On boundary value problems for an ordinary linear differential system.

D. Wexler (Bucharest)

Summary. - *It is shown that some general boundary value problems for an ordinary linear differential system are normally solvable.*

1. - Introduction.

The purpose of this paper is to indicate a way to obtain the adjoint problem of some general b.v.p. for ordinary linear differential systems on compact interval (Section 3). We shall also discuss the relationships between the b.v.p. and its adjoint (Section 4).

In our considerations a significant part play some concepts of distributions theory. The adjoint equation is a differential equation in a distribution space. We discuss such differential equations in Section 2. We use a well known theorem concerning linear equations in Banach spaces: let E, F be Banach spaces and Ω a linear continuous operator $E \to F$. The following assertions are equivalent (see [1] chpt. IV, § 4, or [4], chpt. VI, § 6):

(i) the equation $\Omega x = y$ is normally solvable i.e. this equation has solutions if and only if y is orthogonal to any solution of the adjoint equation ${}^t\Omega f = 0$ (1);

(ii) the equation ${}^t\Omega f = g$ is normally solvable i.e. this equation has solutions if and only if g is orthogonal to any solution of the equation $\Omega x = 0$;

(iii) the range $\mathcal{R}(\Omega)$ is closed in F.

It follows that if $\mathcal{R}(\Omega) = F$, then the kernel $N({}^t\Omega) = \{0\}$.

General b.v.p., where the boundary condition is given by a linear operator, are discussed using other means by R. Conti [3]. If we want to consider the adjoint problem it is suitable this operator be continuous, what we

(1) We design the dual spaces by E', F' and the adjoint operator $F' \to E'$ by ${}^t\Omega$.

shall suppose further on. By this, our framawork is less general than of CONTI. The adjoint problem obtained here includes the adjoint ones found by other means by R. H. COLE [2], and A. HALANAY and A. MORO [5].

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2. - Preliminaries.

1). Let $I=[\alpha, \beta]$ be a compact interval and L^1 be the space of summable on I complex valued functions, with usual norm

$$\|\varphi\|_1 = \int_I |\varphi(t)|\, dt.$$

We denote by $\mathcal{C}_a$ the space of complex valued functions absolutely continuous on I. The functions $\varphi \in \mathcal{C}_a$ are almost every where derivable and $D\varphi \in L^1$. We consider in $\mathcal{C}_a$ the norm

$$\|\varphi\|_a = |\varphi(\bar{t})| + \|D\varphi\|_1,$$

where $\bar{t}$ is a fixed point in I. The topology of $\mathcal{C}_a$ is independent of the choice of $\bar{t}$, for the norm $\|\cdot\|_a$ is equivalent to the norm

$$\|\varphi\| = \sup_{t \in I} |\varphi(t)| + \|D\varphi\|_1 .$$

$\mathcal{C}_a$ is a Banach space. In fact, let (φ_k) be a Cauchy sequence in $\mathcal{C}_a$. It follows that $(\varphi_k(\bar{t}))$ is a convergent numerical sequence, $\lim \varphi_k(\bar{t}) = \lambda$ and $(D\varphi_k)$ is a convergent in L^1 sequence, $\lim D\varphi_k = \psi$. It is easy to verify that $\varphi_k \longrightarrow \varphi$ in $\mathcal{C}_a$, where

$$\varphi(t) = \lambda + \int_{\bar{t}}^{t} \psi(s)ds, \qquad t \in I.$$

It is well-known that the dual space of L^1 may be identified to the space L^∞ of measurable functions essentially bounded on I. The value of a functional $f \in L^\infty$ on $\varphi \in L^1$ is given by

$$< \varphi, f > = \int_I \varphi(t) f(t) dt$$

and the norm of f is

$$\|f\|_\infty = \sup_{\varphi \in L^1, \|\varphi\|_1 \leq 1} |<\varphi, f>| = \operatorname{ess\,sup}_{t \in I} |f(t)|.$$

We design by $\mathcal{C}'_a$ the dual space of $\mathcal{C}_a$. It is a Banach space. We show that $\mathcal{C}'_a$ is an intermediate space between the space $\mathcal{C}'$ of measures on I and the space $\mathcal{C}'^1$ of distributions of order ≤ 1 on I.

