Oscillation Criteria for Second Order Nonlinear Differential Equations. (*)

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Summary. — *Various sufficient conditions are obtained which guarantee that all continuable solutions of (1.1) $y'' + q(t)y' + p(t)f(y) = 0$ are oscillatory. No explicit sign assumptions are made on $p(t)$ although certain integral conditions are assumed to hold with regard to $f(y)$, $p(t)$ and $q(t)$. Examples are given of the form $p(t) = \lambda/t^\mu + (\beta \sin t)/t^\alpha$, $\lambda, \beta, \mu, \alpha > 0$.*

1. — Introduction.

The differential equations to be considered in this paper are of the form

$$(1.1) \quad y'' + q(t)y' + p(t)f(y) = 0$$

or more generally

$$(1.2) \quad (ry')' + p(t)f(y) = 0,$$

where $p(t)$, $q(t)$, $r(t) \in C[0, +\infty)$, $r(t) > 0$ and $f(y) \in C(-\infty, +\infty) \cap C^{(1)}(-\infty, 0) \cap C^{(1)}(0, +\infty)$ with

$$(1.3) \quad yf(y) > 0, \quad f'(y) > 0, \quad y \neq 0.$$

Although equation (1.1) may be transformed into an equation of the form (1.2), the change of variables used makes it more difficult to determine the effect of the separate behaviour of $p(t)$ and $q(t)$; hence, both (1.1) and (1.2) will be considered here.

We shall be primarily concerned with the oscillatory and non-oscillatory behaviour of solutions of (1.1) and (1.2); in particular, when no explicit sign assumption is made on $p(t)$. By definition, a solution of (1.1) or (1.2) which exists on some ray $[t_0 + \infty]$ is said to be oscillatory or nonoscillatory according as it does or does not have arbitrarily large zeros. An important special case of (1.1) is

$$(1.4) \quad y'' + p(t)y^\gamma = 0$$

where γ is the quotient of odd positive integers. Equation (1.4) with the assumption

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$p(t) \geq 0$ has been subject of considerable research. (See [18] for an extensive bibliography for (1.4) and its generalizations). For the sublinear case, $0 < \gamma < 1$, solutions of (1.4) exist on all of $[0, +\infty)$ (cf. [10], p. 30). For the case $\gamma > 1$, $p(t) \geq 0$ and locally of bounded variation, solutions of (1.4) exist on $[0, +\infty)$ also, (See [7]). However, if $p(t)$ becomes negative, there will always exist solutions of (1.4) with a finite maximal interval of existence. (See [12] and [6]). There will, however, exist nontrivial extendible solutions even if $p(t)$ becomes negative, provided certain other restrictions on $p(t)$ are satisfied ([12]). Therefore, we shall restrict attention to solutions of (1.1) and (1.2) which exist on some ray $[t_0, +\infty)$. Our results include those in [3] and extend those in [8], [11], and [13] and are particularly useful for the case when $p(t) = \lambda/t^\mu + (\beta \sin t)/t^\alpha$, $\alpha, \beta, \mu, \lambda$ positive. WALTMAN [16] and BHATIA [2] have also given oscillation criteria for (1.4) and (1.2) with no sign assumption on $p(t)$.

It will also be evident that the results obtained may be extended to a slightly larger class by techniques similar to those in [14]. Results for (1.1) with a nonlinear damping term were obtained by BOBISUD in [4] and [5]. (Note however that Theorem 2 of [4] is incorrect; see [19]).

2. — In this section we shall consider equation (1.1), and determine the effect of positive and negative damping. We begin with two lemmas which give sufficient conditions in order that bounded non-oscillatory solutions of (1.1) are eventually monotone.

LEMMA 2.1. — Let $q \in C^{(1)} [0, +\infty)$ with $q \geq 0$, $q' \leq 0$ for all large t . Assume also that for any $\lambda > 0$ there is a $T(\lambda) \geq 0$ with

$$(H_1) \quad \liminf_{t \rightarrow \infty} \int_T^t p(s) ds \geq \lambda q(T), \quad \text{for all } T \geq T(\lambda).$$

Finally, assume

$$(2.1) \quad \int_{\pm 0}^{\pm 1} \frac{du}{f(u)} < +\infty.$$

Then, if $y(t)$ is any solution of (1.1) with $0 < |y(t)| \leq M$ for $t \geq T_0$ and some $M > 0$, it follows that there exists $T_1 \geq T_0$ with $y(t)y'(t) > 0$, $t \geq T_1$.

