The Boundedness and Extendibality of Differential Systems under Integral Perturbation

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Summary. – *Necessary and sufficient conditions for integral boundedness are given in terms of a Liapunov function are given the construction of the Liapunov function is a modification of the Okamura function. Similar results are also given for the extendability of solutions under integral perturbations.*

Introduction.

Much research has been done concerning the preservation of stability and asymptotic stability under integrable perturbations. In particular, VRKOC [6] defined an unperturbed system to be integrally stable if the solutions of the perturbed equation starting near the origin remain near the origin provided the perturbations are integrable. Simple criteria for integral stability were obtained by VRKOC in terms of the existence of a Liapunov function $V(t, x)$ defined in a closed cylinder about the origin and satisfying a Lipschitz condition in x in which the Lipschitz constant is independent of t and x. Very recently CHOW [2] and CHOW and YORKE [3] have extended Vrkoc's work by using a modification of a function essentially due to OKAMURA [5] and YOSHIZAWA [7]. Not only are their proofs substantially simpler than those of Vrkoc, but they have also enlarged the class of admissible perturbations to include the absolutely diminishing functions.

In this paper we develop the natural analogue of integral stability for the cases of boundedness and extendability on $[t_0, \infty)$. Roughly speaking, integral boundedness (extendability) is the preservation of the uniform boundedness (extendability) of solutions under integrable perturbations. Criteria for integral boundedness (extendability) are obtained in terms of a Liapunov function defined in $R\times R^d$. The construction of the Liapunov function is again a modification of the Okamura function; however, since the domain is no longer a compact set (as in the case of stability) we are not able to use the techniques easily available such Ascoli's Theorem to obtain properties of the Liapunov function.

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For the unperturbed system

$$\dot{x} = f(t, x) , \tag{E}$$

where $f: R \times R^d \to R^d$ is continuous, integral boundedness obviously implies uniform boundedness, but not conversely (see Example 1). However, if all solutions of (E) are uniformly stable, then uniform boundedness and integral extendability are equivalent if f satisfies a global Lipschitz condition.

1. – Definitions and preliminaries.

Let R^d denote Euclidean d-space and $|\cdot|$ will denote any d-dimensional norm. Represent a solution of the unperturbed differential equation (E) through (t_0, x_0) with $t_0 \geqslant 0$ by $x(t, t_0, x_0)$. Consider the perturbed equations

$$\dot{x} = f(t, x) + r(t, x) , \tag{H}$$

$$\dot{x} = f(t, x) + g(t) , \tag{P}$$

and denote solutions of (H) and (P) through (t_0, x_0) by $x_H(t, t_0, x_0)$ and $x_P(t, t_0, x_0)$ respectively.

Except when indicated, assume throughout that $f(\cdot, x)$ is measurable for each x, $f(t, \cdot)$ is continuous for each t, $|f|$ is bounded on compact subsets of $(0, \infty) \times R^d$ (the same conditions hold for $r(t, x)$) and $g: [0, \infty) \to R^d$ is measurable.

Let $V: [0, \infty) \times R^d \to R$ be a Liapunov function. Define the time derivative of V along solutions of (E) as

$$\dot{V}_E(t, x) = \limsup_{h \to 0^+} \frac{1}{h} \Big(V(t + h, x(t + h)) - V(t, x) \Big) .$$

If V is Lipschitzean with respect to x, then it is well known ([7]) that

$$\dot{V}_E(t, x) = \limsup_{h \to 0^+} \frac{1}{h} \Big(V(t + h, x + hf(t, x)) - V(t, x) \Big) .$$

The following definitions will be used:

DEFINITION 1. – Solutions of (E) are integrally bounded if for each $\alpha > 0$ and and $\eta > 0$ there exists $\beta = \beta(\alpha, \eta) > 0$ such that whenever $|x_0| < \alpha$ and

$$\int_{t_0}^{\infty} \sup_{|x| < \beta(\alpha, \eta)} |R(t, x)| \, dt < \eta$$

then $|x_H(t, t_0, x_0)| < \beta$ for all $t \geqslant t_0 \geqslant 0$ where x_H is a solution of (H).

Using techniques similar to those used by HALANAY [4, p. 96] for the case of stability, the following lemma shows that it is sufficient to only consider (P) for integral boundedness.