Let $\mathcal{C}^1$ be the space of functions with continuous first derivative on I, normed by

$$N_1(\varphi) = \sup_{t \in I} |\varphi(t)| + \sup_{t \in I} |D\varphi(t)|.$$

Its dual $\mathcal{C}'^1$ is the space of distributions of order ≤ 1 on I. Obviously $\mathcal{C}^1 \subset \mathcal{C}_a$, the topology of $\mathcal{C}^1$ is stronger that the one induced by $\mathcal{C}_a$ and $\mathcal{C}^1$ is dense in (2) $\mathcal{C}_a$. Then the restriction to $\mathcal{C}^1$ of a functional $f \in \mathcal{C}'_a$ is in $\mathcal{C}'^1$ and f is uniquely determined by this restriction (for $\mathcal{C}^1$ is dense in $\mathcal{C}_a$). We see that the restriction operator establishes an (algebraical) isomorphism between $\mathcal{C}'_a$ and a subspace of $\mathcal{C}'^1$. By identifying $\mathcal{C}'_a$ to this subspace, we can consider $\mathcal{C}'_a \subset \mathcal{C}'^1$.

In the space $\mathcal{C}$ of continuous on I functions we consider the usual norm

$$N_0(\varphi) = \sup_{t \in I} |\varphi(t)|.$$

Its dual $\mathcal{C}'$ is the space of measures on I. Obviously, $\mathcal{C}_a \subset \mathcal{C}$, the topology of $\mathcal{C}_a$ is stronger that the one induced by $\mathcal{C}$ and $\mathcal{C}_a$ is dense in $\mathcal{C}$ (since the space of polynomials is dense in $\mathcal{C}$). Then the restriction operator establishes an isomorphism between $\mathcal{C}'$ and a subspace of $\mathcal{C}'_a$, i.e. $\mathcal{C}' \subset \mathcal{C}'_a$. In the same way, $L^\infty \subset \mathcal{C}$. We conclude

$$L^\infty \subset \mathcal{C}' \subset \mathcal{C}'_a \subset \mathcal{C}'^1.$$

It is easy to see that all inclusions are strict.

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(2) In fact the space $\mathcal{C}$ of continuous on I functions is dense in L^1. Then φ for $\varphi \in \mathcal{C}_a$ there is $\psi_k \in \mathcal{C}$ so that $\psi_k \longrightarrow D\psi$ in L^1. The functions

$$\varphi_k(t) - \varphi(\bar{t}) + \int_{\bar{t}}^{t} \psi_k(s)ds, \; t \in I$$

belong to $\mathcal{C}^1$. By $D\varphi_k = \psi_k$ we deduce

$$\|\varphi - \varphi_k\|_a = \|D\varphi - \psi_k\|_1.$$

It follows $\varphi_k \longrightarrow \varphi$ in $\mathcal{C}_a$ and $\mathcal{C}^1$ is dense in $\mathcal{C}_a$.

2). Suppose $b \in \mathcal{C}_a$. For $\varphi \in \mathcal{C}_a$ we have $b\varphi \in \mathcal{C}_a$ and the linear operator $\varphi \longrightarrow b\varphi$ is continuous. The adjont operator is called the multiplication by b operatar $\mathcal{C}'_a \longrightarrow \mathcal{C}'_a$. In other words, for $f \in \mathcal{C}'_a$ we have $bf \in \mathcal{C}'_a$, the linear operator $f \longrightarrow bf$ is continuous and the functional bf is defined by

$$< \varphi, bf > = < b\varphi, f >, \text{ for } \varphi \in \mathcal{C}_a.$$

If $b \in L^1$, then $bf \in L^\infty$ for any $f \in L^\infty$ and the multiplication by b is a liaear continuous operator $L^\infty \longrightarrow L^\infty$.

The function $b \in L^1$ can be considered also as a multiplicator $\mathcal{C}_a \longrightarrow L^1$. Then the adjont operator $L^\infty \longrightarrow \mathcal{C}'_a$ coincides with the multiplication $L^\infty \longrightarrow L^\infty$ defined above.

