The Cauchy problem for an elliptic parabolic operator*.

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Summary. - *Necessary and sufficient conditions are established for the existence of a solution of a Cauchy problem which is not well posed in the sense of Hadamard.*

1. - Introduction.

If a subsonic flow is given in some domain $\mathfrak{D}$ whose boundary contains a sonic line S then, under certain assumptions, the subsonic flow can be continued in a unique way across S as a supersonic flow without discontinuities. The desired continuation is obtained by solving a CAUCHY problem with data given on S. In fact, by means of a transformation, one may consider a CAUCHY problem for an equation of the form

$$\frac{\partial^2 u}{\partial y^2} + K(y)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where K is a monotone function such that $K(0) = 0$ and $yK(y) > 0$ for $y \neq 0$. Here the sonic line S corresponds to a segment of $y = 0$ and a solution of the CAUCHY problem is sought in some domain contained in $y < 0$ where the equation is of hyperbolic type. It has been shown by BERS [2] and others that this CAUCHY problem for equation (1.1) is well posed in the sense of HADAMARD, that is, a unique solution exists which (in some suitable norm) depends continuously on the CAUCHY data.

It was also pointed out in BERS [2. p. 25] that it would be of interest to obtain results concerning a problem converse to the one discussed there; namely, assume that a supersonic flow is given in a domain whose boundary contains a sonic line S and determine suitable conditions under which the flow can be continued into the subsonic region. However, in the region where $y > 0$ equation (1.1) is of elliptic type and it is well known that a CAUCHY problem in this case, with data given on a segment of $y = 0$, is not well posed in the sense of HADAMARD; in particular, the solution will not in general depend continuously on the data.

Since there are other physically interesting situations which also lead to

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mathematical problems that are not well posed in the sense of Hadamard, there has been in recent years increased interest in problems of this type (see e. g. [3; 4; 6; 8]). However, the main emphasis in the study of non well posed problems up to now has been on the questions of uniqueness and continuous dependence, whereas relatively little work has been done on the more difficult question of existeuce. The present paper represents a contribution towards the resolution of the latter question.

In this paper we will consider an operator T_α of the form

$$T_\alpha u = \frac{\partial^2 u}{\partial y^2} + y^\alpha \frac{\partial^2 u}{\partial x^2}, \quad \alpha > 0. \tag{1.2}$$

Let us note that the equation $T_\alpha u = 0$ is of the form (1.1) for $y > 0$, and that it includes the TRICOMI equation ($\alpha = 1$) as an important special case. We will study the following CAUCHY problem for the operator T_α:

Problem C will consist of determining a function $u = u(x, y)$ which satisfies the equation $T_\alpha u = 0$ in a domain $D = \{(x, y) : 0 < y < y_0\}$, and the prescribed initial conditions $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial y}(x, 0) = g(x)$, $-\infty < x < \infty$.

It is known that Problem C is not well posed in the sense of Hadamard. Moreover, in a recent paper PAYNE and SATHER [7] have established a necessary and sufficient condition for the existence of a periodic solution of Problem C in the special case when f and g are periodic functions. However, their methods, which involve the use of FOURIER series, are not appropriate for the case of non-periodic data.

In order to include a larger class of admissible initial data for Problem C we will seek a solution that only assumes the initial values in some generalized sense. We will say that u is a *generalized solution* in $0 < y < y_0$ of Problem C if (1)

$$T_\alpha u = 0 \quad 0 < y < y_0, \tag{1.3}$$

$$\sup_{0 < y < y_0 - \theta} \int |u(x, y)|^2 \, dx < \infty \tag{1.4}$$

for every θ satisfying $0 < \theta < y_0$, and, as $y \to 0$,

$$\int |u(x, y) - f(x)|^2 dx \to 0 \tag{1.5}$$

(1) Here, and in the sequel, an integral without limitis is taken over $(-\infty, \infty)$

and

$$\int \left| \frac{\partial u}{\partial y}(x, y) - g(x) \right|^2 dx \to 0. \tag{1.6}$$

It is of interest to note that the existence of a solution satisfying a condition of the form (1.4) is often assumed when discussing questions of uniqueness and continuous dependence for problems which are not well posed in the sense of HADAMARD (see e. g., [4; 6]).

Although one may establish the existence of a solution of Problem C by imposing various sufficient conditions on the CAUCHY data (see e. g. [11]), due to the inherent over — prescribed nature of Problem C the most desirable type of existence theorem would seem to be one that imposes conditions on the CAUCHY data which are both necessary and sufficient for the existence of a solution. In Section 3 we establish such a theorem by formulating necessary and sufficient conditions for the existence of a generalized solution of Problem C for the operator T_α. In addition, even though we consider Problem C in this paper for only the operator T_α it will be clear that the method used is appropriate for other operators of the form (1.1) which appear in the literature.

2. - Preliminaries.

Let us begin by presenting some definitions and results which are needed in the main section of the paper (Section 3).

We will require the following basic lemma which is a simple consequence of a theorem of PALEY and WIENER [5, p. 3ff].

LEMMA 1. - Let $\widehat{h}$ denote the FOURIER (PLANCHEREL) transform of $h \in L^2$. Then $\widehat{h}(\xi)e^{(\gamma-\varepsilon)|\xi|} \in L^2$, $0 < \varepsilon < \gamma$, if and only if h is equal a. e. to the restriction to the real axis of a complex valued function $H = H(x + iy)$ such that

(1) H is analytic for $|y| < \gamma$, and

(2) $\sup_{|y|<\gamma-\varepsilon} \int |H(x + iy)|^2 dx < \infty$.

