H-subnormal in L by Theorem 11.5, and L is n-fold G/H-subnormal by assumption. Our statement now follows from 11.4.

THEOREM 11.7. - *Assume $H \leqq N \leqq K \leqq G$ and let N be subnormal in G. If K is n-fold G/H-subnormal then K is subnormal in G. If K is subnormal in G, then K is at least* 1*-fold G/H-subnormal.*

PROOF. - Lemma 11.2 and [4], Theorem 1.7.

The crucial step towards a Jordan-Hölder Theorem is the proof of the relevant Four Subgroup Theorem (Zassenhaus' Lemma). Here we have to restrict ourselves to 1-fold G/H-normality. Also as the Second Isomorphism

Theorem 10.2 does not hold for isomorphy but for pre-isomorphy only, we have to use pre-isomorphy instead of isomorphy in the Four Subgroup Theorem.

THE FOUR SUBGROUP THEOREM 11.8. - *Let K_0, K, L_0, L be subgroups of G containing H. Assume that K_0 is a 1-fold K/H-normal subgroup of K, and that L_0 is a 1-fold L/H-normal subgroup of L. Then*

(1) $(K \cap L_0)K_0$ *is a* 1-*fold* $(K \cap L)K_0/H$-*normal subgroup of* $(K \cap L)K_0$

(2) $(K_0 \cap L)L_0$ *is a* 1-*fold* $(K \cap L)L_0/H$-*normal subgroup of* $(K \cap L)L_0$

(3) *The homogeneous spaces*

$$((K \cap L_0)K_0 : (K \cap L)K_0,\ (K \cap L)K_0, \cdot)$$

$$((K_0 \cap L)L_0 : (K \cap L)L_0,\ (K \cap L)L_0, \cdot)$$

are 2-*fold pre-isomorphic.*

PROOF. - (1) and (2) follow from Lemma 7.4 and [4], Theorem 2.1. Because of

$$(K \cap L_0)K_0 \cap (K \cap L) = (K_0 \cap L)(K \cap L_0) = (K_0 \cap L)L_0 \cap (K \cap L),$$

$$(K \cap L_0)K_0(K \cap L) = (K \cap L)K_0,\ (K_0 \cap L)L_0(K \cap L) = (K \cap L)L_0$$

and the Second Isomorphism Theorem 10.2 the homogeneous space

$$((K_0 \cap L)(K \cap L_0) : K \cap L,\ K \cap L, \cdot)$$

is 2-fold pre-isomorphic to each of the homogeneous spaces

$$((K \cap L_0)K_0 : (K \cap L)K_0,\ (K \cap L)K_0, \cdot),$$

$$((K_0 \cap L)L_0 : (K \cap L)L_0,\ (K \cap L)L_0, \cdot),$$

and (3) follows by Definition 5.3.

An immediate consequence of the Four Subgroup Theorem is

THE REFINEMENT THEOREM FOR 1-FOLD G/H-SUBNORMAL CHAINS 11.9. - *Let*

(i) $$G = K_0 \geqq K_1 \geqq \dots \geqq K_r = L$$

(ii) $$G = L_0 \geqq L_1 \geqq \dots \geqq L_s = L$$

be 1-*fold* G/H-*subnormal chains. Set*

$$K_{i,j} = K_i(K_{i-1} \cap L_j) \quad (i = 1, \dots, r;\ j = 0, \dots, s),$$

$$L_{j,i} = L_j(L_{j-1} \cap K_i) \quad (j = 1, \dots, s;\ i = 0, \dots, r).$$

Then

(1) $$K_{i-1} = K_{i,0} \geqq K_{i,1} \geqq \dots \geqq K_{i,s} = K_i$$

is a 1*-fold* K_{i-1}/H*-subnormal chain for each* $i = 1, \dots, r$.

(2) $$L_{j-1} = L_{j,0} \geqq L_{j,1} \geqq \dots \geqq L_{j,r} = L_j$$

is a 1*-fold* L_{j-1}/H*-subnormal chain for each* $j = 1, \dots, s$.

(3) *The homogeneous spaces*

$$(K_{i,j} : K_{i,j-1}, K_{i,j-1}, \cdot) \text{ and } (L_{j,i} : L_{j,i-1}, L_{j,i-1}, \cdot)$$

are 2*-fold pre-isomorphic for all* $i = 1, \dots, r$ *and all* $j = 1, \dots, s$.

(4) *Joining the chains* (1), *respectively* (2), *together, we obtain refinements of the chains* (i) *and* (ii) *for which*

$$(K_{i,j} : K_{i,j-1}, K_{i,j-1}, \cdot) \longleftrightarrow (L_{j,i} : L_{j,i-1}, L_{j,i-1}, \cdot)$$

is a one-to-one correspondence of their $\mathcal{H}$*-factors such that corresponding* $\mathcal{H}$*-factors are* 2*-fold pre-isomorphic.*

THE THEOREM OF JORDAN AND HÖLDER FOR 1-FOLD G/H-COMPOSITION CHAINS 11.10. - *Let*

$$G = K_0 > K_1 > \dots > K_r = L$$

$$G = L_0 > L_1 > \dots > L_s = L$$

be 1*-fold* G/H*-composition chains. Then*

(1) $r = s$.

(2) *There exists a permutation* π *of* $\{1, \dots, r\}$ *such that*

$$(K_i : K_{i-1}, K_{i-1}, \cdot) \text{ and } (L_{\pi(i)} : L_{\pi(i)-1}, L_{\pi(i)-1}, \cdot)$$

are 2*-fold pre-isomorphic for all* $i = 1, \dots, r$.

The proof immediately follows from the Refinement Theorem 11.9.

12. - Classification of the Homogeneous Spaces by Double Coset Semigroups.

We resume our discussion begun at the beginning of Section 11, and now ask for the relationship between homogeneous spaces, coset semigroups, and double coset semigroups.