LEMMA 1. – Solutions of (E) are integrally bounded if and only if for each $\alpha > 0$ and $\eta > 0$ there exists $\beta = \beta(\alpha, \eta) > 0$ such that whenever $|x_0| < \alpha$ and

$$\int_{t_0}^{\infty} |g(t)|\, dt < \eta$$

then $|x_P(t, t_0, x_0)| < \beta$ for all $t \geqslant t_0 \geqslant 0$ where $x_P(t)$ is a solution of (P).

The proof is the same as in [4, p. 96] with very slight modifications and we shall omit it.

We now define integral extendability of (E) in terms of (P) (the definition in terms of (H) is equivalent as in the case of boundedness).

DEFINITION 2. – Solutions of (E) are integrally extendable if for each $t_0 \geqslant 0$, $T < 0$, $\alpha > 0$, and $\eta > 0$ there exists $\beta = \beta(t_0, T, \alpha, \eta)$ such that whenever $|x_0| < \alpha$ and

$$\int_{t_0}^{\infty} |g(t)|\, dt < \eta$$

then $|x_P(t, t_0, x_0)| < \beta$ for $t \in [t_0, t_0 + T]$ where $x_P(t)$ is a solution of (P).

Examples of integrally bounded systems and integrally extendable systems are $\dot{x} = 0$ and $\dot{x} = x$ respectively. In the next two examples, we show that uniform boundedness does not imply integral boundedness and extendability does not imply integral extendability.

EXAMPLE 1. – Solutions of the scalar equation

$$\text{(S)} \qquad \dot{x} = (x - n)(n + 1 - x) \qquad n < x \leqslant n + 1\,, \qquad n = 1, 2, \ldots$$

are uniformly bounded but we now show that (S) is not integrally bounded by constructing a function $p(t)$ such that some solution of

$$\text{(SP)} \qquad \dot{x} = (x - n)(n + 1 - x) + p(t) \qquad n < x \leqslant n + 1\,, \qquad n = 1, 2, \ldots$$

where $p(t) \geqslant 0$ for $t \in [0, \infty)$, $\int_0^{\infty} p(t)\, dt < \infty$ are not bounded. Pick a point $(0, x_0)$, x_0 not an integer; then there exists $t_1 > 0$ such that $[x_0] + 1 - x(t_1, 0, x_0) < \frac{1}{2}$ ($[\cdot]$ represents the greatest integer function) where $x(\cdot)$ is a solution of (SP) in which

$$p(t) \equiv 0 \qquad t \in [0, t_1],$$
$$p(t) \equiv \tfrac{1}{2} \qquad t \in [t_1, t_1 + 1].$$

Then $x(t_1 + 1, 0, x_0) > [x_0] + 1$ and now define $t_2 > t_1 + 1$ such that $[x(t_1 + 1, 0, x_0)] + 1 - x(t_2, 0, x_0) < \frac{1}{4}$ in which

$$p(t) \equiv 0 \qquad t \in [t_1 + 1, t_2],$$
$$p(t) \equiv \tfrac{1}{4} \qquad t \in [t_2, t_2 + 1].$$

Then $x(t_2 + 1, 0, x_0) > [x_0] + 2$. Continuing this process, we have

$$|x(t, 0, x_0)| \to \infty \qquad \text{as } t \to \infty,$$

where

$$p(t) \equiv 0, \quad t \in \left(\bigcup_{n=1}^{\infty} [t_n + 1, t_{n+1}] \cup [0, t_1]\right),$$

and

$$p(t) \equiv \frac{1}{2^n}, \qquad t \in [t_n, t_n + 1).$$

Hence $\int_0^\infty p(t)dt < \infty$, and solutions are not integrally bounded.

The next example, a slight modification of an example in [1] shows that extendability does not imply integral extendability.

EXAMPLE 2. – Consider the scalar equation

$$\text{(S2)} \qquad \dot{x} = \varphi(t)h(x),$$

where $\varphi: [0, \infty) \to [0, \infty)$ and satisfies $\int_0^\infty \varphi(t)\,dt < \infty$. Define $1/h(x)$ as follows: for each integer $n > 0$ such that $n \leqslant x \leqslant n + 1$,

$$\frac{1}{h(n)} = n, \qquad \frac{1}{h(n+1)} = n + 1,$$
$$\frac{1}{h(x)} = \frac{1}{x^2}, \qquad \text{for } n + \frac{1}{n^2} \leqslant x \leqslant n + 1 - \frac{1}{(n+1)^2};$$

and

$$\frac{1}{h(x)} \quad \text{is linear for } n \leqslant x \leqslant n + \frac{1}{n^2},$$

and for

$$(n+1)-\frac{1}{(n+1)^2}\leq x\leq n+1 .$$

After some elementary computation, it follows that

$$\int_1^\infty \frac{dr}{h(r)}=\infty \qquad \text{and} \qquad \int_1^\infty \frac{dr}{h(r)+1}<\infty .$$