PROOF. - Let us suppose first of all that $\widehat{h}(\xi)e^{(\gamma-\varepsilon)|\xi|} \in L^2$, $0 < \varepsilon < \gamma$. If we define

$$H(z) = (2\pi)^{-1/2} \int \widehat{h}(\xi)e^{i\xi z} d\xi \quad (z = x + iy), \tag{2.1}$$

then the integral converges absolutely and uniformly on compact subsets of $|y| < \gamma$ and, hence, H is analytic for $|y| < \gamma$. By hypothesis there is a $\psi \in L^2$ such that

$$\widehat{h}(\xi)e^{-y\xi} = \psi(\xi)e^{-(\gamma-\varepsilon)|\xi|-y\xi} \equiv \psi(\xi)e^{-\theta|\xi|}, \tag{2.2}$$

where, if $\xi < 0$, $\theta = \gamma - \varepsilon - y$. and, if $\xi > 0$, $\theta = \gamma - \varepsilon + y$. Let y satisfy $|y| < \gamma - \varepsilon$; then $\theta > 0$ and, therefore, $\widehat{h}(\xi)e^{-y\xi}$ belongs to $L^1 \cap L^2$. In particular, by the inversion theorem $H(x) = h(x)$ a. e. on $y = 0$. Moreover, for each, y satisfying $|y| < \gamma - \varepsilon$, it follows that $H(-x + iy)$ is the Fourier transform of $\widehat{h}(\xi)e^{-y\xi}$ and, by the Plancherel theorem (²),

$$\int |H(-x + iy)|^2\, dx = \int |\widehat{h}(\xi)e^{-y\xi}|^2\, d\xi \leq \|\psi\|^2 \tag{2.3}$$

which implies that H satisfies also property (2).

Conversely, let us suppose that h is equal a. e. to the restriction to the real axis of a function $H(x + iy)$ which satisfies properties (1) and (2). For each y satisfying $|y| < \gamma$, let H_y denote the function defined by $H_y(x) = H(x + iy)$. Then $H_0 = h \in L^2$ and, therefore, $\widehat{H}_0 = \widehat{h} \in L^2$. Then one can show by an argument due to PALEY and WEINER (see e. g. [9, p. 130]) that, for each y satisfying $|y| \leq \gamma - \varepsilon$, the FOURIER transform of H_y (which is in L^2 by property (2)) is given a. e. by

$$\widehat{H}_y(\xi) = \widehat{h}(\xi)e^{-y\xi}, \tag{2.4}$$

where $\widehat{h}(\xi)e^{-y\xi}$ belongs to L^2. In particular,

$$\widehat{h}(\xi) = \widehat{H}_{\gamma-\varepsilon}(\xi)e^{(\gamma-\varepsilon)\xi} = \widehat{H}_{-\gamma+\varepsilon}(\xi)e^{-(\gamma-\varepsilon)\xi}. \tag{2.5}$$

Hence, if we define

$$\psi(\xi) = \begin{cases} \widehat{H}_{\gamma-\varepsilon}(\xi) & \xi < 0 \\ \widehat{H}_{-\gamma+\varepsilon}(\xi) & \xi > 0 \end{cases} \tag{2.6}$$

then $\psi \in L^2$ and

$$\widehat{h}(\xi)e^{(\gamma-\varepsilon)|\xi|} = \psi(\xi). \tag{2.7}$$

This completes the proof of the lemma.

Let R_β denote the Riesz kernel of order β which is defined by

$$R_\beta(x) = c\,|x|^{\beta-1}, \quad 0 < \beta < 1, \tag{2.8}$$

(²) Throughout the paper the norm of an element $v \in L^2$ is denoted by $\|v\|$ where $\|v\|^2 = \int |v(t)|^2\, dt$.

where

$$c = \frac{\Gamma\left(\frac{1-\beta}{2}\right)}{2^{\beta}(\pi)^{1/2}\Gamma\left(\frac{\beta}{2}\right)}. \tag{2.9}$$

In addition, let G_β denote the BESSEL kernel of order β as introduced by ARONSZAJN and SMITH [1, p. 414], namely,

$$G_\beta(t) = \frac{1}{2^{(\beta-1)/2}(\pi)^{1/2}\Gamma\left(\frac{\beta}{2}\right)} \mid t \mid^{\frac{\beta-1}{2}} K_{\frac{1-\beta}{2}}(\mid t \mid), \quad \beta > 0, \tag{2.10}$$

where K_v denotes the modified BESSEL function of the third kind. It can be shown that, for $0 < \beta < 1$, R_β is the principal part of G_β at the origin.

In Section 3 we shall require the following two connections between the kernels R_β and G_β. There are positive constants B_1 and B_2 such that, for $\mid t \mid > 0$ and $0 < \beta < 1$,

$$G_\beta(t) \leq B_1 R_\beta(t) \tag{2.11}$$

and

$$\mid G_\beta(t) - R_\beta(t) \mid \leq B_2. \tag{2.12}$$

In fact, by employing well known asymptotic expansions for K_v one can show that [1, p. 416ff]

$$\frac{G_\beta}{R_\beta} \longrightarrow 1 \ \ as \ t \longrightarrow 0, \tag{2.13}$$

and

$$2^{\beta/2}\Gamma\left(\frac{\beta}{2}\right) \mid t \mid^{1-\beta/2} e^{\mid t \mid} G_\beta(t) \longrightarrow 1 \ \ as \ \mid t \mid \longrightarrow \infty. \tag{2.14}$$

Therefore, there are constants such that (2.11) holds if $\mid t \mid$ is either sufficiently small or sufficiently large. Since G_β is also positive one can easily determine a constant so that, in addition, (2.11) holds over any finite set $0 < \delta \leq \mid t \mid \leq N < \infty$. In order to establish (2.12) let us note that a routine calculation involving only the definitions of R_β and G_β shows that $G_\beta - R_\beta$ is a bounded function for $0 < \mid t \mid \leq 1$. Moreover, G_β is a decreasing function of $\mid t \mid$ [1, p. 417] so that

$$\mid G_\beta(t) - R_\beta(t) \mid \leq G_\beta(1) + c \quad \mid t \mid \geq 1,$$

which implies (2.12).