PROOF. — By (1.3), we may assume without loss of generality, that $y(t) > 0$, $t \geq T_0$. Multiplication of (1.1) by $1/f(y(t))$ and an integration by parts for $t \geq T \geq T_0$ produces

$$(2.2) \quad \begin{aligned} \frac{y'(t)}{f(y(t))} + \int_T^t \frac{(y')^2 f'(y(s))}{f(y(s))^2} ds + \int_T^t \frac{yy'}{f(y(s))} ds + \int_T^t p ds \\ = \frac{y'(T)}{f(y(T))}. \end{aligned}$$

Evaluating the second integral in (2.2) we have

$$(2.3) \quad \int_T^t \frac{qy'}{f(y(s))} ds = q(t) \int_{y(T)}^{y(t)} \frac{du}{f(u)} - \int_T^t q' \int_{y(T)}^{y(s)} \frac{du}{f(u)} ds \\ \geq - \int_0^M \frac{du}{f(u)} q(T)$$

since $q(t)$ is positive and decreasing. Hence, using (H_1) and (2.3) in (2.2) we obtain

$$(2.4) \quad \frac{y'(t)}{f(y(t))} < \frac{y'(T)}{f(y(T))}$$

for $t > T$ sufficiently large. Now if $y'(T) < 0$ for some large T , then $y'(t) < 0$ for all $t \geq T_1 \geq T$ so that

$$(2.5) \quad \frac{y'(t)}{f(y(t))} < \frac{y'(T_1)}{f(y(T_1))} < 0$$

for all $t \geq T_2 \geq T_1$. But then integrating (2.5) we obtain

$$(2.6) \quad \int_{T_2}^t \frac{y'}{f(y(s))} ds = \int_{y(T_2)}^{y(t)} \frac{du}{f(u)} < \frac{y'(T_1)}{f(y(T_1))} (t - T_2)$$

and the right hand side tends to $-\infty$, contradicting (2.1).

This proves the Lemma.

LEMMA 2.2. — Let $q \in C^{(1)} [0, +\infty)$ with $q < 0$, $q' \geq 0$ for all large t . Assume, in addition that for any $\lambda > 0$ there is a $T(\lambda) \geq 0$ with

$$(H_2) \quad \liminf_{t \rightarrow \infty} \int_T^t p(s) ds \geq \lambda |q(T)|, \quad \text{for all } T \geq T(\lambda).$$

Then if $y(t)$ is any solution of (1.1) with $0 < |y(t)| \leq M$ for $t \geq T_0$ and some $M > 0$, it follows that there exists $T_1 \geq T$ with $y(t)y'(t) > 0$, $t \geq T_1$.

PROOF. — As in Lemma 2.1, we obtain (2.2) and instead of (2.3) we obtain the estimate:

$$(2.7) \quad \int_T^t \frac{qy'}{f(y(s))} ds \geq \left(\int_{y(T)}^M \frac{du}{f(u)} \right) q(T).$$

Hence, as in Lemma 2.1, it follows that the same inequality (2.4) holds for all large t . Therefore, if $y'(T) \leq 0$ for some large T , then $y'(t) < 0$ for all $t \geq T_1 \geq T$. Moreover, using (H_2) and an elementary argument, we may assume T_1 is such that $\int_{T_1}^t p(s) ds \geq 0$ for all $t \geq T_1$.

Then we have

$$(2.8) \quad \int_{T_1}^t p(s) f(y(s)) ds = f(y(t)) \int_{T_1}^t p(s) ds - \int_{T_1}^t f'(y(s)) y'(s) \int_{T_1}^s p(\sigma) d\sigma ds \geq 0, \quad t \geq T_1.$$

Therefore, integrating (1.1), applying (2.8), and using the fact that $q(t)y'(t) \geq 0$, $t \geq T_1$, we obtain

$$(2.9) \quad y'(t) \leq y'(T_1) < 0, \quad t > T_1$$

which is a contradiction. This proves the Lemma.

We wish to remark that assumption (H_1) or (H_2) implies that $\int_t^\infty p(s) ds$ exists in the extended reals. Hence, either $\int_t^\infty p(s) ds = +\infty$ in which case (H_1) and (or) (H_2) hold trivially, or $\lambda|q(t)| \leq \int_t^\infty p(s) ds < +\infty$ holds for $t \geq T(\lambda)$ (i.e., $q(t) = o(\int_t^\infty p(s) ds)$ as $t \rightarrow \infty$).