Thus, solutions of (S2) are extendable (in fact uniformly bounded); however, some solutions of

$$\text{(S3)} \qquad \dot{x}=\varphi(t)h(x)+\varphi(t)$$

do not exist in the future. This can be seen by picking any point x_0 so large that

$$\int_{x_0}^\infty \frac{dx}{h(x)+1}<\int_0^\infty \varphi(t)\,dt;$$

then the solution $x(t, 0, x_0)$ of (S3) is not extendable on $[0, \infty)$.

2. - Results.

We now present our main results.

THEOREM 1. - Solutions of (E) are integrally bounded if and only if there exists a continuous Liapunov function $V(t, x)$ defined on $[0, \infty)\times R^d$ satisfying

(*a*) $a(|x|)\leq V(t, x)\leq b(|x|)$, $a(r)\to\infty$ as $r\to\infty$ monotonically and $b(r)$ is monotone increasing.

(*b*) $|V(t, x)-V(t, y)|\leq K|x-y|$ for some $K>0$, and for all $(t, x, y)\in[0, \infty)\times \times R^d\times R^d$.

(*c*) $\dot{V}(t, x)\leq 0$.

REMARK. - Condition (*b*) (V satisfies a global Lipschitz condition) is the essential characteristic of the Liapunov function in describing the difference between uniform boundedness and integral boundedness. In the case of uniform boundedness, we can only conclude that the Liapunov function is locally Lipschitz, that is, the Lipschitz constant depends on t and x when f is locally Lipshitz. Even for the case in which f satisfies a global Lipschitz condition, we cannot conclude there exists a Liapunov function satisfying a global Lipschitz condition as Example 1 shows. However, if we assume all solutions of (E) are uniformly bounded and uniformly stable $(|x(t, t_0, x_0) - x(t, t_0, y_0)| \leqslant L|x_0 - y_0|$ for all $t \geqslant t_0 \geqslant 0$ and $x_0 \in R^d$, $y_0 \in R^d)$, then it is not difficult to show $V(t, x) \equiv \sup_{\tau \geqslant 0} |x(t+\tau, t, x)|$ satisfies (a), (b), and (c). Hence, with Theorem 1 we have the following corollary. Observe that no Lipschitz condition is needed on f.

COROLLARY 1. – Assume all solutions of (E) are uniformly stable. Then, solutions of (E) are integrally bounded if and only if they are uniformly bounded.

THEOREM 2. – Solutions of (E) are integrally extendable if and only if there exists a continuous Liapunov function satisfying (b), (c), and

(d) $V(t, x) \to \infty$ as $|x| \to \infty$ uniformly for t in compact sets of $[0, \infty)$.

REMARK. – If f is locally Lipschitz, then the extendability of a solution is equivalent to the existence of a Liapunov function satisfying (c), (d) and a local Lipschitz condition. However, by assuming that f satisfies the Lipschitz condition,

$$|f(t, x_1) - f(t, x_2)| \leqslant \lambda(t)|x_1 - x_2|, \quad \text{where } \lambda(t) \tag{2.1}$$

is continuous, then using the function $V(t, x) = |x|$, which satisfies (a) and (b), we obtain

$$\dot{V} \leqslant \lambda(t) V + |f(t, 0)|, \tag{2.2}$$

From the variation of constants formula, the solutions of (2.2) are extendable. A slight modification of Theorem 2 implies that conditions (a), (b), and (2.2) also yield the integral extendability. We thus obtain the following corollary.

COROLLARY 2. – Assume $|f(t, x_1) - f(t, x_2)| \leqslant \lambda(t)|x_1 - x_2|$, where $\lambda(t)$ is continuous. Then solutions of (E) are integrally extendable.

In Example 2, $h(x)$ does not satisfy a global Lipschitz condition and this agrees with our results.

3. – Proofs.