The differential operator $D : \mathcal{C}_a \longrightarrow L^1$ being linear and continuous, the adjoint ${}^tD : L^\infty \longrightarrow \mathcal{C}'_a$ posesses also these properties. For $f \in L^\infty$ we have ${}^tDf \in \mathcal{C}'_a$ and ${}^tDf$ is defined on $\mathcal{C}_a$ by

$$< \varphi, {}^tDf > = < D\varphi, f >, \qquad \varphi \in \mathcal{C}_a.$$

The derivation presents hese some peculiarities in comparison with the derivations on an open interval. Using the integration by parts for STIELTJES integrals, it is easy to verify that if f is a function of bounded variation, then

$$-{}^tDf = f(\alpha)\delta_\alpha - f(\beta)\delta_\beta + df \text{ (}^3\text{)}, \tag{1}$$

where df is the STIELTJES measure defined by f. If f is absolutely continuous on I, then $df = (Df)dt$ (the product of the summable function Df by the Lebesgue measure dt) and

$$-{}^tDf = f(\alpha)\delta_\alpha - f(\beta)\delta_\beta + Df.$$

If $f(\alpha) = f(\beta) = 0$, then $-{}^tDf = df$ (respectively $-{}^tDf = Df$). On can say that $-{}^tD$ is the derivation operator $L^\infty \longrightarrow \mathcal{C}'_a$.

If $b \in \mathcal{C}_a$ and $f \in L^\infty$, then

$${}^tD(bf) = -(Db)f + b({}^tDf). \tag{2}$$

In fact, we have for $\varphi \in \mathcal{C}_a$

$$< \varphi, {}^tD(bf) > = < D\varphi, bf > = < b(D\varphi), f > = < D(b\varphi) - (Db)\varphi, f > =$$
$$= < D(b\varphi), f > - < (Db)\varphi, f > = < b\varphi, {}^tDf > - < \varphi, (Db)f > =$$
$$= < \varphi, b({}^tDf) > + < \varphi, -(Db)f > = < \varphi, -(Db)f + b({}^tDf) >.$$

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(3) We design by δ_γ the measure of Dirac concentrated in the point $\varphi \in I$: the functional δ_γ is defined by $< \varphi, \delta_\gamma > = \varphi(\gamma)$, for $\varphi \in \mathcal{C}_a$.

3). We shall indicate a way which allows to extend the previous constructions to vector distributions with values in the complex n-dimensional euclidean space C^n. The elements of C^n will be written as row-vectors, the norm of

$$c = (c_1, \ldots, c_n)$$

beeing $|c| = \max |c_j|$. Let us denote by $\mathcal{C}_a(C^n)$ the product of n spaces equal to $\mathcal{C}_a$. With the norm

$$\|\varphi\|_a = \max \|\varphi_j\|_a . \ \varphi = (\varphi_1, \ldots, \varphi_j), \ \varphi_j \in \mathcal{C}_a,$$

$\mathcal{C}_a(C_n)$ is a Banach space. Its dual $\mathcal{C}'_a(C_n)$ can be identified to the space of column-vectors

$$f = \mathrm{col}(f_1, \ldots, f_n), \qquad f_j \in \mathcal{C}'_a,$$

the value of the functional f on a function $\varphi \in \mathcal{C}_a(C^n)$ being

$$< \varphi, f > = \sum_j < \varphi_j, f_j >.$$

$\mathcal{C}'_a(C^n)$ is (algebraically) isomorphic to the product of n spaces equal to $\mathcal{C}'_a$. It is easy to see that the norm of $f \in \mathcal{C}'_a(C^n)$ is

$$\|f\|'_a = \max \|f_j\|'_a .$$

In the same way, one can introduce the spaces $L^1(C^n)$, $\mathcal{C}(C^n)$, $\mathcal{C}^1(C^n)$ and their dual spaces. It is easy to extend the properties mentioned in the previous paragraphs to these spaces.

A linear operator $A : \mathcal{C}_a(C^n) \longrightarrow L^1(C^n)$ can be represented by a $n \times n$ matrix $A = (A_{jk})$, where A_{jk} are linear operators $\mathcal{C}_a \longrightarrow L^1$ and

$$A\varphi = (\sum_j A_{j1}\varphi_j, \ldots, \sum_j A_{jn}\varphi_j) \qquad \text{for } \varphi \in \mathcal{C}_a(C^n).$$

The operator A is continuous if and only if A_{jk} are continuous. Analogously an operator $B : L^\infty(C^n) \longrightarrow \mathcal{C}'_a(C^n)$ can be represented by a $n \times n$ matrix $B = (B_{jk})$,

$$Bf = \mathrm{col}\,(\sum_k B_{1k} f_k, \ldots, \sum_k B_{nk} f_k) \text{ for } f \in L^\infty(C^n).$$

If $A = (A_{jk})$ is a, continuous linear operator $\mathcal{C}_a(C^n) \longrightarrow L^1(C^n)$, then the adjoint operator ${}^tA : L^\infty(C^n) \longrightarrow \mathcal{C}'_a(C^n)$ is represented by the matrix ${}^tA = ({}^tA_{jk})$. The multiplicators $\mathcal{C}_a(C^n) \longrightarrow L^1(C^n)$ are $n \times n$ matrices of summable functions.