Let $(\Omega, G, \cdot)$ be a permutation structure. We set

$$\overline{\Omega} = \{ \Delta \mid \emptyset \neq \Delta \subseteq \Omega \}.$$

LEMMA 12.1. - *$\overline{\Omega}$ is a commutative semigroup with respect to set theorical unions and it is closed under taking unions of arbitrarily many of its elements.*

Let $(\Omega', G', \cdot)$ also be a permutation structure. For every mapping $\varphi : \Omega \to \Omega'$ we define a mapping $\overline{\varphi} : \overline{\Omega} \to \overline{\Omega}'$ by

$$\Delta\overline{\varphi} = \{ \delta\varphi \mid \delta \in \Delta \} \textit{ for all } \Delta \in \overline{\Omega}.$$

Then

$$(\bigcup_{i \in I} \Delta_i)\overline{\varphi} = \bigcup_{i \in I} \Delta_i \overline{\varphi} \textit{ for any } \Delta_i \in \overline{\Omega} \textit{ and any index set } I.$$

DEFINITION 12.2. - *A mapping σ of $\overline{\Omega}$ into $\overline{\Omega}'$ is called a* homomorphism *if*

$$(\bigcup_{i \in I} \Delta_i)\sigma = \bigcup_{i \in I} \Delta_i \sigma \textit{ for any } \Delta_i \in \overline{\Omega} \textit{ and any index set } I.$$

For any mappings σ_i of $\overline{\Omega}$ into $\overline{\Omega}'$, and for any index set I we define a mapping $\bigcup_{i \in I} \sigma_i$ of $\overline{\Omega}$ into $\overline{\Omega}'$ by

$$\Delta(\bigcup_{i \in I} \sigma_i) := \bigcup_{i \in I} \Delta\sigma_i \text{ for all } \Delta \in \overline{\Omega}.$$

LEMMA 12.3. - *Let* End$(\overline{\Omega})$ *be the set of all homomorphisms (in the sense of Definition* 12.2) *of $\overline{\Omega}$ into itself. Then*

(1) $\rho\sigma \in \text{End}(\overline{\Omega})$ *for all* $\rho, \sigma \in \text{End}(\overline{\Omega})$.

(2) $\bigcup_{i \in I} \sigma_i \in \text{End}(\overline{\Omega})$ *for any* $\sigma_i \in \text{End}(\overline{\Omega})$ *and any index set* I.

(3) $\sigma(\bigcup_{i \in I} \sigma_i) = \bigcup_{i \in I} \sigma\sigma_i$ *and* $(\bigcup_{i \in I} \sigma_i)\sigma = \bigcup_{i \in I} \sigma\sigma_i$
for any $\sigma, \sigma_i \in \text{End}(\overline{\Omega})$ *and any index set* I.

For any subgroup H of the group G we set

$$\overline{H:G} = \{ \emptyset \neq X \subseteq G \mid HX = X \},$$

$$\overline{G/H} = \{ \emptyset \neq Y \subseteq G \mid HYH = Y \}.$$

It appears that $\overline{G/H}$ has an analogous structure as $\text{End}(\overline{\Omega})$.

LEMMA 12.4. - (1) *$\overline{G/H}$ is a semigroup with respect to the multiplication*

$$(X,Y) \to XY = \{xy \mid x \in X \text{ and } y \in Y\}.$$

(2) $\bigcup_{i\in I} X_i \in \overline{G/H}$ *for any* $X_i \in \overline{G/H}$ *and any index set* I.

(3) $X(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} XX_i$ *and* $(\bigcup_{i\in I} X_i)X = \bigcup_{i\in I} X_iX$

for any $X, X_i \in \overline{G/H}$ *and any index set* I.

In particular, this lemma holds for $\overline{G} = \overline{G/1}$ with $H = 1$. In order to compare $\text{End}(\overline{\Omega})$ with $\overline{G/H}$ we introduce the following notion.

DEFINITION 12.5. - *Let A and A' be algebraic structures with*

1. *a binary algebraic composition (written as multiplication),*
2. *a composition $\cup$ which is defined for any index set.*

Then a mapping $\omega : A \to A'$ *is called a* homomorphism *of A into A' if*

(1) $(ab)\omega = (a\omega)(b\omega)$ *for all* $a,b \in A$.

(2) $(\bigcup_{i\in I} a_i)\omega = \bigcup_{i\in I} a_i\omega$ *for any* $a_i \in A$ *and any index set* I.

Now we apply these concepts and notations to our permutation structures.

LEMMA 12.6. - *Let* $(\Omega,G,\cdot)$ *be a permutation structure. We denote by* $(\overline{\Omega},\overline{G},\cdot)$ *the algebraic structure which is given by* $\overline{\Omega}$, *(in the sense of* 12.1), $\overline{G}$ *(in the sense of* 12.4), *and the external algebraic composition*

$$(\Delta,X) \to \Delta X = \{\delta x \mid \delta \in \Delta \text{ and } x \in X\}.$$

Then

(1) $(\Delta \cup \Gamma)X = \Delta X \cup \Gamma X$ *for all* $\Delta, \Gamma \in \overline{\Omega}$ *and all* $X \in \overline{G}$.

(2) $\Delta(X \cup Y) = \Delta X \cup \Delta Y$ *for all* $\Delta \in \overline{\Omega}$ *and all* $X, Y \in \overline{G}$.

(3) $(\Delta X)Y = \Delta(XY)$ *for all* $\Delta \in \overline{\Omega}$ *and all* $X, Y \in \overline{G}$

(4) $\Delta 1 = \Delta$ *for all* $\Delta \in \overline{\Omega}$.

DEFINITION 12.7. - *Let* $(\Omega,G,\cdot)$ *and* $(\Omega',G',\cdot)$ *be permutation structures. A pair* (σ,τ) *of mappings*

$$\sigma : \overline{\Omega} \to \overline{\Omega}' \text{ and } \tau : \overline{G} \to \overline{G}'$$

is called a homomorphism *of* $(\overline{\Omega},\overline{G},\cdot)$ *into* $(\overline{\Omega}',\overline{G}',\cdot)$ *if*

(1) σ *is a homomorphism of* $\bar{\Omega}$ *into* $\bar{\Omega}'$ (*in the sense of Definition* 12.2),

(2) τ *is a homomorphism of* $\bar{G}$ *into* $\bar{G}'$ (*in the sense of Definition* 12.5),

(3) $(\Delta X)\sigma = (\Delta\sigma)(X\tau)$ *for all* $\Delta \in \bar{\Omega}$ *and all* $X \in \bar{G}$.

LEMMA 12 8. - *Let* (φ, ψ) *be a homomorphism of the permutation structure* $(\Omega, G, \cdot)$ *into the permutation structure* $(\Omega', G', \cdot)$. *Then* $(\bar{\varphi}, \bar{\psi})$ *is a homomorphism of* $(\bar{\Omega}, \bar{G}, \cdot)$ *into* $(\bar{\Omega}', \bar{G}', \cdot)$.