PROOF OF THEOREM 1. – Assume V satisfies (a), (b), and (c). Then (b) and (c) imply

$$\dot{V}_P(t, x) \leqslant \dot{V}_E(t, x) + K|g(t)| \leqslant K|g(t)|.$$

Hence

$$V(t, x_P(t, t_0, x_0)) \leqslant V(t_0, x_0) + K\int_{t_0}^{t} |g(s)|\, ds;$$

and from (a) for $|x_0| < \alpha$

$$a(|x_p(t, t_0, x_0)|) \leqslant b(\alpha) + K\eta ,$$

where $\int_0^\infty |g(s)| ds < \eta$. This implies

$$|x_p(t, t_0, x_0)| \leqslant a^{-1}(b(\alpha) + K\eta) \equiv \beta(\alpha, \eta) ,$$

thus proving that solutions are integrally bounded.

Conversely, assume solutions are integrally bounded. Let $A(t, x)$ be the set of absolutely continuous functions $\{\varphi(\cdot)\}$ defined on $[0, t]$ satisfying $\varphi(0) = 0$ and $\varphi(t) = x$. Define

$$V(t, x) = \inf_{\varphi \in A(t,x)} \int_0^t |\varphi'(s) - f(s, \varphi(s))| ds .$$

This is the Okamura function and many of the techniques we use now have been developed by YOSHIZAWA [7, p. 5-8] for the case of uniqueness of the zero solution. Thus, for some parts of the proof, only references will be indicated. As mentioned before, since the domain of the Liapunov function is $[0, \infty) \times R^d$ instead of $[0, \infty) \times \times \{x: |x| < \varrho\}$ (as in stability and uniqueness) certain techniques will be developed.

Let $p_1(t_1, x_1)$ and $p_2(t_2, x_2)$, $t_1 < t_2$, be any two points in R^d and denote by $\psi_{p_1 p_2}$ the family of all absolutely continuous functions $\{\varphi(t)\}$ defined on $[t_1, t_2]$ satisfying $\varphi(t_1) = x_1$ and $\varphi(t_2) = x_2$. Define for $t_2 > t_1$

$$v(p_1, p_2) = \inf_{\varphi \in \psi_{p_1 p_2}} \int_{t_1}^{t_2} |\dot{\varphi}(t) - f(t, \varphi(t))| dt .$$

If $t_1 = t_2$, let $v(p_1, p_2) = |x_1 - x_2|$.

Let $\{\varphi_k(t)\}$ be a sequence such that

$$v(p_1, p_2) = \lim_{k \to \infty} \int_{t_1}^{t_2} |\dot{\varphi}_k(t) - f(t, \varphi_k(t))| dt . \tag{3.1}$$

We show $\{\varphi_k(t)\}$ is uniformly bounded on $[t_1, t_2]$. Define

$$g_k(t) = \begin{cases} \dot{\varphi}_k(t) - f(t, \varphi_k(t)) & t_1 \leqslant t \leqslant t_2 \\ 0 & t_2 < t . \end{cases}$$

Then from (3.1), $\int_0^\infty |g_k(t)| dt < M_1$ for some $M_1 > 0$ and for all k; moreover, let $x_k(t)$ for $t \in [t_1, \infty)$ be a solution of

$$\dot{x} = f(t, x) + g_k(t) ,$$

in which $x_k(t)=\varphi_k(t)$ for $t\in[t_1,t_2]$. Since solutions of (E) are integrally bounded, $|x_k(t)|\leqslant\beta(|x_1|, M_1)$ for all k and $t\geqslant t_1$. Hence $\{\varphi_k(t)\}$ is uniformly bounded. Using this, we now show the existence of $\bar{K}$ such that

$$|v(p_1, p_2)-|x_1-x_2||\leqslant\bar{K}(t_2-t_1)\,, \tag{3.2}$$

where $\bar{K}$ depends on β and the interval $[t_1, t_2]$ (This is Lemma 1.2 in [7]). Since $\varphi_k(t)$ is uniformly bounded there exists a $\bar{K}$ such that $|f(t,\varphi_k(t)|\leqslant\bar{K}$ for $t\in[t_1, t_2]$ by the hypotheses on f. Therefore,

$$\int_{t_1}^{t_2}|\dot\varphi_k(t)-f(t,\varphi_k(t))|\,dt\geqslant\int_{t_1}^{t_2}|\dot\varphi_k(t)|\,dt-\int_{t_1}^{t_2}|f(t,\dot\varphi_k(t)|\,dt\geqslant|x_2-x_1|-\bar{K}(t_2-t_1)\,.$$