Similar remarks are valid for operator $L^1(C^n) \longrightarrow L^1(C^n)$, $\mathcal{C}_a(C^n) \longrightarrow \mathcal{C}_a(C^n)$ and so on.

4). Let us consider the equation

$$Du + uA = 0, \qquad (u = \text{row-vector}), \tag{3}$$

where A is a $n \times n$ matrix whose elements are summable functions i.e. A is a multiplicator $\mathcal{C}_a(C^n) \to L^1(C^n)$. The solutions of (3) shall be understood as Carathéodory solutions. Let us denote by $U(t)$, $t \in I$, the fundamental matrix of solutions, $U(\alpha) = E$, where E is the unit matrix. Let $\mathfrak{U}$ be the Cauchy-operator $C_n \to \mathcal{C}_a(C^n)$, $(\mathfrak{U}u_0)(t) = u_0 U(t)$, $t \in I$. The solutions of (3) are $\mathfrak{U}u_0$, $u_0 \in C^n$.

We look now for the solutions in $L^\infty(C^n)$ of the equation

$$^tDy + Ay = g, \tag{4}$$

where $g \in \mathcal{C}'_a(C^n)$. Let us denote by $\mathfrak{B}_1$ the operator $\mathcal{C}_a(C_n) \to L^1(C^n)$, $\mathfrak{B}_1\varphi = D\varphi + \varphi A$. Then (3) can be written as $\mathfrak{B}_1 u = 0$, and (4) can be written as $^t\mathfrak{B}_1 y = g$. The range $\mathfrak{R}(\mathfrak{B}_1) = L^1(C^n)$. By the theorem of Section 1 it follows that (4) has solutions if and only if g is orthogonal to any solution of (3), i.e. $< \mathfrak{U}u_0, g > = 0$ for $u_0 \in C^n$. At the same time if this condition holds, then (4) has a unique solution.

Suppose now that g is a measure, i.e. $g \in \mathcal{C}'(C^n)$. By a well known theorem of F. RIESZ there exists a function $\chi : I \to C^n$ with bounded variation on I, oontinuous at the right on $[\alpha, \beta)$ so that g is the STIELTJES measure $d\chi$. We also suppose that g is orthogonal to any solution of (3) i.e.

$$\int_\alpha^\beta U(t) d\chi(t) = 0. \tag{5}$$

Let us make the substitution $y = U^{-1}z$. By (2) and by $DU^{-1} = AU^{-1}$ we obtain for z the equivalent to (4) equation

$$^tDz = Ug, \tag{6}$$

whose unique solution is

$$z(s) = -\int_\alpha^s U(t) d\chi(t), \quad s \in I. \tag{7}$$

In fact it is easy to see that this function has bounded variation on I and is continuous at the right on $[\alpha, \beta)$. The STIELTJES measure dz defined by z is

$$dz = -U\,d\chi = -Ug.$$

By (1) and (5) it follows that the function (7) is the unique solution in $L^{\infty}(C^n)$ of (6). We see so, that if $g = d\chi$ verifies (5), then the unique solution in $L^{\infty}(C^n)$ of (4) is

$$y(s) = -\int_{a}^{s} U^{-1}(s)U(t)d\chi(t), \quad s \in I.$$

5). Let Λ be a Banach space, Λ' its dual and $< l, \lambda >$ the value of a functional $\lambda \in \Lambda'$ on $l \in \Lambda$. Let us consider a continuous linear operator $\mathcal{B}_2 : \mathcal{C}_a(C^n) \longrightarrow \Lambda$ and its adjoint ${}^t\mathcal{B}_2 : \Lambda' \longrightarrow \mathcal{C}'_a(C^n)$. Obviously for $\lambda \in \Lambda'$, the superposition $\lambda \circ \mathcal{B}_2$ coincides with ${}^t\mathcal{B}_2\lambda$.