Our intention is to link each homogeneous space with an isomorphy class of double coset semigroups. For this reason we introduce a concept of endomorphism. This concept will show to be more important for the structure $(\bar{\Omega}, \bar{G}, \cdot)$ than the analogous concept for the permutation structure $(\Omega, G, \cdot)$ itself. Yet, to be complete, we discuss briefly its meaning for the homogeneous spaces first.

DEFINITION 12.9. - *A homomorphism* (φ, ψ) *of a permutation structure* $(\Omega, G, \cdot)$ *into itself is called an* endomorphism *of* $(\Omega, G, \cdot)$ *if* $\psi = i_G$ (*the identity mapping of* G). *We denote by* $\mathcal{E}(\Omega, G, \cdot)$ *the set of all mappings* $\varphi : \Omega \to \Omega$ *such that* (φ, i_G) *is an endomorphism of* $(\Omega, G, \cdot)$.

$\mathcal{E}(\Omega, G, \cdot)$ is the set of all mappings $\varphi : \Omega \to \Omega$ such that $(\alpha g)\varphi = (\alpha\varphi)g$ holds for all $\alpha \in \Omega$ and all $g \in G$. $\mathcal{E}(\Omega, G, \cdot)$ is a semigroup with respect to the composition of mappings; i_Ω is its unit element.

THEOREM 12.10. - *Let* $(\Omega, G, \cdot)$ *be a homogeneous space and* $\alpha \in \Omega$. *Then*

$$N_\alpha = \{ g \in G \mid g^{-1} G_\alpha g \leqq G_\alpha \}$$

is a subsemigroup of the group G *for which the following hold.*

(1) *For every* $\varphi \in \mathcal{E}(\Omega, G, \cdot)$ *there exists an element* $g \in N_\alpha$ *such that*

$$\varphi : \alpha x \to \alpha g^{-1} x \quad (x \in G).$$

(2) *For every* $g \in N_\alpha$ *the mapping*

$$\varphi_g : \alpha x \to \alpha g^{-1} x \quad (x \in G)$$

is an element of $\mathcal{E}(\Omega, G, \cdot)$.

(3) *The mapping*

$$\vartheta : g \to \varphi_g$$

is an epimorphism of the semigroup N_α *onto the semigroup* $\mathcal{E}(\Omega, G, \cdot)$ *such that*

$$\operatorname{Ker} \vartheta = G_\alpha .$$

Every $\varphi \in \mathcal{E}(\Omega, G, \cdot)$ *is a surjective mapping which acts fixed point freely on* Ω *if* $\varphi \neq i_\Omega$.

PROOF. I. - For every $\varphi \in \mathcal{E}(\Omega, G, \cdot)$ there exists a $g \in G$ such that $\alpha\varphi = \alpha g^{-1}$. The mapping φ is uniquely determined by g because of

$$(\alpha x)\varphi = (\alpha\varphi)x = (\alpha g^{-1})x = \alpha(g^{-1}x) \text{ for all } x \in G.$$

For every $g_\alpha \in G_\alpha$ we have

$$\alpha g^{-1}\, g_\alpha = (\alpha\varphi)g_\alpha = (\alpha g_\alpha)\varphi = \alpha\varphi = \alpha g^{-1}$$

and hence $g^{-1}\, g_\alpha\, g \in G_\alpha$ which means $g \in N_\alpha$.

II. Take any $g \in N_\alpha$. If $\alpha x = \alpha y$ for $x, y \in G$, then $x = g_\alpha y$ for some $g_\alpha \in G_\alpha$, and therefore

$$\alpha g^{-1} x = \alpha g^{-1} g_\alpha y = \alpha(g^{-1} g_\alpha g) g^{-1} y = \alpha g^{-1} y.$$

It follows that

$$\varphi_g : \alpha x \to \alpha g^{-1} x$$

is a well defined mapping, and it is easy to see that $\varphi_g \in \mathcal{E}(\Omega, G, \cdot)$.

III. $$\varphi_{gh} = \varphi_g\, \varphi_h \text{ for all } g, h \in N_\alpha$$

shows that ϑ is a homomorphism of N_α into $\mathcal{E}(\Omega, G, \cdot)$. ϑ is an epimorphism because of (1). $\varphi_g = i_\Omega$ if and only if $g \in G_\alpha$. The rest of the statements is easily proved.

COROLLARY 12.11. - *If* $(\Omega, G, \cdot)$ *is a homogeneous space such that* Ω *is a finite set, then* $N_\alpha = \mathfrak{N}_G(G_\alpha)$ *is the normalizer of* G_α *in* G, *therefore it is a group, and* $\mathcal{E}(\Omega, G, \cdot)$ *is isomorphic to the group* $\mathfrak{N}_G(G_\alpha)/\,G_\alpha$, *and hence is a group itself.*

Theorem 12.10. shows that there will be many homogeneous spaces where the endomorphism semigroup $\mathcal{E}(\Omega, G, \cdot)$ reduces to the identity mapping i_Ω, for instance if G_α is not normal in G, and if there does not exist any subsemigroup of G properly between G and G_α (i. e. $G\pi_\Omega$ is a strongly primitive permutation group on Ω in the sense of [7], Definition 8.5 and 8.6b). But the endomorphism structure of $(\bar{\Omega}, \bar{G}, \cdot)$, as we shall see, still has some significance even in cases where $\mathcal{E}(\Omega, G, \cdot) = \{ i_\Omega \}$.

DEFINITION 12.12. - *Let* $(\Omega, G, \cdot)$ *be a permutation structure. A homomorphism* (σ, τ) *of* $(\bar{\Omega}, \bar{G}, \cdot)$ *into itself (in the sense of Definition* 12.7) *is called an* endomorphism *of* $(\bar{\Omega}, \bar{G}, \cdot)$ *if* $\tau = i_{\bar{G}}$ *(the identity mapping of* $\bar{G}$). *We denote by* $\mathcal{E}(\bar{\Omega}, \bar{G}, \cdot)$ *the set of all mappings* $\sigma : \bar{\Omega} \to \bar{\Omega}$ *such that* $(\sigma, i_{\bar{G}})$ *is an endomorphism of* $(\bar{\Omega}, \bar{G}, \cdot)$.