THEOREM 2.3. – Let the hypotheses of Lemma (2.1) hold and assume, in addition, that

$$(2.10) \quad \limsup_{t \rightarrow \infty} \int_t^\infty sp(s) ds = +\infty.$$

Then all bounded solutions of (1.1) are oscillatory.

PROOF. – If not, let $y(t)$ be a bounded non-oscillatory solution which we may assume satisfies $0 < y(t) \leq M$, $y'(t) > 0$, $t \geq T$. Multiplying (1.1) by $t/f(y(t))$ and integrating by parts we have

$$(2.11) \quad \frac{ty'(t)}{f(y(t))} + \int_T^t \frac{(sq(s) - 1)y'}{f(y(s))} ds + \int_T^t sp(s) ds < \frac{Ty'(T)}{f(y(T))}.$$

Since $q \geq 0$ for $t \geq T$, the first integral in (2.11) is bounded below by $-\int_{y(T)}^M du/f(u)$, so that (2.10) implies $(ty'(t))/f(y(t)) < 0$ at some points t sufficiently large. This is a contradiction and prove the Theorem.

For the case of negative damping, we need a slightly stronger assumption.

THEOREM 2.4. – Let the hypotheses of Lemma 2.2 hold and assume, in addition, that

$$(2.12) \quad \limsup_{t \rightarrow \infty} \int_x^t (sp(s) + \lambda q(s)) ds = +\infty$$

for any $\lambda > 0$. Then all bounded solutions of (1.1) are oscillatory.

PROOF. – If not, we proceed as in Theorem 2.3 to obtain (2.11). Then we have

$$(2.13) \quad \begin{aligned} \int_x^t \frac{(sq(s) - 1)y'}{f(y(s))} ds &\geq (tq(t) - 1) \int_{v(T)}^M \frac{du}{f(u)} - \int_x^t (sq(s))' \int_{v(T)}^{v(s)} \frac{du}{f(u)} \\ &\geq K \left(tq(t) - 1 - \int_x^t sq'(s) ds \right) \\ &= K \left(Tq(T) - 1 + \int_x^t q(s) ds \right) \end{aligned}$$

where $K = \int_{v(T)}^M du/f(u)$. Therefore, (2.13) and (2.12) along with (2.11) yield a contradiction, as in Theorem 2.3.

If in Lemma 2.1 and 2.2 we assume in addition, that

$$(2.14) \quad \int_{\pm 1}^{\pm \infty} \frac{du}{f(u)} < +\infty,$$

this it follows that the conclusion of Lemma 2.1 or 2.2 holds for all non-oscillatory solutions. Consequently, we have

COROLLARY 2.5. – In addition to the assumption of Theorem 2.3 or Theorem 2.4, let condition (2.14) hold. Then all solutions of (1.1) are oscillatory.

REMARK 2.6. – Corollary 2.5 extends to equation (1.1) the well-known result of ATKINSON [1] for equation (1.4) in which $p(t) \geq 0$ and $\gamma > 1$.

Also, Corollary 2.5 shows that Theorems 2 and 3 of BOBISUD [3] are extraneous in that the conclusion of Theorems 2 or 3 hold for Theorem 1 of [3] without any additional assumptions. Moreover, we see that the condition $\int_0^\infty tp(t) dt = +\infty$ of Theorem 1 of [3] may be replaced by $\limsup_{t \rightarrow \infty} \int_x^t sp(s) ds = +\infty$.

For the case when (1.1) is linear and $q \equiv 0$, we have

COROLLARY 2.7. – All solutions of

$$(2.15) \quad y'' + p(t)y = 0$$

are unbounded (and nonoscillatory) if $\limsup_{t \rightarrow \infty} \int_t^t sp(s) ds = +\infty$ and $0 < t \int_t^\infty p(s) ds < \frac{1}{4}$ for all large t .

PROOF. – Apply Lemma 2.2, Theorem 2.4, and a non-oscillation criterion due to MOORE ([15], Theorem 6).

The next Theorem describes the solutions of (1.1) under slightly different hypotheses.

THEOREM 2.8. – Assume $\int_t^\infty tp(t) dt = +\infty$ and let $tq(t) \leq 1$, $(tq(t))' \geq 0$ for all large t . Then all bounded solutions of (1.1) are oscillatory or tend monotonically to zero.