Next we define certain standard functional spaces (treated, for example, in detail in [1, Chap. 2, Sec. 2]) which in turn will be used in Section 3 to describe the classes of functions which are admissible as initial data. Let us denote by H_β the set consisting of (equivalence classes of) L^2 functions for which

$$|u|_\beta^2 = \int (1 + |\xi|^{2\beta}) |\hat{u}(\xi)|^2 \, d\xi < \infty.$$

H_β is a HILBERT space under the norm $|u|_\beta$; the class C_0^∞(smooth functions with compact support) is dense (in norm) in H_β.

For future reference, we list at this point the FOURIER transforms of the kernels R_β (see e. g, [9, p. 182]) and G_β (see [1, p. 410]); namely,

$$\widehat{R}_\beta(\xi) = (2\pi)^{-\frac{1}{2}} \int R_\beta(x) e^{-i\xi x} dx = (2\pi)^{-\frac{1}{2}} |\xi|^{-\beta} \tag{2.15}$$

and

$$\widehat{G}_\beta(\xi) = (2\pi)^{-\frac{1}{2}} (1 + |\xi|^2)^{-\frac{\beta}{2}}. \tag{2.16}$$

The integral in (2.15) exists only for $0 < \beta < 1$ and, then only as an improper RIEMANN integral.

3. - The Existence Theorem.

In this section we establish the main result of the paper.

By use of the method of separation of variables, particular solutions of the equation

$$\frac{\partial^2 u}{\partial y^2} + y^\alpha \frac{\partial^2 u}{\partial x^2} = 0, \quad y > 0, \tag{3.1}$$

of the form $v(y)\, w(x)$ are easily determined; namely, set $w(x) = e^{i\xi x}$ and let v be a solution of the equation

$$\frac{d^2 v}{dy^2} - \xi^2 y^\alpha v = 0, \quad y > 0 \text{ and } \xi \neq 0. \tag{3.2}$$

It is easily seen that solutions of (3.2) are functions of the single variable $\zeta = |\xi|^\beta y$ where $\beta = \dfrac{2}{\alpha + 2}$; in fact, $v(y) = u(|\xi|^\beta y)$ $(\xi \neq 0)$ satisfies (3.2) if and only if $u = u(\zeta)$ satisfies

$$\frac{d^2u}{d\zeta^2} - \zeta^\alpha u = 0.$$

If we set $\beta = \dfrac{2}{\alpha + 2}$ and $\rho = \beta\,|\xi|\,i$, the substitions $V = y^{-1/2}\,v(y)$ and $t = \rho y^{1/\beta}$ transform (3.2) into Bessels' equation. We choose the real linearly independent solutions:

$$\lambda(|\xi|^\beta y) = c_\beta(\beta\,|\xi|\,y^{1/\beta})^{\beta/2} K_{\frac{\beta}{2}}(\beta\,|\xi|\,y^{1/\beta}), \tag{3.3}$$

$$\mu(|\xi|^\beta y) = d_\beta(\beta\,|\xi|\,y^{1/\beta})^{\beta/2} I_{\frac{\beta}{2}}(\beta\,|\xi|\,y^{1/\beta}) + \lambda(|\xi|^\beta y), \tag{3.4}$$

where

$$c_\beta = \frac{2^{\beta/2}\beta^{1-\beta}}{\Gamma\left(1 - \frac{\beta}{2}\right)}. \tag{3.5}$$

$$d_\beta = 2^{\beta/2}\beta^{1-\beta}\Gamma\left(\frac{\beta}{2}\right). \tag{3.6}$$

Here I_ν and K_ν denote the modified BESSEL functions of the first and third kind (see e. g., [10, p. 96 and p. 77ff.]).

The following properties of λ and μ are required in the sequel.

(1) By using the well known formulas [10, p. 79]

$$\frac{d}{dz}\{z^\nu K_\nu(z)\} = -z^\nu K_{1-\nu}(z), \tag{3.7}$$

$$\frac{d}{dz}\{z^\nu I_\nu(z)\} = z^\nu I_{\nu-1}(z), \tag{3.8}$$

we obtain

$$\lambda'(|\xi|^\beta y) = -c_\beta \beta^{\beta-1} z^{1-\beta/2} K_{1-\beta/2}(z), \tag{3.9}$$

$$\mu'(|\xi|^\beta y) = d_\beta \beta^{\beta-1} z^{1-\beta/2} I_{\frac{\beta}{2}-1}(z) + \lambda'(|\xi|^\beta y), \tag{3.10}$$

where $z = \beta\,|\xi|\,y^{1/\beta}$ and the prime denotes differentiation with respect to $|\xi|^\beta y$. Therefore $\lambda'(0) = -1$, $\mu'(0) = 1$ and

$$\lambda_0 = \lambda(0) = \mu(0) = \left(\frac{\beta}{2}\right)^{1-\beta} \frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(1 - \frac{\beta}{2}\right)}. \tag{3.11}$$

(2) $\lambda(|\xi|^\beta y) \to 0$ and $\mu_{\_}(|\xi|^\beta y) \to \infty$ as $|\xi|^\beta y \to \infty$ [10, p. 202ff.].