The elements of the product $\mathcal{C}_a(C_n) \times \Lambda$ will be denoted (φ, l) where $\varphi \in \mathcal{C}_a(C^n)$, $l \in \Lambda$. With the norm

$$\| (\varphi, l) \| = \max (\| \varphi \|_a, \| b \|)$$

this product is a Banach space. Its dual $\mathcal{C}'_a(C^n) \times \Lambda'$ is also a Banach space, the value of a functional

$$\begin{pmatrix} f \\ \lambda \end{pmatrix}, \quad f \in \mathcal{C}'_a(C^n), \quad \lambda \in \Lambda',$$

on $(\varphi, l) \in \mathcal{C}_a(C^n) \times \Lambda$ beeing

$$< (\varphi, l), \begin{pmatrix} f \\ \lambda \end{pmatrix} > = < \varphi, f > + < l, \lambda >.$$

We shall also consider the product $L^1(C^n) \times \Lambda$ and its dual $L^{\infty}(C^n) \times \Lambda'$.

3. - Boundary value problems.

We consider now the following b.v.p : given $\varphi \in L^1(C^n)$ and $l \in \Lambda$, determine $x \in \mathcal{C}_a(C^n)$ such that

$$\mathcal{B}_1 x = \varphi, \ \mathcal{B}_2 x = l. \tag{8}$$

Let us consider the operator $\mathcal{B} : \mathcal{C}_a(C^n) \longrightarrow L^1(C^n) \times \Lambda$, $\mathcal{B}\psi = (\mathcal{B}_1\psi, \mathcal{B}_2\psi)$. Then the b.v.p. can be written

$$\mathcal{B}x = (\varphi, l). \tag{8'}$$

The domain of the linear operator $\mathcal{B}_{2U} = \mathcal{B}U$ is C^n. Then the range $\mathcal{R}(\mathcal{B}_{2U})$ is a finite-dimensional subspace of Λ and $\mathcal{B}_{2U}$ can be represented by a $n \times m$ matrix, where $m = \dim \mathcal{R}(\mathcal{B}_{2U})$. Note that $m \leq n$.

For $\psi \in L^1(C^n)$ we define the function $J\psi$,

$$(J\psi)(t) = \int_{a}^{t} \psi(s)U^{-1}(s)U(t)ds, \ t \in I.$$

It is easy to see that $J\psi \in \mathcal{C}(C^n)$ and that the linear operator $J : L^1(C^n) \to \mathcal{C}_a(C^n)$ is continuous.

The solutions of the equation $\mathfrak{B}_1 x = \varphi$ are

$$x = \mathfrak{U}x_0 + J\varphi, \quad x_0 \in C^n.$$

Such a function x verifies $\mathfrak{B}_2 x = l$ if and only if x_0 verifies

(9) $$\mathfrak{B}_{2U} x_0 = l - \mathfrak{B}_2 J\varphi.$$

Thus the b.v.p. (8) has solntions if and only if (9) has solutions.

We show that the range $\mathfrak{R}(\mathfrak{B})$ is closed in $L^1(C^n) \times \Lambda$. In fact, the operator

$$V : L^1(C^n) \times \Lambda \to \Lambda, \; V(\varphi, l) = l - \mathfrak{B}_2 J\varphi,$$

is linear and continuous. The equation (9) has solutions if and only if

$$V(\varphi, l) \in \mathfrak{R}(\mathfrak{B}_{2U}) \text{ i.e. } (\varphi, l) \in V^{-1}\mathfrak{R}(\mathfrak{B}_{2U}).$$

This yields $\mathfrak{R}(\mathfrak{B}) = V^{-1}\mathfrak{R}(\mathfrak{B}_{2U})$. The space $\mathfrak{R}(\mathfrak{B}_{2U})$ is closed in Λ for it is finite-dimensional. [4] Then $V^{-1}\mathfrak{R}(\mathfrak{B}_{2U})$ in closed in $L^1(C^n) \times \Lambda$, for V is continuous.

It folllows by the theorem mentioned in Section 1 that the *b.v.p.* (8) *is normally solvable.* In other words, if we denote by ${}^t\mathfrak{B}$ the adjoint operator $L^\infty(C^n) \times \Lambda' \to \mathcal{C}'_a(C^n)$, then the b.v.p. (8) has solutions (in $\mathcal{C}_a(C^n)$) if and only if

$$< (\varphi, l), \begin{pmatrix} y \\ \lambda \end{pmatrix} > = 0$$

for any solution $\begin{pmatrix} y \\ \lambda \end{pmatrix}$ (in $L^\infty(C^n) \times \Lambda'$) of the adjoint equation

(10) $${}^t\mathfrak{B}\begin{pmatrix} y \\ \lambda \end{pmatrix} = 0.$$

4. - The adjoint problem.