In addition

- a) If $\int_{\pm 1}^{\pm \infty} du/f(u) < +\infty$, then the conclusion remains valid for all solutions of (1.1).
- b) If $\int_{0^\pm}^{\pm 1} du/f(u) < +\infty$, then all bounded solutions of (1.1) are oscillatory.
- c) If $\int_{0^\pm}^{\pm \infty} du/f(u) < +\infty$, then all solutions of (1.1) are oscillatory.

PROOF. – Suppose first that $y(t)$ is a bounded non-oscillatory solution of (1.1) with $0 < y(t) \leq M$, $t \geq T$. As in Theorem 2.3, we obtain (2.11). Then

$$\begin{aligned} \int_T^t \frac{(sq(s)-1)y'}{f(y(s))} ds &= (tq(t)-1) \int_{y(T)}^{y(t)} \frac{du}{f(u)} - \int_T^t (sq(s))' \left(\int_{y(T)}^{y(s)} \frac{du}{f(u)} \right) ds \\ &\geq K \left[tq(t) - 1 - \int_T^t (sq(s))' ds \right] \\ &= K(Tq(T) - 1), \end{aligned}$$

where $K = \int_{y(T)}^M du/f(u)$. Therefore,

$$(2.16) \quad \frac{ty'(t)}{f(y(t))} + K(Tq(T) - 1) + \int_T^t sp(s) ds < \frac{Ty'(T)}{f(y(T))}$$

and since $\int_T^t sp(s) ds \rightarrow \infty$, we obtain

$$(2.17) \quad \frac{y'(t)}{f(y(t))} < \frac{-k}{t}, \quad t \geq T_1 \geq T,$$

for some $k > 0$. Hence, an integration of (2.17) shows that $y(t) \rightarrow 0$. The proof of statements *a*), *b*), and *c*) of the Theorem are clear.

The next result shows that the conclusion of the first part of Theorem 2.8 may be strengthened provided $p(t)$ and $q(t)$ are restricted slightly.

COROLLARY 2.9. — Assume $\int_0^\infty t p(t) dt = +\infty$, $q(t) \leq 0$, and $(tq(t))' \geq 0$ for all large t . In addition, let there exist a sequence $T_n \rightarrow \infty$ such that

$$(2.18) \quad \int_{T_n}^t p(s) ds \geq 0, \quad t \geq T_n.$$

Then all bounded solutions of (1.1) are oscillatory and if $\int_{\pm 1}^{\pm \infty} du/f(u) < +\infty$, then all solutions of (1.1) are oscillatory.

PROOF. — If not, we may assume that $y(t)$ is a solution of (1.1) with $y(t) > 0$, $y'(t) < 0$, $t \geq T_n$, for some $n \geq 1$. Since $q(t)y'(t) \geq 0$, we integrate (1.1) to obtain

$$(2.19) \quad y'(t) - y'(T_n) + \int_{T_n}^t p(s) f(y(s)) ds \leq 0, \quad t \geq T_n.$$

As in the proof of Lemma 2.2, the integral in (2.19) is non-negative so $y'(t) \leq y'(T_n) < 0$, $t \geq T_n$, and this is a contradiction.

We now give a result complementary to Corollary 2.9 and which may be considered as an extension of the necessary and sufficient condition for the existence of bounded non-oscillatory solutions of

$$(2.20) \quad y'' + p(t) f(y) = 0$$

(See [18], for example). It is also related to a result of KIGURADZE ([12], Theorem 2). First we define the (extended) real-valued numbers

$$\begin{aligned} \varrho^+ &\equiv \int_0^\infty t p^+(t) dt \\ \varrho^- &\equiv \int_0^\infty t p^-(t) dt \end{aligned}$$

where $p^+(t) = \max\{p(t), 0\}$, $p^-(t) = \max\{-p(t), 0\}$. Then we have

THEOREM 2.10. — Assume $\varrho^+ < +\infty$, $\varrho^- < +\infty$. Then (2.20) has a bounded non-oscillatory solution $y(t)$ with $\liminf_{t \rightarrow \infty} |y(t)| > 0$ iff $\varrho^- < +\infty$, $\varrho^+ < +\infty$.