(3) Since λ and μ are also solutions of (3.2) they cannot have an inflection point for $y > 0$ and, hence, λ is a decreasing positive function of y, and μ and μ' are both increasing positive functions of y, $y > 0$.

(4) Let y satisfy $0 < y_1 \leq y \leq y_2$. It follows from the asymptotic expansions for large z of $K_\nu(z)$ and $I_\nu(z)$ [10, p. 202ff.] that there are positive constants M_i, depending on y_1 and y_2 but *not* on ξ, such that for $y_1 \leq y \leq y_2$

$$\lambda(|\xi|^\beta y) \leq M_1 e^{-|\xi|\beta y^{1/\beta}}, \quad -\infty < \xi < \infty, \tag{3.12}$$

$$M_2 |\xi|^{\frac{\beta-1}{2}} e^{|\xi|\beta y^{1/\beta}} \leq \mu(|\xi|^\beta y) \leq M_3 e^{|\xi|\beta y^{1/\beta}}, \quad |\xi| \geq N, \tag{3.13}$$

$$\mu'(|\xi|^\beta y) \leq M_4 |\xi|^{\frac{1-\beta}{2}} e^{|\xi|\beta y^{1/\beta}}, \quad -\infty < \xi < \infty. \tag{3.14}$$

Let us define for $\varepsilon > 0$

$$\varphi_\varepsilon(x) = \varepsilon^{1-\beta} G_\beta(\varepsilon x), \quad \beta = \frac{2}{\alpha + 2}, \tag{3.15}$$

where G_β is the BESSEL kernel given by (2.10). For the particular choice of $\beta = \frac{2}{\alpha+2} (0 < \beta < 1)$, let us denote the RIESZ kernel by φ instead of R_β. The usefulness of the kernels φ_ε will be seen to stem from the fact that φ_ε and $\widehat{\varphi}_\varepsilon$ simultaneously approximate φ and $\widehat{\varphi}$. The basic relations between φ and φ_ε are exhibited in the following two inequalities: there are constants B_1 and B_2, which are independent of ε, such that

$$\varphi_\varepsilon(x) \leq B_1 \varphi(x) \tag{3.16}$$

and

$$|\varphi_\varepsilon(x) - \varphi(x)| \leq B_2 \varepsilon^{1-\beta}. \tag{3.17}$$

Since φ is homogeneous of degree $\beta - 1$ the inequalities (3. 16) and (3.17) follow immediately from (2.11) ano (2.12), and the identity $\varphi_\varepsilon(x) - \varphi(x) = \varepsilon^{1-\beta}[G_\beta(\varepsilon x) - \varphi(\varepsilon x)]$.

It is well known that if $\psi \in L^1$ and $g \in L^2$ then the convolution

$$\psi * g(x) = \int \psi(x - y) g(y) dy$$

is defined a. e., belongs to L^2, and has the Fourier transform

$$(\psi * g)^\frown = (2\pi)^{1/2} \widehat{\psi}\widehat{g}. \tag{3.18}$$

Although the RIESZ kernel φ is only locally integrable we now show that if g is suitably restricted then (3.18) holds even when ψ is replaced by φ [3].

LEMMA 2. - If $g \in \mathfrak{L} \cap L^2$ then either $\varphi * g \in L^2$ or $\widehat{\varphi g} \in L^2$ implies that $(\varphi * g)^{\frown} = (2\pi)^{1/2} \widehat{\varphi g}$.

PROOF. - It is sufficient to consider the case when g is non-negative. Let us first show that $\widehat{\varphi g} \in L^2$ implies $\varphi * g \in L^2$. Since $\varphi_\varepsilon \in L^1$ it follows from (2.16) and (3.15) that

$$(3.19) \qquad (\varphi_\varepsilon * g)^{\frown}(\xi) = (2\pi)^{1/2} \widehat{\varphi}_\varepsilon(\xi) \widehat{g}(\xi) = \frac{\widehat{g}(\xi)}{(\varepsilon^2 + \xi^2)^{\beta/2}}$$

and, hence, by the Plancherel theorem

$$(3.20) \qquad \| \varphi_\varepsilon * g \|^2 = \int \frac{|\widehat{g}(\xi)|^2}{(\varepsilon^2 + \xi^2)^\beta} d\xi.$$

Moreover, since $g \in \mathfrak{L}$, an immediate consequence of (3.17) is that

$$(3.21) \qquad \lim_{\varepsilon \to 0} \varphi_\varepsilon * g(x) = \varphi * g(x).$$

It follows from (3.20) and Fatou's lemma that

$$(3.22) \qquad \int | \varphi * g |^2 \, dx \leq \varliminf_{\varepsilon \to 0} \| \varphi_\varepsilon * g \|^2 \leq 2\pi \| \widehat{\varphi g} \|^2$$

which implies $\varphi * g \in L^2$.

On the other hand, let us suppose that $\varphi * g \in L^2$. Then (3.16) implies

$$(3.23) \qquad \varphi_\varepsilon * g \leq B_1 \varphi * g$$

and, thus, (3.21) and an application of Lebesgue's theorem yield

$$(3.24) \qquad \lim_{\varepsilon \to 0} \| \varphi_\varepsilon * g - \varphi * g \| = 0.$$

Therefore, $\| \varphi_\varepsilon * g \| \leq$ Const., and an application of Fatou's lemma to (3.20) implies

$$(3.25) \qquad 2\pi \int | \widehat{\varphi g} |^2 \, d\xi < \infty .$$

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(3) See also [12]. Let $\mathfrak{L}$ denote the class of measurable functions h such that $\int (1 + |\xi|)^{\beta - 1} |h(\xi)| d\xi < \infty$. If $h \in \mathfrak{L} \cap L^2$ then $R_\beta * h$ is defined a.e. and is locally square integrable.