$\mathcal{E}(\bar{\Omega},\bar{G},\cdot)$ is the set of all mappings $\sigma:\bar{\Omega}\to\bar{\Omega}$ such that $(\bigcup_{i\in I}\Delta_i)\sigma=\bigcup_{i\in I}\Delta_i\sigma$ for any $\Delta_i\in\bar{\Omega}$ and any index set I, and $(\Delta X)\sigma=(\Delta\sigma)X$ for all $\Delta\in\bar{\Omega}$ and all $X\in\bar{G}$.

PROPOSITION 12.13. – *Let* $(\Omega,G,\cdot)$ *be a permutation structure. Set* $\mathcal{E}=\mathcal{E}(\Omega,G,\cdot)$ *and* $\bar{\mathcal{E}}=\mathcal{E}(\bar{\Omega},\bar{G},\cdot)$. *Then*

(1) $\sigma\tau\in\bar{\mathcal{E}}$ *for every* $\sigma,\ \tau\in\bar{\mathcal{E}}$.

(2) $\bigcup_{i\in I}\tau_i\in\bar{\mathcal{E}}$ *for any* $\tau_i\in\bar{\mathcal{E}}$ *and every index set* I.

(3) $\tau(\bigcup_{i\in I}\tau_i)=\bigcup_{i\in I}\tau\tau_i$ *and* $(\bigcup_{i\in I}\tau_i)\tau=\bigcup_{i\in I}\tau_i\tau$ *for any* $\tau,\tau_i\in\bar{\mathcal{E}}$ *and any index set* I.

(4) $\bar{\varphi}\in\bar{\mathcal{E}}$ *for all* $\varphi\in\mathcal{E}$.

This remark shows that $\mathcal{E}(\bar{\Omega},\bar{G},\cdot)$ is an algebraic structure of the same type as $\overline{G/H}$ (Lemma 12.4), and now we come to the object of this section.

THEOREM 12.14. – *Let* $(\Omega,G,\cdot)$ *be a homogeneous space and* $\alpha\in\Omega$. *Then* $\mathcal{E}(\bar{\Omega},\bar{G},\cdot)$ *is isomorphic to* $\overline{G/G_\alpha}$ (*in the sense of Definition* 12.5).

PROOF. – For every $\Delta\in\bar{\Omega}$ there exists an $X\in\bar{G}$ such that

$$\Delta=\alpha X,$$

since $(\Omega,G,\cdot)$ is a homogeneous space. Clearly

$$X_\Delta=G_\alpha X$$

is the largest element of the set $\{Y\in\bar{G}\,|\,\Delta=\alpha Y\}$ with respect to the set theoretical inclusion, that is $X_\Delta=\{g\in G\,|\,\alpha g\in\Delta\}$. Note that

$$X_\Delta\in\overline{G_\alpha:G}.$$

For every $\tau\in\mathcal{E}(\bar{\Omega},\bar{G},\cdot)$ we set

$$A_\tau=X_{\alpha\tau}.$$

From

$$\alpha A_\tau G_\alpha=(\alpha\tau)G_\alpha=(\alpha G_\alpha)\tau=\alpha\tau=\alpha A_\tau$$

and the maximality property of $X_{\alpha\tau}$ it follows that

$$A_\tau G_\alpha=G_\alpha A_\tau=A_\tau$$

and therefore

$$A_\tau\in\overline{G/G_\alpha}.$$

τ is uniquely determined by A_τ since

$$\Delta\tau = (\alpha X_\Delta)\tau = (\alpha\tau)X_\Delta = \alpha A_\tau X_\Delta \text{ for all } \Delta \in \bar{\Omega}.$$

Therefore

$$\omega : \tau \to A_\tau^{-1} = \{g^{-1} \mid g \in A_\tau\}$$

is an injective mapping of $\mathcal{E}(\bar{\Omega}, \bar{G}, \cdot)$ into $\overline{G/G_\alpha}$. ω is surjective as well, for if we take any $B \in \overline{G/G_\alpha}$, then $B^{-1} \in \overline{G/G_\alpha}$ and

$$\tau : \Delta \to \alpha B^{-1} X_\Delta$$

is an element of $\mathcal{E}(\bar{\Omega}, \bar{G}, \cdot)$ such that $A_\tau = B^{-1}$ and $\tau\omega = B$. Furthermore

$$\alpha A_{\sigma\tau} = \varphi\sigma\tau = (\alpha A_\sigma)\tau = (\alpha\tau)A_\sigma = \alpha A_\tau A_\sigma \text{ for all } \sigma,\tau \in \mathcal{E}(\bar{\Omega}, \bar{G}, \cdot).$$

It follows that $A_\tau A_\sigma \subseteqq A_{\sigma\tau}$. Because of $G_\alpha A_\tau A_\sigma = A_\tau A_\sigma$ the product $A_\tau A_\sigma$ is the largest of the subsets Y of G such that $\alpha A_{\sigma\tau} = \alpha Y$, and hence we have

$$A_{\sigma\tau} = A_\tau A_\sigma,$$

$$A_{\sigma\tau}^{-1} = A_\sigma^{-1} A_\tau^{-1}.$$

Also

$$\alpha A_{\bigcup_{i \in I} \tau_i} = \alpha(\bigcup_{i \in I} \tau_i) = \bigcup_{i \in I} \alpha\tau_i = \bigcup_{i \in I} \alpha A_{\tau_i} = \alpha \bigcup_{i \in I} A_{\tau_i}$$

for any $\tau_i \in \mathcal{E}(\bar{\Omega}, \bar{G}, \cdot)$ and any index set I. By the same reasoning as before we obtain

$$A_{\bigcup_{i \in I} \tau_i}^{-1} = \bigcup_{i \in I} A_{\tau_i}^{-1}.$$

We have proved that ω is an isomorphism of $\mathcal{E}(\bar{\Omega}, \bar{G}, \cdot)$ onto $\overline{G/G_\alpha}$ according to Definition 12.5.

$\overline{G/G_\alpha}$ is already completely determined by the double coset semigroup G/G_α. Since the category of all double coset semigroups contains the category of all groups, it seems more elegant to deal with double coset semigroups in the following. We recall the definition of the morphisms of the category of all double coset semigroups ([3], Definition 2.1).

