Specialization of a Generic Hyperplane Section through a Rational Point of an Algebraic Variety.

WEI-EIHN KUAN (East Lansing, Michigan) (U.S.A. (*))

Summary. — Let V/k be an irreducible algebraic variety of dimension ≥ 3 , defined over a field k of characteristic 0, passing through the origin (O) of the ambient affine n -space. Let $\mathfrak{p}$ be the prime ideal of V in the polynomial ring $k[X_1, \dots, X_n]$. If V/k is k -normal at (O) , then for almost all hyperplanes $H_a: a_1X_1 + \dots + a_nX_n = 0$ with $a_1, \dots, a_n \in k$ the ideal $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ is a prime ideal. If the generic hyperplane section of V through (O) is normal over the ground field of the generic section of V , then the section $V \cap H_a$ is k -normal at (O) for almost all H_a .

1. — Introduction.

Let V/k be an irreducible affine algebraic variety defined over a field k passing through a rational point, which we assume to be the origin, (0) , of the ambient affine space, A^n . Let $\mathfrak{p}$ be the prime ideal of V in the polynomial ring $k[X] = k[X_1, \dots, X_n]$. Let H_u be a generic hyperplane with the defining polynomial $u_1X_1 + \dots + u_nX_n$ in the polynomial ring $k(u)[X] = k(u_1, \dots, u_n)[X_1, \dots, X_n]$, where $u_1, \dots, u_n$ are indeterminates over $k(V)$, the function field of V over k . According to [2; Theorem 1, p. 397], if V is of dimension $r \geq 2$ then $(\mathfrak{p}, u_1X_1 + \dots + u_nX_n)k(u_1, \dots, u_n)[X_1, \dots, X_n] = \mathfrak{p}_u \cap \mathfrak{q}$, where $\mathfrak{p}_u$ is a prime ideal and $\mathfrak{q}$ is a primary ideal with $(X_1, \dots, X_n) \cdot k(u_1, \dots, u_n)[X_1, \dots, X_n]$ as the radical. Let $v \cap H_u$ be the variety of p_u : We have shown in [2], among other results, the following, (1) if V/k is an absolutely irreducible variety of dimension $r \geq 3$ defined over an infinite field k , then $V \cap H_u/k(u)$ is an absolutely irreducible variety [2, Theorem 3, p. 399]; (2) if V/k is normal at (0) , and if the local ring $k[V]_{(0)}$ is a Cohen-Macaulay ring, then $V \cap H_u/k(u)$ is normal at (0) , [2; Theorem 8, p. 404].

Specializing $(u_1, \dots, u_n)$ to $(a_1, \dots, a_n)$, where $a_1, \dots, a_n \in k$, we prove in this paper that if k is of characteristic 0, then the above results hold for almost all hyperplane sections of V , $V \cap H_a$, where H_a is the hyperplane defined by $a_1X_1 + \dots + a_nX_n = 0$, with $a_1, \dots, a_n \in k$. Moreover $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ is prime almost always even if the condition of absolute irreducibility on V/k is removed. In the process of proof, the following result is crucial. If k is of characteristic 0, then for almost all hyperplanes $a_1X_1 + \dots + a_nX_n = 0$, with $a_1, \dots, a_n \in k$, it is true that any singular zero of the ideal $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ is also a singular zero for the ideal $\mathfrak{p}$. However,

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this result turned out to be false when k is of characteristic $p > 0$. A counter example by taking $\mathfrak{p} = (X_2 + X_1^{p+1})$ in $k[X_1, X_2]$ is communicated to us by Professor A. SEIDENBERG. I would like to thank him for doing so.

2. – Notation and terminology.