Combining (3.22) and (3.25) we see that without loss of generality one may assume for the remainder of the proof that both $\varphi * g \in L^2$ and $\widehat{\varphi}\widehat{g} \in L^2$.

Let us now establish the desired extension of (3.18). It follows from (2.15) and (3.19) that $\widehat{\varphi}_\varepsilon \leq \widehat{\varphi}$ and

$$\lim_{\varepsilon \to 0} (\varphi_\varepsilon * g)^\frown(\xi) = (2\pi)^{1/2}\widehat{\varphi}(\xi)\widehat{g}(\xi). \tag{3.26}$$

Therefore, by Lebesgue's theorem, $\| \widehat{\varphi}_\varepsilon\widehat{g} - \widehat{\varphi}\widehat{g} \| \to 0$ as $\varepsilon \to 0$. But, by the PLANCHEREL theorem, $\| \varphi_\varepsilon * g - \varphi * g \| \to 0$ implies $\| (\varphi_\varepsilon * g)^\frown - (\varphi * g)^\frown \| \to 0$ and, hence $(\varphi_\varepsilon * g)^\frown$ converges in L^2 to both $(\varphi * g)^\frown$ and $(2\pi)^{1/2}\widehat{\varphi}\widehat{g}$. This completes the proof of Lemma 2.

We turn next to the definition of the functional spaces of CAUCHY data. The admissible initial values (f) of a solution are assumed to be L^2 functions while the admissible initial values (g) of its normal derivative are more restrictive.

Let us denote by V the subset of L^2 consisting of (equivalence classes of) functions g for which $|\xi|^{-\beta} g^\frown$ belongs to L^2 and then introduce a norm $|g|_V$ on V by setting

$$|g|_V^2 = \|g\|^2 + \| |\xi|^{-\beta}\widehat{g} \|^2.$$

The function $I: V \to H_\beta$ obtained by setting

$$Ig = v \textit{ if and only if } \widehat{g} = |\xi|^\beta \widehat{v}, \quad v \in H_\beta$$

is an isometry:

$$|v|_\beta^2 = \int (1 + |\xi|^{2\beta}) |\widehat{v}(\xi)|^2 d\xi$$
$$= \|\widehat{v}\| + \| |\xi|^\beta \widehat{v} \|^2$$
$$= \| |\xi|^{-\beta}\widehat{g} \|^2 + \|g\|^2 = |g|_V^2.$$

In addition, if $v \in H_\beta$ then $|\xi|^\beta \widehat{v} \in L^2$ whereby there is a (unique) $g \in L^2$ with $\widehat{g} = |\xi|^\beta \widehat{v}$. Thus I is an isometry of V *onto* H_β. In particular, V is a Hilbert space; moreover, V may be regarded as a subspace of L^2 since $|g|_V = 0$ if and only if $\|g\| = 0$.

REMARK 1. - If $g \in \mathfrak{L} \cap V$ then $\varphi * g \in L^2$ and $(\varphi * g)^\frown = (2\pi)^{1/2}\widehat{\varphi}\widehat{g}$. This result is an immediate consequence of Lemma 2 since both g and $\widehat{\varphi}\widehat{g}$ are in L^2 when $g \in V$.

REMARK 2. - If $\alpha > 2$ then $0 < \beta < \frac{1}{2}$ and $\mathfrak{L} \cap L^2 \subset \mathfrak{L} \cap V$. In fact, if $0 < \beta < \frac{1}{2}$ and $g \in \mathfrak{L} \cap L^2$ then $\widehat{g}$ is an L^2 function bounded by $\| g \|_1 = \int | g(x) | \, dx$ and

$$\int | \xi |^{-2\beta} | \widehat{g} |^2 d\xi \leq \| g \|_1^2 \int_{|\xi| \leq 1} | \xi |^{-2\beta} d\xi + \int_{|\xi| > 1} | \widehat{g} |^2 d\xi < \infty .$$

We have the following existence theorem for Problem C.

THEOREM. - Suppose that $f \in L^2$ and $g \in \mathfrak{L} \cap V$. Let u be given by

$$u(x, y) = (2\pi)^{-1/2} \int [a(\xi) \lambda(| \xi |^\beta y) + b(\xi) \mu(| \xi |^\beta y)] e^{i\xi x} d\xi, \tag{3.27}$$

where

$$a = \frac{1}{2\lambda_0} (\widehat{f} - (2\pi)^{1/2} \lambda_0 \widehat{\widehat{\varphi g}}) \quad \text{and} \quad b = \frac{1}{2\lambda_0} (\widehat{f} + (2\pi)^{1/2} \lambda_0 \widehat{\widehat{\varphi g}}).$$

Then u is a generalized solution of Problem C in $0 < y < \left(\frac{\gamma}{\beta}\right)^\beta$ if and only if the function

$$h(x) = f(x) + \lambda_0 \varphi * g(x) \tag{3.28}$$

is equal a. e. to the restriction to the real axis of a complex valued function $H(x + iy)$ such that

(1) H is analytic for $| y | < \gamma$ and

(2) $\sup_{|y| < \gamma - \delta} \int | H(x + iy) |^2 \, dx < \infty$ for every δ satisfying $0 < \delta < \gamma$.

It is convenient to carry out the proof in several stages.