Let G and G' be groups, let H be a subgroup of G, and let H' be a subgroup of G'. Then a mapping $\eta : G/H \to G'/H'$ is called a *homomorphism* of the double coset semigroup G/H into the double coset semigroup G'/H' if

(1) $(XY)\eta = (X\eta)(Y\eta)$ *for all* $X, Y \in G/H$.

(2) *For every HgH, $g \in G$, there exists a $H'g'H'$, $g' \in G'$, such that*

$$(HgH)\eta = H'g'H' \text{ and } (Hg^{-1}H)\eta = H'g'^{-1}H'.$$

(3) $X\eta = \bigcup_{HgH \subseteq X} (HgH)\eta$ *for all* $X \in G/H$.

Every homomorphism η of G/H into G'/H' can be uniquely extended to a homomorphism $\bar{\eta}$ of $\overline{G/H}$ into $\overline{G'/H'}$ in the sense of Definition 12.5 by defining

$$X\bar{\eta} = \bigcup_{HgH \subseteq X} (HgH)\eta \text{ for all } X \in \overline{G/H}$$

(cf. [3], Proposition 2.2). But such a homomorphism $\bar{\eta}$ has, apart from the properties of Definition 12.5, the further property that it maps every double coset HgH onto a double coset $H'g'H'$, and $Hg^{-1}H$ onto $H'g'^{-1}H'$. In the following we call a mapping $\zeta: \overline{G/H} \rightarrow \overline{G'/H'}$ a *homomorphism* if and only if, in addition to 12.5, this condition is satisfied as well, that is if and only if there exists a homomorphism η of G/H into G'/H' such that $\zeta = \bar{\eta}$. In particular, two double coset semigroups G/H and G'/H' are isomorphic if and only if $\overline{G/H}$ and $\overline{G'/H'}$ are isomorphic in the sense just defined.

Now we apply our remarks to the endomorphisms of homogeneous spaces. Let $(\Omega, G, \cdot)$ be a homogeneous space. By Theorem 12.14 $\mathscr{E}(\bar{\Omega}, \bar{G}, \cdot)$ and $\overline{G/G_\alpha}$ are isomorphic in the sense of Definition 12.5. The proof of Theorem 12.14 shows that for every double coset $G_\alpha g G_\alpha$ there exists a $\tau \in \mathscr{E}(\bar{\Omega}, \bar{G}, \cdot)$ such that

$$\tau\omega = G_\alpha g^{-1} G_\alpha$$

and that this holds if and only if

$$\alpha\tau = \alpha G_\alpha g G_\alpha = (\alpha g) G_\alpha .$$

Let $(\Omega', G', \cdot)$ be a homogeneous space and assume that there exists a homomorphism η of G/G_α into $G'/G'_{\alpha'}$ for $\alpha' \in \Omega'$. If we denote by ω' the isomorphism of $\mathscr{E}(\bar{\Omega}', \bar{G}', \cdot)$ onto $\overline{G'/G'_{\alpha'}}$ which is given by Theorem 12.14, then for every $G_\alpha g G_\alpha$, $g \in G$, there exist σ', $\tau' \in \mathscr{E}(\bar{\Omega}', \bar{G}', \cdot)$ and $g' \in G'$ such that

$$\sigma'\omega' = (G_\alpha g^{-1} G_\alpha)\eta = G'_{\alpha'} g'^{-1} G'_{\alpha'},$$
$$\tau'\omega' = (G_\alpha g G_\alpha)\eta = G'_{\alpha'} g' G'_{\alpha'}.$$

This, again by the proof of Theorem 12.14, holds if and only if

$$\alpha'\sigma' = \alpha' G'_{\alpha'} g' G'_{\alpha'} = (\alpha' g') G'_{\alpha'} .$$
$$\alpha'\tau' = \alpha' G'_{\alpha'} g'^{-1} G'_{\alpha'} = (\alpha' g'^{-1}) G'_{\alpha'} .$$

The orbits $(\alpha g)G_\alpha$ and $(\alpha g^{-1})G_\alpha$ and also the orbits $(\alpha' g')G'_{\alpha'}$ and $(\alpha' g'^{-1})G'_{\alpha'}$ are paired orbits of G_α and of $G'_{\alpha'}$ respectively (WIELANDT [6], p. 45, Definition, and [7], 10.9).

We recall briefly the concept of paired orbits. By Proposition 2.5 for $n = 2$ (that is [7], 10.6) we have, for fixed $\alpha \in \Omega$, the following one-to-one correspondence between double cosets, orbits of stabilizers, and binary G-relations.

$$G_\alpha g G_\alpha \longleftrightarrow (\alpha g)G_\alpha \longleftrightarrow (\alpha,\alpha g)G$$

For any binary relation R on Ω we denote by

$$R^* = \{(\beta, \gamma) | (\gamma, \beta) \in R\}$$

its converse relation. But

$$((\alpha,\alpha g)G)^* = (\alpha,\alpha g^{-1})G$$

and therefore we have the analogous correspondence

$$(G_\alpha g\, G_\alpha)^{-1} = G_\alpha\, g^{-1}\, G_\alpha \longleftrightarrow (\alpha g^{-1})G_\alpha \longleftrightarrow (\alpha,\alpha g^{-1})\,G = ((\alpha,\alpha g)\,G)^*$$

which gives the orbit $(\alpha g^{-1})G_\alpha$ as the *reflexion* of the orbit $(\alpha g)G_\alpha$ by α ([6], p. 44), and $(\alpha g)G_\alpha$ and $(\alpha g^{-1})G_\alpha$ are called *paired*. We write $((\alpha g)G_\alpha)^* = (\alpha g^{-1})G_\alpha$.