We find an explicit form for (10). For $\begin{pmatrix} f \\ \lambda \end{pmatrix} \in L^\infty(C^n) \times \Lambda'$ and $\psi \in \mathcal{C}^a(C^n)$, we have

$$< \psi, {}^t\mathfrak{B}\begin{pmatrix} f \\ \lambda \end{pmatrix} > = < \mathfrak{B}\psi, \begin{pmatrix} f \\ \lambda \end{pmatrix} > = < (\mathfrak{B}_1\psi, \mathfrak{B}_2\psi), \begin{pmatrix} f \\ \lambda \end{pmatrix} > =$$

$$= < \mathfrak{B}_1\psi, f > + < \mathfrak{B}_2\psi, \lambda > = < \psi, {}^t\mathfrak{B}_1 f > + < \psi, {}^t\mathfrak{B}_2\lambda > =$$

$$= < \psi, {}^t\mathfrak{B}_1 f + {}^t\mathfrak{B}_2\lambda >.$$

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[4] We note that here some difficulties arise if we wish to discuss b.v.p for differential equations in a Banach space, for then $\mathfrak{R}(\mathfrak{B}_{2U})$ is not finite-dimensional.

This yields

$$ {}^t\mathfrak{B}\binom{f}{\lambda} = {}^t\mathfrak{B}_1 f + {}^t\mathfrak{B}_2\lambda. $$

By ${}^t\mathfrak{B}_1 f = {}^tDf + Af$ it follows that (10) can be written as

$$ {}^tDy + Ay + {}^t\mathfrak{B}_2\lambda = 0 \tag{10'} $$

and the adjoint problem is: determine the solutions $\binom{y}{\lambda}$, $y \in L^\infty(C^n)$, $\lambda \in \Lambda'$ of (10′). Remember that ${}^t\mathfrak{B}_2\lambda = \lambda \circ \mathfrak{B}_2$.

We recognize that for fixed $\lambda \in \Lambda'$ the equation (10′) is of type studied in Section 1. Hence it has a solution if and only if $< \mathfrak{U}u_0,\ {}^t\mathfrak{B}_2\lambda > = 0$ for $u_0 \in C^n$, i.e.

$$ < \mathfrak{B}_{2U}u_0,\ \lambda > = 0, \text{ for } u_0 \in C^n. \tag{11} $$

In other words, (10′) has a solution if and only if λ belong to the orthogonal complement $\mathfrak{R}(\mathfrak{B}_{2U})^0$ (in Λ') of $\mathfrak{R}(\mathfrak{B}_{2U})$. At the same time, if this condition holds, it follows by the assertion mentioned in Section 1 that (for fixed $\lambda \in \mathfrak{R}(\mathfrak{B}_{2U})^0$) the solution y is uniquely determined.

Returing to the adjoint problem we deduce that its solution are $\binom{y}{\lambda}$, where $\lambda \in \mathfrak{R}(\mathfrak{B}_{2U})^0$ and y is the corresponding solution of (10′).

Since $\dim \mathfrak{R}(\mathfrak{B}_{2U}) < \infty$, there is a closed in Λ subspace Λ_1 such that Λ can be represented as a topological sum

$$ \Lambda = \mathfrak{R}(\mathfrak{B}_{2U}) \dot{+} \Lambda_1 $$

(see [1] chpt. II, § 3). Then Λ' can be represented by the topological sum

$$ \Lambda' = \mathfrak{R}(\mathfrak{B}_{2U})^0 + \Lambda_1^0. \tag{12} $$

The orthogonal complement Λ_1^0 of Λ_1 is the space of functionals which are continuous and linear on Λ, vanishing on Λ_1. We see that Λ_1^0 is isomorphic to the space of functionals which are continuous and linear on $\mathfrak{R}(\mathfrak{B}_{2U})$. Then

$$ \dim \Lambda_1{}^0 = \dim \mathfrak{R}(\mathfrak{B}_{2U}) = m. $$

By (12) we deduce

$$ \dim \mathfrak{R}(\mathfrak{B}_{2U})^0 = \dim \Lambda' - \dim \mathfrak{R}(\mathfrak{B}_{2U}). \tag{13} $$