PROOF. – If both ϱ^- and ϱ^+ are finite, then a slight modification of the standard successive approximation argument for solutions of the integral equation equivalent to (2.20) yields the result (See [18]). Conversely, assume $y(t)$ is a non-oscillatory solution with $y(t) > 0$, $t \geq T$, $\liminf_{t \rightarrow \infty} y(t) = m < y(t) < M$, $t \geq T$, and suppose to be specific, that $\varrho^+ < +\infty$. Let $\varphi(t) \equiv ty'(t) - y(t)$, $t \geq T$. Then

$$(2.21) \quad \varphi'(t) = -tp(t)f(y(t)) \geq -tp^+(t)f(M) + tp^-(t)f(m).$$

Now if $\varrho^- = +\infty$, then integrating (2.21) we have

$$\varphi(t) \geq k + f(m) \int_T^t sp^-(s) ds, \quad t \geq T$$

for some constant $k > 0$. Hence, $ty'(t) > 1$ for all large t so that $y(t) \rightarrow \infty$, a contradiction. A similar argument holds if $\varrho^- < +\infty$.

The next Theorem is based on a linear comparison technique and avoids the integrability assumption (2.14).

THEOREM 2.11. – Assume there exists $\lambda > 0$ and $K > 0$ such that

$$(2.22) \quad f'(y) \geq \frac{f(y)}{y} \geq \lambda, \quad |y| \geq K$$

and such that the equation

$$(2.23) \quad y'' + \lambda py = 0$$

is oscillatory. Assume further that $q(t) \geq 0$,

$$(2.24) \quad \int_0^\infty \frac{ds}{r(s)} = +\infty,$$

$$(2.25) \quad \liminf_{t \rightarrow \infty} \int_T^t r(s)p(s) ds \geq 0, \quad \text{for all large } T,$$

and

$$(2.26) \quad \limsup_{t \rightarrow \infty} \left(\int_0^t sr(s)p(s) ds - \lambda r(t) \right) = +\infty, \quad \text{for all } \lambda > 0,$$

where $r(t) = \exp \left(\int_0^t q(s) ds \right)$.

Then all solutions of (1.1) are oscillatory.

PROOF. — If not, let $y(t)$ be a non-oscillatory solution of (1.1) with $y(t) > 0$, $t \geq T$. Rewrite (1.1) in the form

$$(2.27) \quad (ry')' + rpf(y) = 0, \quad r = \exp \left(\int_x^t q(s) ds \right).$$

Then (2.27) yields

$$(2.28) \quad \frac{r(t)y'(t)}{f(y(t))} + \int_x^t rps ds \leq \frac{r(T)y'(T)}{f(y(T))}, \quad t \geq T.$$

Hence, if $y'(T) \leq 0$ for some large T , then by (2.25) we have that $y'(t) < 0$ for $t \geq T_1 \geq T$.

(The case $y'(t) \equiv 0$ for all large t implies $p \equiv 0$ for all large t so that (2.23) is non-oscillatory by the STURM comparison Theorem; hence, this case does not occur).

By (2.25) we may assume that $\int_{T_1}^t r(s)p(s)ds \geq 0$, $t \geq T_1$, so that

$$(2.29) \quad \int_{T_1}^t r(s)p(s)f(y(s))ds \geq 0, \quad t \geq T_1,$$

as in (2.8). Hence, integrating (2.27) we obtain by (2.29)

$$y'(t) \leq \frac{r(T_1)y'(T_1)}{r(t)} < 0, \quad t \geq T_1$$

so that $y(t) \rightarrow -\infty$ by (2.24).

Therefore, we may assume that $y'(t) > 0$, $t \geq T$.

Assume next that $y(t) \leq K$ for all large t . Multiply (2.27) by $t/f(y(t))$, $t \geq T$, and integrate by parts to obtain

$$(2.30) \quad \frac{tr(t)y'(t)}{f(y(t))} + \int_x^t sr(s)p(s)ds \leq \frac{Tr(T)y'(T)}{f(y(T))} + \int_x^t \frac{r(s)y'(s)}{f(y(s))}.$$

Then

$$\begin{aligned} \int_x^t \frac{r(s)y'(s)}{f(y(s))} ds &\leq r(t) \int_{y(T)}^K \frac{du}{f(u)} - \int_x^t q(s)r(s) \int_{y(T)}^{y(s)} \frac{du}{f(u)} ds \\ &\leq r(t) \int_{y(T)}^K \frac{du}{f(u)}. \end{aligned}$$

Using (2.26) we conclude $(tr(t)y'(t))/f(y(t)) < 0$ for some large t .

This contradiction shows we may assume $y(t) > K$ for $t \geq T$. Let $z(t)$ denote the solution of (2.23) with $z(T) = 0$, $z'(T) = 1$ and let $t_0 > T$ be such that $z'(t_0) = 0$ with $z' > 0$ on $[T, t_0]$.