For this reason we introduce the following involutionary antiautomorphism $\tau \rightarrow \tau^*$ of $\mathcal{E}(\overline{\Omega},\overline{G},\cdot)$. For every $\tau \in \mathcal{E}(\overline{\Omega},\overline{G},\cdot)$ there exists one and only one $\tau^* \in \mathcal{E}(\overline{\Omega},\overline{G},\cdot)$ such that for the isomorphism ω of Theorem 12.14

$$\tau^*\omega = (\tau\omega)^{-1}$$

holds. It is clear that

$$\tau^{**} = \tau,\ (\bigcup_{i\in I}\tau_i)^* = \bigcup_{i\in I}\tau_i^*,\ (\sigma\tau)^* = \tau^*\sigma^* \text{ for all } \sigma,\tau,\tau_i \in \mathcal{E}(\overline{\Omega},\overline{G},\cdot).$$

Let us return to the homomorphism $\eta : G/G_\alpha \rightarrow G'/G'_{\alpha'}$. The mapping $\varepsilon = \omega\overline{\eta}\omega'^{-1}$ is a homomorphism of $\mathcal{E}(\overline{\Omega},\overline{G},\cdot)$ into $\mathcal{E}(\overline{\Omega}',\overline{G}',\cdot)$ in the sense of Definition 12.5 with the additional property that for $\tau \in \mathcal{E}(\overline{\Omega},\overline{G},\cdot)$

$$\alpha\tau = \beta G_\alpha \text{ for some } \beta \in \Omega$$

implies

$$\alpha'(\tau\varepsilon) = \beta' G'_{\alpha'} \text{ and } \alpha'(\tau^*\varepsilon) = (\beta' G'_{\alpha'})^* \text{ for some } \beta' \in \Omega'.$$

This property, which relates the orbits of stabilizers, is an essential part of permutation structure. Therefore it is essential to define an own notion of homomorphy for the endomorphism structures of homogeneous spaces such that it corresponds with the homomorphy of double coset semigroups mentioned above ([3], Definition 2.1).

DEFINITION 12.15. - *Let* $(\Omega, G, \cdot)$ *and* $(\Omega', G', \cdot)$ *be homogeneous spaces. A mapping* ε *of* $\mathcal{S}(\bar{\Omega}, \bar{G}, \cdot)$ *into* $\mathcal{S}(\bar{\Omega}', \bar{G}', \cdot)$ *is called a* homomorphism *if*

(1) ε *satisfies* 12.5 (1) *and* (2),

(2) *There exist letters* $\alpha \in \Omega$ *and* $\alpha' \in \Omega'$ *such that for* $\tau \in \mathcal{S}(\bar{\Omega}, \bar{G}, \cdot)$

$$\alpha\tau = \beta G_\alpha \text{ for some } \beta \in \Omega$$

implies

$$\alpha'(\tau\varepsilon) = \beta' G'_{\alpha'} \text{ and } \alpha'(\tau^* \varepsilon) = (\beta' G'_{\alpha'})^* \text{ for some } \beta' \in \Omega'.$$

As a consequence of our discussion we can state the following.

PROPOSITION 12.16. - *Let* $(\Omega, G, \cdot)$ *and* $(\Omega', G', \cdot)$ *be homogeneous spaces. For any* $\alpha \in \Omega$ *and any* $\alpha' \in \Omega'$ *we denote by* ω_α *and* $\omega'_{\alpha'}$ *the related isomorphisms of Theorem* 12.14. *Then*

(1) *For every homomorphism* $\varepsilon : \mathcal{S}(\bar{\Omega}, \bar{G}, \cdot) \rightarrow \mathcal{S}(\bar{\Omega}', \bar{G}', \cdot)$ *(in the sense of Definition* 12.15) *the mapping* $\eta : X \rightarrow X\omega_\alpha^{-1} \varepsilon \omega'_{\alpha'}$ *is a homomorphism of the double coset semigroup* G/G_α *into the double coset semigroup* $G'/G'_{\alpha'}$.

(2) *For every homomorphism* $\eta : G/G_\alpha \rightarrow G'/G'_{\alpha'}$ *the mapping* $\varepsilon = \bar{\omega}_\alpha \eta \bar{\omega}'_{\alpha'}$ *is a homomorphism of* $\mathcal{S}(\bar{\Omega}, \bar{G}, \cdot)$ *into* $\mathcal{S}(\bar{\Omega}', \bar{G}', \cdot)$.

(3) $\mathcal{S}(\bar{\Omega}, \bar{G}, \cdot)$ *and* $\mathcal{S}(\bar{\Omega}', \bar{G}', \cdot)$ *are isomorphic if and only if* G/G_α *and* $G'/G'_{\alpha'}$ *are isomorphic.*

Now we have arrived at that point where we can classify the homogeneous spaces by double coset semigroups.

DEFINITION 12.17. - *Let* $(\Omega, G, \cdot)$ *be a homogeneous space and* $\alpha \in \Omega$. *The class* $[G/G_\alpha]$ *of all double coset semigroups isomorphic to* G/G_α *is called the* type *of* $(\Omega, G, \cdot)$.

Since all stabilizers of a single letter are conjugate in G, the type of a homogeneous space $(\Omega, G, \cdot)$ is independent of $\alpha \in \Omega$. In order to get an idea of this classification we look at the following example. We take any group G which has a non-normal subgroup H such that

$$G = H \cup HgH, \; g \in G - H.$$

Then the double coset semigroup $T_2 = G/H$ has three elements H, HgH, G and the multiplication table

	H	HgH	G
H	H	HgH	G
HgH	HgH	G	G
G	G	G	G

LEMMA 12.18. - *Let* $(\Omega,G,\cdot)$ *be a homogeneous space such that* $|\Omega|>2$. *Then* G *acts* 2-*fold transitively on* Ω *if and only if* $(\Omega,G,\cdot)$ *is of type* $[T_2]$.

Does a homomorphism (φ,ψ) of a homogeneous space $(\Omega,G,\cdot)$ into a homogeneous space $(\Omega',G',\cdot)$ always induce a homomorphism ε of $\mathcal{E}(\bar{\Omega},\bar{G},\cdot)$ into $\mathcal{E}(\bar{\Omega}',\bar{G},\cdot)$, and, equivalently, a homomorphism η of the double coset semigroup G/G_α into the double coset semigroup $G'/G'_{\alpha\varphi}$? What does «induce» mean in this context? The homomorphy of (φ,ψ) implies

$$((\alpha g)G_\alpha)\varphi=(\alpha\varphi)(g\psi)(G_\alpha\psi) \text{ and } G_\alpha\psi\leqq G'_{\alpha\varphi}.$$

Therefore if $\tau\in\mathcal{E}(\bar{\Omega},\bar{G},\cdot)$ has the property

$$\alpha\tau=(\alpha g)G_\alpha \text{ for some } g\in G$$

then certainly we have to define $\tau\varepsilon$ as the uniquely determined element $\tau'\in\mathcal{E}(\bar{\Omega}',G',\cdot)$ such that

$$(\alpha\varphi)\tau'=(\alpha\varphi)(g\psi)G'_{\alpha\varphi}$$

holds. Equivalently we have to define

$$(G_\alpha gG_\alpha)\eta=G'_{\alpha\varphi}(g\psi)G'_{\alpha\varphi},$$

and therefore

$$X\eta=G'_{\alpha\varphi}(X\psi)G'_{\alpha\varphi} \text{ for all } X\in G/G_\alpha.$$