Throughout this paper, all fields are of characteristic 0, unless otherwise stated. Let V/k be an irreducible r algebraic variety of dimension defined over a field k in an affine n -space A^n over k passing through the origin (0) of A^n , i.e. V is a subset of A^n consisting of all zeros of a finite set of polynomials $f_1, \dots, f_s$ in $k[X] = k[X_1, \dots, X_n]$ generating a prime ideal $\mathfrak{p}$ contained in the maximal ideal $(X_1, \dots, X_n) \cdot k[X_1, \dots, X_n]$. Let $(\xi) = (\xi_1, \dots, \xi_n)$ be a generic point of V over k , let $Q = (q_1, \dots, q_n)$ be a point on V with $\mathfrak{q}$ as its prime ideal in the coordinate ring $k[\xi] = k[\xi_1, \dots, \xi_n]$. Let $k[\xi]_{\mathfrak{q}}$ be the local ring of V at Q in the function field $k(\xi)$, V is k -normal at Q if the local ring $k[\xi]_{\mathfrak{q}}$ is integrally closed in $k(\xi)$. V is k -simple at Q if the local ring $k[\xi]_{\mathfrak{q}}$ is regular or equivalently (over perfect fields) the rank of the Jacobian $(\partial f_i / \partial X_i)_{\mathfrak{q}}$, $(\partial f_i / \partial X_i)$ evaluated at Q , is $n - r$. Q is singular on V if V is not k -simple at Q . Let $u_1, \dots, u_n$ be n indeterminates over $k(\xi)$; the hyperplane H_u defined by $u_1 X_1 + \dots + u_n X_n = 0$ is then called a generic hyperplane through (0). As we pointed out in the introduction, we denote the variety of $\mathfrak{p}_u$ by $V \cap H_u$.

Let E be a property which can be asserted or denied for each point $(a_1, \dots, a_n)$ with $a_1, \dots, a_n \in k$ either of the affine n -space A^n over k or the projective $(n-1)$ -space P^{n-1} over k . We say that E holds for almost all points of A^n or P^{n-1} if it holds for all point $(a_1, \dots, a_n)$ with $a_1, \dots, a_n \in k$, except perhaps those lying on a proper algebraic variety of A^n or of P^{n-1} .

3. – Singularities of $(\mathfrak{p}, a_1 X_1 + \dots + a_n X_n)$.

We start with the following well known lemma, see for instance [7; Theorem 7.4, p. 19].

LEMMA 1. – Let $F_1, \dots, F_t$ be t nonzero polynomials in $k[X_1, \dots, X_n]$. Let $S \subset k$ be an infinite subset of k . Then there exist $x_1, \dots, x_n \in S$ such that $F_i(x_1, \dots, x_n) \neq 0$ for $i = 1, 2, \dots, t$.

[9; Bertini's theorem, p. 138] stated: A variety V_{r-1} which varies in a linear system on a variety V_r cannot have variable singular points outside the singular locus of V_r and outside the base locus of the linear system. With regard to this theorem, we prove the following lemma.

LEMMA 2. – Let V/k be an irreducible variety of dimension r defined over a field k with $\mathfrak{p}$ as the prime ideal in $k[X]$. If (0) is a simple zero for $\mathfrak{p}$, then for almost

all hyperplanes $a_1X_1 + \dots + a_nX_n = 0$, $a_1, \dots, a_n \in k$, (0) is a simple zero for $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$.

PROOF. — Let $f_1, \dots, f_s$ be a set of generators of $\mathfrak{p}$, $\text{rank}(\partial f_i / \partial x_j)_0 = n - r$. If (0) were a singular zero of $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ for all $(a_1, \dots, a_n) \in k^n$, then $\text{rank}(\partial f_i / \partial X_j, a_j)_{(0)} < n - r + 1$ for all $(a_1, \dots, a_n) \in k^n$. Thus the determinant of every $(n - r + 1)$ -square submatrix of $(\partial f_i / \partial X_j, u_j)_{(0)}$ is identically zero. Thus $\text{rank}(\partial f_i / \partial x_j, u_j)_{(0)} < n - r + 1$, and nontrivial linear relations among u_i 's are therefore obtained. This contradicts to that $u_1, \dots, u_n$ are indeterminants. So

$$\left\{ (a_1, \dots, a_n) \mid \text{rank} \left(\frac{\partial f_i}{\partial X_j}, a_j \right)_{(0)} < n - r + 1 \right\}$$

is contained in a proper closed subset of A^n .

THEOREM 1. — Let V/k be an irreducible variety of dimension r with $\mathfrak{p}$ as its prime ideal in $k[X]$. Then for almost all hyperplanes $a_1X_1 + \dots + a_nX_n = 0$, $a_1, \dots, a_n \in k$, it is true that any singular zero of the ideal $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ is also singular for the ideal $\mathfrak{p}$.