Then

$$\begin{aligned} 0 &\leq \int_T^{t_0} \left(\frac{f(y)}{y} - \lambda \right) (z')^2 dt = \lambda \int_T^{t_0} p z^2 \left(\frac{f(y)}{y} - \lambda \right) dt \\ &\quad - \int_T^{t_0} z z' \frac{y'}{y} \left(f'(y) - \frac{f(y)}{y} \right) dt \\ &\leq \lambda \int_T^{t_0} p z^2 \left(\frac{f(y)}{y} - \lambda \right) dt \end{aligned}$$

by (2.22). Since $q(t) > 0$, $y'(t) > 0$, $t \geq T$, we have $y'' = p(t)f(y) < 0$, $t \geq T$ so that

$$\begin{aligned} \int_T^{t_0} p z^2 \left(\frac{f(y)}{y} - \lambda \right) dt &= \int_T^{t_0} \frac{z}{y} (z p f(y) - \lambda p z y) dt \\ &\leq \int_T^{t_0} \frac{z}{y} (z' y - y' z)' dt \\ &= \frac{-y'(t_0)}{y(t_0)} z(t_0)^2 - \int_T^{t_0} \left(\frac{z' y - y' z}{y} \right)^2 dt \\ &< 0, \end{aligned}$$

and this contradiction proves the Theorem.

3. - Remarks and examples.

Theorem 2.11 is a complementary result to Theorem 4 of [8] in that oscillation criteria for (1.1) are thereby extended to the case when (2.23) is conditionally oscillatory (i.e., oscillatory for large $\lambda > 0$ and non-oscillatory for $\lambda > 0$ and sufficiently small). In addition, there is an analogue of Theorem 2.11 which shows that all solutions of (1.1) are oscillatory in the case when the assumptions of Theorem 2.3 hold, condition (2.10), and (2.22) hold, and (2.23) is oscillatory. The analogue of Theorem (2.11) for equation (1.2) is left to the reader to formulate.

We now give several examples which illustrate a few of the results obtained:

EXAMPLE 3.1. - Let $q(t) \equiv 0$, $f(y) = y'$, $\gamma > 0$ the quotient of odd integers, and $p(t) = \lambda/t^2 + (\beta \sin t)/t$, $\lambda > 0$, β real. Then

- a) If $\gamma > 1$ all solutions of (1.1) are oscillatory by Corollary 2.9.
- b) If $\gamma = 1$, all solutions of (1.1) are non-oscillatory, provided $\lambda < \frac{1}{4} - \beta^2/2$, (see WILLET [17]), and hence are unbounded.
- c) If $0 < \gamma < 1$ all bounded solutions of (1.1) are oscillatory by Corollary 2.9.

EXAMPLE 3.2. - Let $q(t) \equiv 0$, and $p(t) = \lambda/t^{1+\alpha} + (\beta \sin t)/t^\alpha$ where λ, β, α are all positive and satisfy $\lambda > \frac{1}{4} - \beta^2/2$, $0 < \alpha < 1$, and $\beta\alpha < \lambda < \beta(1-\alpha)$. By a result of WILLET [17] the equation

$$y'' + \left(\frac{\lambda}{t^2} + \frac{\beta \sin t}{t} \right) y = 0$$

is oscillatory if $\lambda > \frac{1}{4} - \beta^2/2$. Hence by Corollary 2.2 of [9], the equation

$$y'' + t^{1-\alpha} \left(\frac{\lambda}{t^2} + \frac{\beta \sin t}{t} \right) y = 0$$

is oscillatory for $\alpha > 1$ and $\lambda > \frac{1}{4} - \beta^2/2$.

The fact that $\beta\alpha < \lambda$ implies

$$\int_t^\infty p(t) dt \geq 0, \quad \text{for all large } t.$$

In addition, we have, since $\lambda < \beta(1-\alpha)$,

$$-\infty = \liminf_{t \rightarrow \infty} \int_t^t sp(s) ds < \limsup_{t \rightarrow \infty} \int_t^t sp(s) ds = +\infty.$$

Therefore, if $f(y) = y'(\log(1+|y|))^\delta$ we have

- a) $0 < \gamma < 1$, $\delta = 0$ implies all bounded solutions of (1.1) are oscillatory by Theorem 2.3.
- b) $\gamma \geq 1$, $\delta \geq 0$ implies all solutions of (1.1) are oscillatory by Theorem 2.11.

Finally, we wish to remark that the oscillatory nature of the examples cited above may not be concluded from the results of any of the references.

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