We have established above that $\mathfrak{R}(\mathfrak{B}_{2U})^0$ is isomorphic to the space of solutions of the adjoint problem ${}^t\mathfrak{B}\binom{y}{\lambda} = 0$. Then the kernel $\mathfrak{N}({}^t\mathfrak{B})$ satisfies

$$ \dim \mathfrak{N}({}^t\mathfrak{B}) = \dim \mathfrak{B}(\mathfrak{B}_{2U})^0 $$

and, by (13)

$$\dim \mathfrak{N}({}^t\mathcal{B}) = \dim \Lambda' - \dim \mathcal{R}(\mathcal{B}_{2U}). \tag{14}$$

Thus the *space* $\mathfrak{N}({}^t\mathcal{B})$ *of solutions of the adjoint problem is finite-dimensional if and only if* $\dim \Lambda < \infty$.

We note that the space $\mathfrak{N}(\mathcal{B})$ of solutions of the homogeneous b.v.p. is always finite-dimensional. It is the space of solutions $\mathfrak{U}u_0$ of (3) whose initial condition u_0 verifies $\mathcal{B}_{2U}u_0 = 0$. For the Cauchy operator $\mathfrak{U}$ is an isomorphism, we have

$$\dim \mathfrak{N}(\mathcal{B}) = \dim \mathfrak{N}(\mathcal{B}_{2U}).$$

Since the domain of $\mathcal{B}_{2U}$ is C^n, we have

$$\dim \mathfrak{N}(\mathcal{B}_{2U}) + \dim \mathcal{R}(\mathcal{B}_{2U}) = n \tag{15}$$

and then by (14)

$$\dim \mathfrak{N}({}^t\mathcal{B}) = \dim \Lambda' + \dim \mathfrak{N}(\mathcal{B}) - n. \tag{16}$$

It follows that $\dim \mathfrak{N}({}^t\mathcal{B}) = \dim \mathfrak{N}(\mathcal{B})$, if and only if $\dim \Lambda = n$.

The b.v.p. (8) has solutions for any $(\varphi, l) \in L^1(C^n) \times \Lambda$, if and only if $\mathfrak{N}({}^t\mathcal{B}) = \{0\}$. By (14) this condition is equivalent to

$$\dim \Lambda < \infty \text{ and } \dim \mathcal{R}(\mathcal{B}_{2U}) = \dim \Lambda,$$

or by (16) it is equivalent to

$$\dim \Lambda < \infty \text{ and } \dim \Lambda = n - \dim \mathfrak{N}(\mathcal{B}).$$

The unicity condition for solution of b.v.p. (8) is $\mathfrak{N}(\mathcal{B}) = \{0\}$, or by (15), $\dim \mathcal{R}(\mathcal{B}_{2U}) = n$. This condition implies the existence of solutions for any $(\varphi, l) \in L^1(C^n) \times \Lambda$, provided that $\dim \Lambda = n$.

It follows that, generally, for b.v.p. (8) the alternative of Fredholm ([5]) does not hold. Note also that the alternative holds always if $\Lambda = C^n$.

5. - Examples.

1. Take now $\Lambda = C^p$. Then $\mathcal{B}_2$ can be represented by a $n \times p$ matrix $\mathcal{B}_2 = (\mathcal{B}_2^{jk})$, where $\mathcal{B}_2^{jk} \in \mathcal{C}'_a$ and

$$\mathcal{B}_2\varphi = (\sum_j < \varphi_j, \mathcal{B}_2^{j1} >, \ldots, \sum_j < \varphi_j, \mathcal{B}_2^{jp} >), \text{ for } \varphi \in \mathcal{C}_a(C^n).$$

([5]) I.e. the b.v.q. (8) has solutions for any $(\varphi, l) \in L^1(C^n) \times \Lambda$, if and only if tho homogeneous b.v q. $\mathcal{B}x = 0$ has only the zero solution.

Let us designate this matrix also by $\mathcal{B}_2$. The operator $\mathcal{B}_{2U}: C^n \longrightarrow C^p$ may be represented a $n \times p$ matrix $\mathcal{B}_{2U} = (\mathcal{B}_{2U}^{jk})$.