We present here two proofs to this theorem.

PROOF. 1. — Follows from [9; Bertini's theorem, § 5, p. 138] and lemma 2.

The second proof itself is of some interest. We proceed with some preliminaries as follows.

Let V/k be of dimension r . Let $P = (p_1, \dots, p_n)$ be a point on V . Let $\partial f_i / \partial p_j$ be $\partial f_i / \partial x_j$ evaluated at P . Let $T(P, V)$ be the linear variety defined by

$$(X_1 - p_1) \partial f_i / \partial p_1 + \dots + (X_n - p_n) \partial f_i / \partial p_n = 0, \quad i = 1, 2, \dots, s,$$

over $k(P)$, P is k -simple if and only if $T(P, V)$ is of dimension r . A hyperplane $a_0 + a_1X_1 + \dots + a_nX_n = 0$, $a_0, a_1, \dots, a_n \in k$, going through P and containing $T(P, V)$ is called a tangent hyperplane to V at P . Two observations follow from the definition:

O_1 : $a_0 + a_1X_1 + \dots + a_nX_n = 0$ defines a tangent hyperplane to V at P if and only if $(\mathfrak{p}, a_0 + a_1X_1 + \dots + a_nX_n)$ has P as a singular zero, and

O_2 : $a_0 + a_1X_1 + \dots + a_nX_n = 0$ is a tangent hyperplane to V at P if and only if there exist $c_1, \dots, c_s \in k$, such that $c_j = c_1 \partial f_1 / \partial p_j + \dots + c_s \partial f_s / \partial p_j$ for $j = 1, 2, \dots, n$ and $-a_0 = \sum_{i=1}^n c_i \left(\sum_{j=1}^s p_j \partial f_i / \partial p_j \right)$. It follows from O_2 easily the following corollary.

COROLLARY. — If V is of dimension $r \geq 1$ and k -simple at (0) then the generic hyperplane defined by $u_1X_1 + \dots + u_nX_n = 0$ going through (0) is not a tangent hyperplane to V at (0).

It also follows from our observations the following theorem.

THEOREM 2. — Let V/k be a variety of dimension r defined over a field k passing through (0). Let (ξ) be a generic point of V over k . A necessary and sufficient condition for all the tangent hyperplanes to V at (ξ) go through (0) is that V is a cone with vertex at (0).

PROOF. — Let $\mathfrak{p} = (f_1, \dots, f_s)$ be the prime ideal of V in $k[X]$. If all the tangent hyperplanes to V at (ξ) go through (0), then $\xi_1 \partial \xi_1 + \dots + \xi_n \partial f_i / \partial \xi_n = 0$ for $i = 1, 2, \dots, s$. Thus $X_1 \partial f_i / \partial X_1 + \dots + X_n \partial f_i / \partial X_n \in \mathfrak{p}$, for $i = 1, 2, \dots, s$. Let $f_{i,j}$ be the homogeneous part of f_i of degree j , $1 \leq j \leq d_i$, where $d_i = \deg f_i$. We want to show each $f_{i,j}$ is in $\mathfrak{p}$. By Euler's identity $X_1 \partial f_{i,j} / \partial X_1 + \dots + X_n \partial f_{i,j} / \partial X_n = j f_{i,j}$ for $i = 1, 2, \dots, s$ and $1 \leq j \leq d_i$. Hence

$$\sum_{j=0}^{d_i} (X_1 \partial f_{i,j} / \partial X_1 + \dots + X_n \partial f_{i,j} / \partial X_n) = \sum_{j=0}^{d_i} j f_{i,j}, \quad \text{for } i = 1, 2, \dots, s.$$

Let $F_i = \sum_{j=0}^{d_i} j f_{i,j}$, since

$$\sum_{j=0}^{d_i} (X_1 \partial f_{i,j} / \partial X_1 + \dots + X_n \partial f_{i,j} / \partial X_n) = X_1 \partial F_i / \partial X_1 + \dots + X_n \partial F_i / \partial X_n,$$

therefore, $F_i \in \mathfrak{p}$, for $i = 1, 2, \dots, s$. Repeating the above arguments on f_i to F_i , and continue on, we have that $f_{i,j}$, for each i and $1 \leq j \leq d_i$, are solutions of the system of linear equations $X_{i,1} + 2^m X_{i,2} + \dots + d_i^m X_{i,d_i} \equiv 0 \pmod{\mathfrak{p}}$, where $m = 1, 2, \dots, d_i$. Since the determinant of the matrix (j^m) , $m, j = 1, 2, \dots, d_i$, is distinct from zero, we have $f_{i,j} \in \mathfrak{p}$ for each i and $j = 1, 2, \dots, d_i$. Thus $\mathfrak{p}$ is homogeneous, and V is a cone with vertex at (0).