PART 1 - The condition that the particular combination of the data given by (3.28) has an analytic extension can be replaced by a second necessary and sufficient condition concerning the FOURIER transform of (3.28). Since $\varphi * g \in L^2$ by Remark 1, it is clear that $h \in L^2$. Lemma 1 asserts that h has an analytic extension H satisfying properties (1) and (2) in the statement of the theorem if and only if $\widehat{h}(\xi) e^{(\gamma - \varepsilon)|\xi|}$ belongs to L^2 for *every* ε satisfying $0 < \varepsilon < \gamma$. Moreover, an immediate consequence of Remark 1 is that $\widehat{h} = \widehat{f} + (2\pi)^{1/2} \lambda_0 \widehat{\widehat{\varphi g}}$, and hence, by the definition of b, $\widehat{h} = 2\lambda_0 b$. Thus h has an analytic extension H satisfying properties (1) and (2) if and only if $b(\xi) e^{(\gamma - \varepsilon)|\xi|}$ belongs to L^2 for every ε satisfying $0 < \varepsilon < \gamma$. Therefore, in order to complete the proof, it is sufficient to show that u is a generalized solution in $0 < y < \left(\frac{\gamma}{\beta}\right)^{\beta}$ if and only if $b(\xi)e^{(\gamma-\varepsilon)|\xi|}$ belongs to L^2 for every ε satisfying $0 < \varepsilon < \gamma$.

PART 2. - Let us suppose that u is a generalized solution of Problem C in $0 < y < \left(\frac{\gamma}{\beta}\right)^{\beta}$. Then, by definition, the integral in (3.27) converges for $0 < y < \left(\frac{\gamma}{\beta}\right)^{\beta}$ and all x. Moreover, the function u_y given by $u_y(x) = u(x, y)$ belongs to L^2 so that $\widehat{u}_y$ is in L^2 and is given pointwise, for almost all ξ (see e. g. [9, p. 84]), by

$$\widehat{u}_y(\xi) = (2\pi)^{-1/2} \lim_{A \to \infty} \int_{-A}^{A} \left(1 - \frac{|x|}{A}\right) u_y(x) e^{-i\xi x} dx. \tag{3.29}$$

In addition, since a and b belong to L^2 and λ, μ are continuous, $a\lambda + b\mu$ is locally integrable and, obviously, u_y is locally integrable. Thus it follows, by a theorem on the uniqueness of FOURIER integrals (see e. g. [9, p. 164]), that for almost all ξ

$$\begin{aligned} a(\xi)\lambda(|\xi|^{\beta}y) &+ b(\xi)\mu(|\xi|^{\beta}y) \\ &= (2\pi)^{-1/2} \lim_{A \to \infty} \int_{-A}^{A} \left(1 - \frac{|x|}{A}\right) u_y(x) e^{-i\xi x} dx. \end{aligned} \tag{3.30}$$

Upon comparing (3.29) and (3.30) we see that for almost all ξ

$$\widehat{u}_y = a\lambda + b\mu. \tag{3.31}$$

Let ε satisfy $0 < \varepsilon < \gamma$ and set $y = \left(\frac{\gamma - \varepsilon/2}{\beta}\right)^{\beta}$. It follows from (3.13) that

$$\mu(|\xi|^{\beta}y) \geq M_2(|\xi|^{\frac{\beta-1}{2}} e^{\frac{\varepsilon}{2}|\xi|}) e^{(\gamma-\varepsilon)|\xi|} \geq M_2 e^{(\gamma-\varepsilon)|\xi|}$$

holds for sufficiently large $|\xi|$. Hence, since $\mu(|\xi|^{\beta}y) \geq \lambda_0$, there is a number M, depending on y but not on ξ, such that

$$e^{(\gamma-\varepsilon)|\xi|} \leq M\mu(|\xi|^{\beta}y). \tag{3.32}$$

Combining (3.12), (3.31) and (3.32) we obtain

$$|b| e^{(\gamma-\varepsilon)|\xi|} \leq M(|\widehat{u}_y| + M_1 |a|), \; y = \left(\frac{\gamma - \varepsilon/2}{\beta}\right)^{\beta}. \tag{3.33}$$

Thus, $be^{(\gamma-\varepsilon)|\xi|} \in L^2$ for every ε satisfying $0 < \varepsilon < \gamma$.

PART 3. - Conversely, let us suppose that $be^{(\gamma-\varepsilon)|\xi|} \in L^2$ for every ε satisfying $0 < \varepsilon < \gamma$. Let y satisfy $0 < y < \left(\frac{\gamma}{\beta}\right)^\beta$ and set $\varepsilon = \frac{1}{2}(\gamma - \beta y^{1/\beta})$. Then $\gamma - \varepsilon - \beta y^{1/\beta} = \varepsilon > 0$ and it follows from (3.12) and (3.13) that there are positive constants M_1 and M such that

$$|a|\lambda \leq M_1 |a| e^{-|\xi|\beta y^{1/\beta}} \tag{3.34}$$

and

$$|b|\mu \leq M|b| e^{(\gamma-\varepsilon)|\xi|} e^{-\varepsilon|\xi|}. \tag{3.35}$$

Hence $(a\lambda + b\mu) \in L^1 \cap L^2$ so that for each y satisfying $0 < y < \left(\frac{\gamma}{\beta}\right)^\beta$ the integral in (3.27) converges for all x. We will show that if u is defined by (3.27) then u is a generalized solution of Problem C in $0 < y < \left(\frac{\gamma}{\beta}\right)^\beta$.