With that definition $X\to X\eta$ is a mapping of G/G_α into $\overline{G'/G'_{\alpha\varphi}}$ such that the conditions (2) and (3) of a double coset semigroup homomorphism are satisfied (p. 271). The only question now is whether

$$(XY)\eta=(X\eta)(Y\eta) \text{ for all } X,Y\in G/G_\alpha$$

holds as well in which case $X\eta\in G'/G'_{\alpha\varphi}$ holds for every $X\in G/G_\alpha$, and condition (1) of a double coset semigroup homomorphism is satisfied too. We start with a sufficient condition, which is «almost» necessary as we shall soon see.

THEOREM 12.19. - *Let* (φ,ψ) *be a homomorphism of the homogeneous space* $(\Omega,G,\cdot)$ *into the homogeneous space* $(\Omega',G',\cdot)$ *which satisfies* $(\mathfrak{S}_2)$, *and* $\alpha\in\Omega$.

Then the mapping

$$\eta : X \to G'_{\alpha\varphi}(X\psi)G'_{\alpha\varphi}$$

is a homomorphism of the double coset semigroup G/G_α into the double coset semigroup $G'/G'_{\alpha\varphi}$.

PROOF. - $(\mathfrak{S}_2)$ implies

$$(\alpha\varphi)(g\psi)(G_\alpha\psi) = ((\alpha g)G_\alpha)\varphi = ((\alpha g)\varphi)G'_{\alpha\varphi} = (\alpha\varphi)(g\psi)G'_{\alpha\varphi} \text{ for all } g \in G.$$

This is equivalent to

$$G'_{\alpha\varphi}(g\psi)(G_\alpha\psi) = G'_{\alpha\varphi}(g\psi)G'_{\alpha\varphi} \text{ for all } g \in G,$$

and therefore for all $X, Y \in G/G_\alpha$ we obtain

$$\begin{aligned}(XY)\eta = G'_{\alpha\varphi}(XY)\psi G'_{\alpha\varphi} &= G'_{\alpha\varphi}(X\psi)(G_\alpha\psi)(Y\psi)G'_{\alpha\varphi}\\ &= (G'_{\alpha\varphi}(X\psi)G'_{\alpha\varphi})(G'_{\alpha\varphi}(Y\psi)G'_{\alpha\varphi})\\ &= (X\eta)(Y\eta).\end{aligned}$$

We have proved that η is a homomorphism of G/G_α into $G'/G'_{\alpha\varphi}$.

THEOREM 12.20. - *Let (φ,ψ) be a 1-fold homomorphism of the homogeneous space $(\Omega, G, \cdot)$ into the homogeneous space $(\Omega', G', \cdot)$ (i.e. φ is a surjective mapping (Lemma 6.3)), and $\alpha \in \Omega$. Then the mapping*

$$\eta : X \to G'_{\alpha\varphi}(X\psi)G'_{\alpha\varphi}$$

is a homomorphism of the double coset semigroup G/G_α into the double coset semigroup $G'/G'_{\alpha\varphi}$ if and only if (φ,ψ) satisfies $(\mathfrak{S}_2)$.

PROOF. - Assume that η is a homomorphism of the double coset semigroup G/G_α into the double coset semigroup $G'/G'_{\alpha\varphi}$, and set $K = \mathrm{Ker}_\alpha(\varphi,\psi)$. Note that $K \in \overline{G/G_\alpha}$ because of $G_\alpha \leqq K$ (8.2(3)).

$$\bar{K}\eta = G'_{\alpha\varphi}(K\psi)G'_{\alpha\varphi} = G'_{\alpha\varphi}$$

is the unit element of $G'/G'_{\alpha\varphi}$ by 8.2(5). Therefore $K \leqq \mathrm{Ker}\,\eta$ where the kernel of η is defined as the union of those $G_\alpha g G_\alpha$, $g \in G$, which are mapped by η onto the unit element of $G'/G'_{\alpha\varphi}$ ([3], Definition 2.6). Conversely $g \in \mathrm{Ker}\,\eta$ implies

$$G'_{\alpha\varphi}(g\psi)G'_{\alpha\varphi} = (G_\alpha g G_\alpha)\eta = G'_{\alpha\varphi},$$

and hence $g \in K$. Therefore

$$K = \mathrm{Ker}_\alpha(\varphi,\psi) = \mathrm{Ker}\,\eta\,.$$

But $\operatorname{Ker}\eta$ is a G/G_α-normal subgroup of G ([3], Theorem 2.7), hence K is a 1-fold G/G_α-normal subgroup of G (Lemma 7.4). Then (φ_K, i_G) is a 2-fold epimorphism of $(G_\alpha : G, G, \cdot)$ onto $(K : G, G, \cdot)$ (Theorem 8.6).

$$\varphi_\alpha : \alpha g \to G_\alpha g$$

is a bijective mapping of Ω onto $G_\alpha : G$, and (φ_α, i_G) is an isomorphism of $(\Omega, G, \cdot)$ onto $(G_\alpha : G, G, \cdot)$. Also

$$\varphi_0 : Kg \to (\alpha g)\varphi = (\alpha\varphi)(g\psi)$$

is an injective mapping of $K : G$ into Ω', and (φ_0, ψ) is a homomorphism of $(K : G, G, \cdot)$ into $(\Omega', G' \cdot)$.