Conversely, if V is a cone with (0) as the vertex, then $\mathfrak{p}$ is homogeneous. So for each i , $i = 1, 2, \dots, s$, $f_{i,j} \in \mathfrak{p}$. Since $\sum_{i=1}^{d_i} X_1 \partial f_{i,j} / \partial X_i = j \cdot f_{i,j}$, we have $\sum_{i=1}^{d_i} X_i \partial f_i / \partial X_i \in \mathfrak{p}$, hence $\sum_{i=1}^{d_i} \xi_i \partial \xi_i / \partial \xi_i = 0$.

$$(X_1 - \xi_1) \partial f_i / \partial \xi_1 + \dots + (X_n - \xi_n) \partial f_i / \partial \xi_n = 0, \quad i = 1, 2, \dots, s,$$

passes through (0), the vertex. Let $a_0 + a_1 X_1 + \dots + a_n X_n = 0$ be a tangent hyperplane to V at (ξ) , we have $-a_0 = \sum_{i=1}^s c_i \left(\sum_{j=1}^{d_i} \xi_j \partial f_i / \partial \xi_j \right)$ for some $c_1, \dots, c_s \in k$. Hence $a_0 = 0$, i.e. every tangent hyperplane to V at (ξ) passes through (0).

NOTE. — The above theorem is false if k is of characteristic $p > 0$. For example, in A^2 , we take the curve $X_1^{p+1} + X_1 + X_2 = 0$ [5; § 4, p. 374].

Let $V \subset A^n$ be an affine variety of dimension r defined over a field k . Tangent

hyperplanes to V correspond to points in a projective n -space P^n , one may identify tangent hyperplane to V with a point in P^n .

[8; pp. 322.323] did show that if not all tangent hyperplanes to V pass through the origin, (0) , of A^n , then the set of all tangent hyperplanes of V form an irreducible variety V' of dimension at most equal to $n-1$ in P^n .

Now we give the second proof to theorem 1.

PROOF 2. — Assume firstly that V is not a cone with (0) as the vertex. If all the ideals $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ carries a singular zero which is a simple zero of $\mathfrak{p}$, then $a_1X_1 + \dots + a_nX_n = 0$ is a tangent hyperplane to V . By Theorem 2, not all tangent hyperplanes pass through (0) , whence $V' \not\subset v(X'_0 = 0)$, where $X'_0, \dots, X'_n$ is a coordinate system for P^n , and $\dim(V' \cap v(X'_0 = 0)) \leq n-2$. On the other hand, since $(0, a_1, \dots, a_n) \in V'$ for every $(a_1, \dots, a_n) \in k^n$, therefore $\dim(V' \cap v(X'_0 = 0)) = n-1$. This is a contradiction. Therefore not all $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ carry a singular zero which is a simple zero of $\mathfrak{p}$. For those $a_1X_1 + \dots + a_nX_n = 0$, $a_1, \dots, a_n \in k$, such that $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ carry a singular zero which is a simple zero of $\mathfrak{p}$, we have $(0, a_1, \dots, a_n) \in V'$. Theorem is thus proved for this case.

Now, assume that V is a cone with (0) as vertex, then $\mathfrak{p}$ is a homogeneous ideal in $k[X]$. Then by [6; theorem 1, p. 359] we have for almost all hyperplanes $a_1X_1 + \dots + a_nX_n = 0$, $a_1, \dots, a_n \in k$, it is true that any singular zero $P \neq (0)$ of the ideal $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ is also a singular zero of $\mathfrak{p}$, and the same follows from lemma when $P = (0)$.