Prooceding as in Lemma 1.2 of [7], we obtain (3.2). Similarly, Lemmas 1.3 and 1.4 in [7] hold and thus

$$v(p_1,p_3)\leqslant v(p_1,p_2)+v(p_2,p_3) \tag{3.3}$$

and

$$|v(p_1,p_2)-v(p_1,p_3)|\leqslant|x_2-x_3|+K_1(t_3-t_2) \tag{3.4}$$

for any points $p_1, p_2, p_3\in R^d$ and $t_1\leqslant t_2\leqslant t_3$, where K_1 depends on the interval $[t_1, t_3]$ as well as on a bound on a minimizing sequence $\{\varphi_k\}$ defined in (3.1) for $v(p_1, p_3)$.

For $p_1=(0, 0)$ and $p_2=(t, x)$, then $V(t,x)=v(p_1,p_2)$. From (3.4),

$$|V(t,x)-V(t,x')|\leqslant|x-x'|\,; \tag{3.5}$$

hence V satisfies (*b*). If $x(t)$ is a solution of (E), then from (3.3) it follows that V is non-increasing along solutions since $v(p_2,p_3)=0$ for $p_2=(t,x(t))$ and $p_3=(t+h, x(t+h))$ for $h>0$. Hence $\dot V\leqslant 0$, thus satisfying (*c*).

Let $\bar{x}$ be a solution of (E) satisfying $\bar{x}(0)=0$. Since $|\bar{x}(t)|\leqslant C$ for some $C>0$ and for all $t\geqslant 0$, then from (3.5),

$$|V(t,x)-V(t,\bar{x}(t))|\leqslant|x-\bar{x}(t)|\leqslant|x|+C\,.$$

Since $V(t,\bar{x}(t))\equiv 0$ for all $t\geqslant 0$, then

$$|V(t,x)|\leqslant|x|+C\,,$$

thus satisfying the right hand inequality in (*a*). It thus remains to prove the left hand inequality of (*a*). Assume the inequality is not true. Then there exists some number $K_1>0$, a sequence of points $\{x_k\}$ with $|x_k|\to\infty$, as $k\to\infty$, and a sequence of points $\{t_k\}$ such that

$$V(t_k, x_k) \leq K_2 .$$

For each $k > 0$, there exists $\varphi_k \in A(t_k, x_k)$ satisfying

$$\int_0^{t_k} |\varphi_k(t) - f(t, \varphi_k(t)| \, dt < K_2 + 1 .$$

Define

$$g_k(t) = \begin{cases} \dot{\varphi}_k(t) - f(t, \varphi_k(t)) & 0 \leq t \leq t_k \\ 0 & t > t_k . \end{cases}$$

Let $x_k(t)$ for $t \in [0, \infty)$ be a solution of

$$\dot{x}(t) = f(t, x) + g_k(t)$$

such that $x_k(t) = \varphi_k(t)$ for $t \in [0, t_k]$. Since $\int_0^\infty |g_k(t)| \, dt < K_2 + 1$, we obtain from the integral boundedness of (E) that $|x_k(t)| \leq \beta(0, K_2 + 1)$ for all $t \geq 0$ and for all k. In particular, $|x_k(t_k)| = |\varphi_k(t_k)| = |x_k| \leq \beta(0, K_2 + 1)$. This is a contradiction, thus concluding the proof of Theorem 1.

PROOF OF THEOREM 2. – Assume V satisfies (b), (c), and (d). Then as in the proof of Theorem 1,

$$\begin{aligned} V(t, x_p(t, t_0, x_0)) &\leq V(t_0, x_0) + K\eta \\ &\leq V_\alpha(t_0) + K\eta \end{aligned} \tag{3.6}$$

for all $|x_0| < \alpha$ where $V_\alpha(t_0) = \sup\limits_{|x_0| \leq \alpha} V(t_0, x_0)$. Observing that the right hand side of (3.6) depends on t_0, α, and η and that V satisfies (d), then it follows for each $\Gamma > 0$ that there exists $\beta = \beta(t_0, \alpha, \eta, T)$ such that $|x_p(t, t_0, x_0)| < \beta$ for $t \in [t, t_0 + T]$. This proves solutions are integrally extendable.

Conversely, if solutions are integrally extendable, then we may use the same Liapunov function as in Theorem 1. The proof that V satisfies (b), (c), and (d) is essentially the same one used in Theorem 1 and we omit the details.

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