Let us note first of all that

$$u = (\breve{a}\lambda + \breve{b}\mu)^{\frown}, \tag{3.36}$$

where $\breve{a}(\xi) = a(-\xi)$ and $\breve{b}(\xi) = b(-\xi)$. Moreover, since $(a\lambda + b\mu) \in L^2$ it follows from (3.36) that

$$\widehat{u}_y = a\lambda + b\mu, \tag{3.37}$$

and, hence, by the PLANCHEREL theorem

$$\|u_y\| = \|a\lambda + b\mu\|. \tag{3.38}$$

If δ satisfies $0 < \delta < \gamma$ then, for all y satisfying $0 < y < \left(\frac{\gamma-\delta}{\beta}\right)^\beta = \tilde{\gamma}$,

$$|b(\xi)\mu(|\xi|^\beta y)| \leq |b(\xi)|\,|\mu(|\xi|^\beta \tilde{\gamma})| \leq M|b(\xi)|\, e^{(\gamma-\delta)|\xi|}, \tag{3.39}$$

where M is independent of y. From (3.38) and (3.39) we obtain

$$\sup_{0<y<\tilde{\gamma}} \|u_y\| \leq \lambda_0 \|a\| + M\|be^{(\gamma-\delta)|\xi|}\| < \infty. \tag{3.40}$$

Therefore, u satisfies condition (1.4).

Let K be a compact subset of the strip $0 < y < \left(\frac{\gamma}{\beta}\right)^\beta$ and let $\rho > 0$ be such that K is contained in the strip $S_\rho : \left(\frac{\rho}{\beta}\right)^\beta \leq y \leq \left(\frac{\gamma - 2\rho}{\beta}\right)^\beta$. Upon using (3.12) and (3.13) as above we obtain estimates like (3.34) and (3.35) (with ε replaced by ρ) which are valid for y in S_ρ. It follows that $a\lambda + b\mu$ is bounded uniformly with respect to y in S_ρ by an integrable function and, therefore, the integral in (3.27) converges uniformly on K. Hence, by a standard argument employing a theorem of Harnack, u is a solution of $T_\alpha u = 0$ in $0 < y < \left(\frac{\gamma}{\beta}\right)^\beta$.

Thus, in order to complete the proof, it remains only to show that u assumes the boundary values in the sense of (1.5) and (1.6). For this purpose we introduce the function

(3.41) $$\Lambda_\nu(t) = t^\nu K_\nu(t),$$

where $\nu > 0$ and K_ν denotes the modified BESSEL function of the third kind. Let us note that, see (3.3) and (3.9),

(3.42) $$\Lambda_{\beta/2}(\beta \mid \xi \mid y^{1/\beta}) = \frac{1}{c_\beta} \lambda(\mid \xi \mid^\beta y),$$

(3.43) $$\Lambda_{1-\frac{\beta}{2}}(\beta \mid \xi \mid y^{1/\beta}) = -\frac{\beta^{1-\beta}}{c_\beta} \lambda'(\mid \xi \mid^\beta y).$$

Moreover,

(3.44) $$\lim_{t \to 0} \Lambda_\nu(t) = 2^{\nu-1}\Gamma(\nu).$$

and (see e. g., [10, p. 172])

(3.45) $$\Lambda_\nu(t) = \frac{2^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{(\pi)^{1/2}} \int^\infty \frac{\cos tu}{(1 + u^2)^{\nu+1/2}} du.$$

By inversion we obtain

(3.46) $$(2\pi)^{-\frac{1}{2}} \int \Lambda_\nu(\beta \mid \xi \mid y^{1/\beta}) e^{i\xi x} d\xi = 2^{\nu-\frac{1}{2}} (\pi)^{1/2} \Gamma(\nu) P_y^\nu(x),$$

where

(3.47) $$P_y^\nu(x) = \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{(\pi)^{1/2}\Gamma(\nu)} \frac{(\beta y^{1/\beta})^{2\nu}}{[(\beta y^{1/\beta})^2 + x^2]^{\nu+1/2}} \qquad \nu > 0,\ y > 0.$$

Let us note that the FOURIER inversion theorem also implies

$$\int P_y^\nu(x)\,dx = \frac{\Lambda(0)}{2^{\nu-1}\Gamma(\nu)} = 1. \tag{3.48}$$

With these preliminary calculations it is not difficult to show that the family $P_y^\nu(y > 0)$ is an approximate identity in L^2, and, for reference, we state formally the result as

LEMMA 3. – The family $P_y^\nu(y > 0)$ defined by (3.47) is an approximate identity in L^2, that is, for every $v \in L^2$,

$$\lim_{y \to 0} \| P_y^\nu * v - v \| = 0.$$

Let us recall that $a = \frac{1}{2\lambda_0}(\widehat{f} - (2\pi)^{1/2}\widehat{\varphi g})$. Since $2^{\beta/2-1}\Gamma\left(\frac{\beta}{2}\right)c_\beta = \lambda_0$ it follows from Lemma 2 and the PARSEVAL theorem that

$$\begin{aligned}\int a(\xi)\lambda(|\xi|^\beta y)e^{i\xi x}d\xi &= c_\beta \int a(\xi)\Lambda_{\beta/2}(\beta\,|\xi|\,y^{1/\beta})e^{i\xi x}d\xi \\ &= (2\pi)^{1/2}P_y^{\beta/2}*\left[\frac{1}{2}(f - \lambda_0\varphi * g)\right](x).\end{aligned} \tag{3.49}$$

Consequently, by Lemma 3, as $y \to 0$

$$(2\pi)^{-1/2}\int a\lambda e^{i\xi x}d\xi \to \frac{1}{2}(f - \lambda_0\varphi * g) \text{ in } L^2. \tag{3.50}$$