NOTE. — Theorem 1 is false, when k is of characteristic $p > 0$. As a counterexample ⁽¹⁾, we consider the algebraic curve $X_2 + X_1^{p+1} = 0$. Working over an algebraically closed base field k of characteristic $p > 0$, one finds that for every a , $(X_2 + X_1^{p+1}, X_2 + aX_1)$ has a singular zero, whereas $X_2 + X_1^{p+1} = 0$ is free of singularities.

There are two corollaries to Theorem 1.

COROLLARY 1. — If V/k is of dimension r and if V is k -normal at (0) , then for almost all hyperplanes $a_1X_1 + \dots + a_nX_n = 0$, $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ is an intersection of prime ideals of dimension $r-1$.

PROOF. — By [2; Corollary 2 to Theorem 1, p. 398], $(\mathfrak{p}, H_u)$ is a prime ideal of dimension $r-1$. By theorem of KRULL [6; p. 366], $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ is unmixed of dimension $r-1$ almost always. Let $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$ be unmixed, and assume that there is a non-prime primary component $\mathfrak{q}$. Let (q) be a generic point of $\sqrt{\mathfrak{q}}$, the radical of $\mathfrak{q}$. Then (q) is a singular zero of $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$, hence it is singular for the ideal $\mathfrak{p}$. By [2; proposition 1, p. 394], $q \notin (X_1, \dots, X_n)$. Since V carries only finitely many number of singular subvarieties of dimension $r-1$, the above situation would not happen if $a_1X_1 + \dots + a_nX_n = 0$ contains none

⁽¹⁾ This example is due to Professor A. SEIDENBERG.

of them entirely and this requirement places algebraic inequalities on the a_i 's. So the corollary is proved.

Recall the definition that V/k is locally free of s -dimensional singularities at P if every s -dimensional subvariety of V containing P is k -simple on V .

COROLLARY 2. — Let V/k be a variety of dimension r , locally free of $(r-1)$ -dimensional singularities at (0) . Then for almost all the hyperplanes $a_1X_1 + \dots + a_nX_n = 0$, $a_1, \dots, a_n \in k$, the hyperplane section of V by $a_1X_1 + \dots + a_nX_n = 0$ is locally free of $(r-2)$ -dimensional singularities at (0) .

PROOF. — Follows immediately from Theorem 1.

4. — Irreducibility of the algebraic variety of $(\mathfrak{p}, a_1X_1 + \dots + a_nX_n)$.

Let R be an integral domain, let $S \subset R$ be a multiplicative system and $0 \notin S$. Let R_s be the ring of quotients of R relative to S . Let $\mathfrak{A}$ be an ideal of R and denote $\mathfrak{A} \cdot R_s$ by $\mathfrak{A}^e$. It is well known that every ideal $\mathfrak{B} \subset R_s$ is of the form $\mathfrak{A}^e$ for some ideal $\mathfrak{A}$ in R . For $\mathfrak{B} \subset R_s$, we denote $\mathfrak{B} \cap R$ by $\mathfrak{B}^e$. If $\mathfrak{B} \subset R_s$ has an irredundant primary decomposition $\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s$ with $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ as the associated prime ideals, we have an irredundant primary decomposition $\mathfrak{B}^e = \mathfrak{q}_1^e \cap \dots \cap \mathfrak{q}_s^e$ with $\mathfrak{p}_1^e, \dots, \mathfrak{p}_s^e$ as the associated prime ideals. These notations are needed for the following local version of a Theorem of KRULL [6; Theorem of Krull, p. 366].

THEOREM 3 (KRULL). — Let $R = k[X_1, \dots, X_n]$ be a polynomial ring in n indeterminates over an infinite field k of arbitrary characteristic, let $\mathfrak{p} \subset R$ be a prime ideal. Let $(\tau) = (\tau_1, \dots, \tau_m)$ be m indeterminates, let $R_\tau = k(\tau_1, \dots, \tau_m)[X_1, \dots, X_n]$ and let $\mathfrak{p} \cdot k(\tau)[X] = \mathfrak{p}_\tau$. Let $\mathfrak{A} \subset (R_\tau)_{\mathfrak{p}_\tau}$ be an ideal, and let

$$\overline{\mathfrak{A}} = \{g(a, X) | g(\tau, X) \in k[\tau, X] \cap \mathfrak{A}^e\} \cdot R_{\mathfrak{p}}, \quad \text{where } (a) = (a_1, \dots, a_m) \in k^m.$$

If $\mathfrak{A}$ is unmixed of dimension r then $\overline{\mathfrak{A}}$ is unmixed of dimension r almost always.