We take any $g \in G$ and any

$$y' \in G'_{\alpha\varphi}(g\psi)G'_{\alpha\varphi}.$$

Since φ is surjective there exists $h \in G$ such that

$$(\alpha\varphi)(h\psi) = (\alpha h)\varphi = (\alpha\varphi)y'$$

and hence

$$G'_{\alpha\varphi}(h\psi) = G'_{\alpha\varphi}y'.$$

But then

$$(G_\alpha h G_\alpha)\eta = G'_{\alpha\varphi}(h\psi)G'_{\alpha\varphi} = G'_{\alpha\varphi}(g\psi)G'_{\alpha\varphi} = (G_\alpha g G_\alpha)\eta$$

By [3], Theorem 2.7 this implies

$$KhK = KgK.$$

There exist $k, k' \in K$ such that $h = kgk'$ and hence

$$y' \in G'_{\alpha\varphi}y' = G'_{\alpha\varphi}(k\psi)(g\psi)(k'\psi) \subseteq G'_{\alpha\varphi}(g\psi)(K\psi)$$

which shows that

$$G'_{\alpha\varphi}(g\psi)G'_{\alpha\varphi} \subseteq G'_{\alpha\varphi}(g\psi)(K\psi).$$

The converse inclusion is trivial because of $K\psi \leqq G'_{\alpha\varphi}$ (8.2(5)), and therefore

$$\begin{aligned}(KgK)\varphi_0 &= (Kg)\varphi_0(K\psi) = (\alpha\varphi)(g\psi)(K\psi) = (\alpha\varphi)G'_{\alpha\varphi}(g\psi)(K\psi)\\ &= (\alpha\varphi)G'_{\alpha\varphi}(g\psi)G'_{\alpha\varphi} = (\alpha\varphi)(g\psi)G'_{\alpha\varphi}\\ &= (K\varphi_0)(g\psi)G'_{K\varphi_0} \quad \text{for all } g \in G.\end{aligned}$$

It follows that (φ_0, ψ) satisfies $(\mathfrak{S}_2)$ since all stabilizers of one letter are conjugate in G because $(K: G, G, \cdot)$ is a homogeneous space. Our homomorphism

$$(\varphi,\psi) = (\varphi_\alpha, i_G)(\varphi_K, i_G)(\varphi_0, \psi)$$

is the product of 3 homomorphisms each of which satisfies $(\mathfrak{S}_2)$. Therefore (φ,ψ) satisfies $(\mathfrak{S}_2)$ (4.2(2)) and, taking Theorem 12.19 into account, our theorem is proved.

COROLLARY 12.21. - *For fixed* $\alpha \in \Omega$ *the pair of mappings*

$$(\Omega, G, \cdot) \rightarrow G/G_\alpha \text{ and } (\varphi,\psi) \rightarrow \eta$$

is a functor of the category $\mathfrak{H}_2$ *into the category of all double coset semigroups. For different choices of* α *we get naturally equivalent functors.*

If we apply Corollary 12.21 to the Theorem of Jordan and Hölder for 1-fold G/H-composition chains 11.10, then we obtain the Theorem of Jordan and Hölder for G/H-composition chains [4], 3.3, with the double coset semigroups L_{i-1}/L_i as factors.

Finally we ask for those homomorphisms of a homogeneous space which induce isomorphisms of the double coset semigroups.

THEOREM 12.22. - *Let* (φ,ψ) *be a* 2-*fold pre-isomorphism of the homogeneous space* $(\Omega, G, \cdot)$ *into the homogeneous space* $(\Omega', G', \cdot)$ *and* $\alpha \in \Omega$. *Then*

$$\eta : X \rightarrow G'_{\alpha\varphi}(X\psi)G'_{\alpha\varphi}$$

is an isomorphism of the double coset semigroup G/G_α *onto the double coset semigroup* $G'/G'_{\alpha\varphi}$.

PROOF. - Because of theorem 12.19 all what we need to show is that η is a bijective mapping. φ is a bijective mapping by Definition 5.1. Therefore for every $g' \in G'$ there exists a $g \in G$ such that

$$(\alpha\varphi)(g\psi) = (\alpha g)\varphi = (\alpha\varphi)g',$$

and hence

$$(G_\alpha g G_\alpha)\eta = G'_{\alpha\varphi}(g\psi)G'_{\alpha\varphi} = G'_{\alpha\varphi}g'G'_{\alpha\varphi}.$$

This proves that η is surjective.

If $(G_\alpha g G_\alpha)\eta = (G_\alpha h G_\alpha)\eta$ then, using $(\mathfrak{S}_2)$,

$$G'_{\alpha\varphi}(g\psi)G'_{\alpha\varphi} = G'_{\alpha\varphi}(h\psi)G_{\alpha\varphi} = G_{\alpha\varphi}(h\psi)(G_\alpha\psi).$$

There exist $x' \in G'_{\alpha\varphi}$ and $y \in G_\alpha$ such that

$$g\psi = x'(h\psi)(y\psi).$$

This implies $x' \in G\psi$. Therefore there exists $x \in G$ such that $x' = x\psi$. Furthermore

$$(\alpha x)\varphi = (\alpha\varphi)(x\psi) = \alpha\varphi$$

means that $x \in G_\alpha$ because φ is bijective. Since ψ is a monomorphism by Definition 5.1, we have

$$g = xhy \in G_\alpha h G_\alpha ,$$

and hence $G_\alpha g G_\alpha = G_\alpha h G_\alpha$. Therefore η is an isomorphism.

COROLLARY 12.23. - *2-fold pre-isomorphic homogeneous spaces are of the same type.*

So far our investigations have dealt with the relationship between homogeneous spaces and double coset semigroups. The coset semigroups did not appear explicitly. For every homogeneous space $(\Omega, G, \cdot)$ and $\alpha \in \Omega$ the pair $(\varepsilon_\alpha, i_G)$ with

$$\varepsilon_\alpha : \alpha g \to G_\alpha g$$

is an isomorphism of $(\Omega, G, \cdot)$ onto the homogeneous space $(G_\alpha : G, G, \cdot)$. Therefore we can take $\Omega = G_\alpha : G$ without loss of generality. But then $\overline{G_\alpha : G}$ is a semigroup with respect to the complex multiplication, and its subsemigroup which is generated by the cosets $G_\alpha g$, $g \in G$, is the coset semigroup of G modulo G_α.

If H is a subgroup of the group G, and H' is a subgroup of the group G', then a mapping ϑ of the coset semigroup of G modulo H into the coset semigroup of G' modulo H' is called a *homomorphism* if

(1) $(XY)\vartheta = (X\vartheta)(Y\vartheta)$,

(2) *For every Hg, $g \in G$, there exists a $H'g'$, $g' \in G'$, such that*

$$(Hg)\vartheta = H'g' \text{ and } (Hg^{-1})\vartheta = H'g'^{-1},$$

(3) $X\vartheta = \bigcup_{Hg \subseteq X} (Hg)\vartheta$.

It is now clear how one may proceed with these concepts, but we leave any further discussion.

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