It is well known that the linear functionals on C^p may be identified to the vectors $\lambda \in C^p$, the value of the fuctional λ on the vector l being $l \cdot \lambda'$ (we denote by λ' the transposed vector). It is easy to see that ${}^t\mathcal{B}_2\lambda'$ may be represented by the product ${}^t\mathcal{B}_2\lambda' = \mathcal{B}_2\lambda'$.

Then the adjoint problem is: determine $y \in \mathcal{C}'_a(C^n)$ and $\lambda \in C^p$ so that

$$ {}^tDy + Ay + \mathcal{B}_2\lambda' = 0. \tag{17} $$

The necessary and sufficient condition (11), that equation (17) posesses solution, is $\mathcal{B}_{2U}\lambda' = 0$. If $\mathcal{B}_{2U}^g$ is the pseudo-inverse of PENROSE [6], then the vectors $\lambda \in C^p$, which verifie $\mathcal{B}_{2U}\lambda' = 0$, are

$$ \lambda' = (E_p - \mathcal{B}_{2U}^g \mathcal{B}_{2U})w, \quad w \in C_p, \tag{18} $$

where E_p is the $p \times p$ unit matrix. The corresponding to λ solution y of (17) is uniquely determined.

Suppose now that $\mathcal{B}_2$ is a Stieltjes measure, $\mathcal{B}_2 = dM$, i.e.

$$ \mathcal{B}_2\varphi = \int_\alpha^\beta \varphi(t)dM(t), \text{ for } \varphi \in \mathcal{C}_a(C^n), $$

where M is a $n \times p$ matrix, whose entries are functions of bounded variation on I, continuous on $[\alpha, \beta)$. If λ has the form (18), by Section 1 it follows that the unique solution y of (17) is the function of bounded variation on I, continuous at the right on $[\alpha, \beta)$,

$$ y(s) = \left[\int_\alpha^s U^{-1}(s)U(t)dM(t) \right] \lambda', \quad s \in I, \tag{19} $$

Thus the solutions of the adjoint problem are $\binom{y}{\lambda}$, where λ is given by (18) and y by (19). The b.v.p. has solutions if and only if (φ, l) is orthogonal to any solution $\binom{y}{\lambda}$, i.e. if and only if

$$ \left[l + \int_\alpha^\beta \varphi(s)ds \int_\alpha^s U^{-1}(s)U(t)dM(t) \right] \left| E_p - \mathcal{B}_{2U}^g \mathcal{B}_{2U} \right| = 0. $$

2. Let v be a vector of C^n. Let us consider the following b.v.p.: given $\varphi \in L^1(C^n)$ and $l \in L^1$, determine $x \in \mathcal{C}_a(C^n)$ such that

$$ \mathcal{B}_2 x = \varphi, \quad x \cdot v' = l \tag{20} $$

Here $\Lambda = L^1$ and $\mathcal{B}_2$ is the operator $\mathcal{C}_a(C^n) \longrightarrow L^1$, $\mathcal{B}_2\psi = \psi v'$. The adjoint operator ${}^t\mathcal{B}_2 : L^\infty \longrightarrow \mathcal{C}'_a(C^n)$ is defined by ${}^t\mathcal{B}_2\lambda = \lambda v'$. Note that $\mathcal{R}({}^t\mathcal{B}_2) \subset L^\infty(C^n)$. The adjoint problem is: find $\lambda \in L^\infty$ and $y \in \mathcal{C}'_a(C^n)$, such that

$$(21) \qquad {}^tDy + Ay + \lambda v' = 0.$$

By Section 1 follows that this problem has a solution if and only if

$$(22) \qquad \int_\alpha^\beta \lambda(t)U(t)v'dt = 0.$$

If λ verifies (22) then the component y of the solution $\begin{pmatrix} y \\ \lambda \end{pmatrix}$ is

$$(23) \qquad y(s) = \int_\alpha^s U^{-1}(s)U(t)\lambda(t)v'dt, \quad s \in I.$$

We see that the b.v.p. problem (20) has solutions if and only if

$$\int_\alpha^\beta l(t)\lambda(t) + \int_\alpha^\beta \varphi(s)y(s)ds = 0,$$

for any solution $\begin{pmatrix} y \\ \lambda \end{pmatrix}$ of the adjoint problem. Note that this condition is equivalent to

$$\int_\alpha^\beta \lambda(t)\left[l(t) + \int_t^\beta \varphi(s)U^{-1}(s)U(t)v'ds\right]dt = 0$$

for any $\lambda \in L^\infty$ satisfying (22).

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