Since $be^{(\gamma-\varepsilon)|\xi|} \in L^2$ by hypothesis, it follows that $b(\xi)\,|\xi|^\beta = \frac{1}{2\lambda_0}(f(\xi)\,|\xi|^\beta + \lambda_0\widehat{g}(\xi)$ belongs to L^2. Thus $g \in L^2$ implies that there is a $\psi \in L^2$ such that $\widehat{\psi}(\xi) = \widehat{f}(\xi)\,|\xi|^\beta$ belongs to L^2 and

$$a(\xi)\,|\xi|^\beta = \frac{1}{2\lambda_0}(\widehat{\psi}(\;) - \lambda_0\widehat{g}(\xi)). \tag{3.51}$$

Another application of the PARSEVAL theorem yields

$$\begin{aligned}&\int a(\xi)\,|\xi|^\beta\lambda'(|\xi|^\beta y)e^{i\xi x}d\xi \\ &= -c\;\beta^{\beta-1}\int a(\xi)\,|\xi|^\beta\Lambda_{1-\beta/2}(\beta\,|\xi|\,y^{1/\beta})e^{i\xi x}d\xi \\ &= -(2\pi)^{1/2}P_y^{1-\beta/2}*\left[\frac{1}{2\lambda_0}(\psi - \lambda_0 g)\right](x).\end{aligned} \tag{3.52}$$

Therefore, by Lemma 3, as $y \to 0$

(3.53) $$(2\pi)^{-1/2} \int a(\xi) \,|\, \xi \,|^{\beta} \lambda'(\,|\, \xi \,|^{\beta} y) e^{i\xi x} d\xi \qquad \frac{1}{2\lambda_0}(\lambda_0 g - \psi) \quad \text{in } L^2.$$

Next we define, for convenience, the functions

(3.54) $$A_y(x) = (2\pi)^{-1/2} \int \mathrm{b}(\xi) \mu(\,|\, \xi \,|^{\beta} y) e^{i\xi x} d\xi,$$

and

(3.55) $$B_y(x) = (2\pi)^{-1/2} \int b(\xi) \,|\, \xi \,|\, \mu'(\,|\, \xi \,|\, y) e^{i\xi x} d\xi.$$

Then in order to establish (1.5) aud (1.6) it is sufficient to show that, as $y \to 0$,

(3.56) $$A_y \to \frac{1}{2}(f + \lambda_0 \varphi^* g) \quad \text{in } L^2$$

and

(3.57) $$B_y \to \frac{1}{2\lambda_0}(\lambda_0 g + \psi) \quad \text{in } L^2,$$

since these limits and (3.50) and (3.53) imply that, as $y \to 0$,

(3.58) $$u \to \frac{1}{2}(f - \lambda_0 \varphi^* g) + \frac{1}{2}(f + \lambda_0 \varphi^* g) = f \quad \text{in } L^2$$

and

(3.59) $$\frac{\partial u}{\partial y} \to \frac{1}{2\lambda_0}(\lambda_0 g - \psi) + \frac{1}{2\lambda_0}(\lambda_0 g + \psi) = g \quad \text{in } L^2.$$

It follows from (3.13) that, for $0 \leq y \leq \bar{\gamma} = \left(\frac{\gamma}{4\beta}\right)^{\beta}$,

(3.60) $$|\, b(\xi) \,|\, \mu(\,|\, \xi \,|^{\beta} y) \leq |\, b(\xi) \,|\, \mu(\,|\, \xi \,|^{\beta} \bar{\gamma}) \leq M \,|\, b(\xi) \,|\, e^{\frac{\gamma}{2}|\xi|} e^{-\frac{\gamma}{4}|\xi|}$$

and, hence, $be^{\frac{\gamma}{2}|\xi|} \in L^2$ implies that $b\mu \in L^1 \cap L^2$ for $0 \leq y \leq \bar{\gamma}$. Therefore

(4.61) $$\widehat{A}_y = b\mu, \quad 0 \leq y \leq \gamma,$$

and by the PLANCHEREL theorem

(3.62) $$\| A_y - A_0 \|^2 = \| \widehat{A}_y - \widehat{A}_0 \|^2 = \int |\, \mathrm{b}\mu - \lambda_0 b \,|^2 d\xi.$$

An application of Lebesque's theorem then yields

$$\lim_{y \to 0} \| A_y - A_0 \| = 0. \tag{3.63}$$

Since

$$A_0(x) = (2\pi)^{-1/2} \int \lambda_0 b(\xi) e \;\; x d\xi \tag{3.64}$$

and $\lambda_0 b = \frac{1}{2}(\widehat{f} + (2\pi)^{1/2}\lambda_0 \widehat{\varphi g})$ belongs to L^1, we have by inversion

$$A_0 = \frac{1}{2}(f + \lambda_0 \varphi * g) \tag{3.65}$$

which together with (3.63) implies (3.56). A similar argument employing (3.14) and the function ψ defined by (3.51) yields (3.57). This completes the proof of the theorem.

Let us remark that in Part 2 of the above proof we showed that $be^{(\gamma-\varepsilon)|\xi|} \in L^2$, $0 < \varepsilon < \gamma$, and it follows that $|\xi|^\beta \widehat{f} = 2\lambda_0 |\xi|^\beta b - \lambda_0 \widehat{g}$ also belongs to L^2. Thus, a second necessary condition that Problem C has a generalized solution (3.27) is that f must belong to H_β.

In conclusion we wish to emphasize that the above theorem gives necessary and sufficient conditions for the existence of a generalized solution of Problem C in a prescribed strip instead of some indeterminate neighborhood of the initial line.

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