PROOF. — Let $\mathfrak{A} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s$ be an irredundant primary decomposition with $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ as the associated prime ideals. Let $\mathfrak{B} = \mathfrak{A}^e$, $\mathfrak{B} = \mathfrak{q}_1^e \cap \dots \cap \mathfrak{q}_s^e$ is an irredundant primary decomposition with $\mathfrak{p}_1^e, \dots, \mathfrak{p}_s^e$ as the associated prime ideals, and $\dim \mathfrak{p}_i^e = \dim \mathfrak{p}_i + \dim \mathfrak{p}_\tau$, for $i = 1, 2, \dots, s$. So if $\mathfrak{A}$ is unmixed of $\dim r$, then $\mathfrak{B}$ is unmixed of $\dim(r + r')$, where $r' = \dim \mathfrak{p}_\tau$. Let $\tilde{\mathfrak{B}} = \{g(a, x) | g(\tau, X) \in k[\tau, X] \cap \mathfrak{B}\}$. By [6; Theorem of Krull, p. 366], $\tilde{\mathfrak{B}}$ is unmixed of $\dim(r + r')$ almost always, it follows that $\tilde{\mathfrak{B}}^e$ is unmixed of $\dim r$ almost always. Hence $\overline{\mathfrak{A}} = \tilde{\mathfrak{B}}^e$ is unmixed of $\dim r$ almost always.

Recall that V/k is absolutely irreducible if k is algebraically closed in $k(\xi)$ and $k(\xi)$ is separable over k , where (ξ) is a generic point of V over k . A prime ideal $\mathfrak{A}$ in $k[X]$ is absolutely irreducible if $\mathfrak{A} \cdot \bar{k}[X]$ is a prime ideal, where $\bar{k}$ is the algebraic closure

of k . It is well known that the prime ideal of V/k in $k[X]$ is absolutely irreducible if and only if V/k is.

For the definition of ground form of an unmixed ideal, and its properties we refer the readers to [2; p. 395].

THEOREM 4. — If V/k is an absolutely irreducible variety of dimension $r \geq 3$, and if V is k -normal at (0) , then for almost all hyperplane $a_1 X_1 + \dots + a_n X_n = 0$, $a_1, \dots, a_n \in k$, the ideal $(\mathfrak{p}, a_1 X_1 + \dots + a_n X_n)$ is prime and absolutely irreducible.

PROOF. — Let H_τ be the generic hyperplane through (0) defined by $\tau_1 X_1 + \dots + \tau_n X_n = 0$. By [2; Theorem 3, p. 399]. $(\mathfrak{p}, \tau_1 X_1 + \dots + \tau_n X_n)$ is a prime ideal and absolutely irreducible. By [4; theorem 15, p. 259] (which states that an ideal $\mathfrak{A}$ is prime and absolutely irreducible if and only if its ground form is prime and absolutely irreducible), $E(z_1, \dots, z_r; \tau, u)$ is absolutely irreducible. By [6; lemma 7; p. 372], $E(Z_1, \dots, Z_r; a; u)$ is absolutely irreducible almost always. Since $E(Z_1, \dots, Z_r; a; u) \in (\mathfrak{p}, a_1 X_1 + \dots + a_n X_n)$ almost always, the ground form of $(\mathfrak{p}, a_1 X_1 + \dots + a_n X_n)$ is either $E(Z_1, \dots, Z_r; a; u)$ or differs from $E(Z_1, \dots, Z_r; a; u)$ by a multiple of unit, i.e. the ground form of $(\mathfrak{p}, a_1 X_1 + \dots + a_n X_n)$ is $E(Z_1, \dots, Z_r; a; u)$ almost always. Since $E(Z_1, \dots, Z_r; a; u)$ is absolutely irreducible almost always, $(\mathfrak{p}, a_1 X_1 + \dots + a_n X_n)$ is a prime ideal and absolutely irreducible almost always.

THEOREM 5. — Let V/k be an irreducible variety of dimension $r \geq 3$ defined over a field k with $\mathfrak{p}$ as its prime ideal in $k[X]$. If V/k is k -normal at (0) , then for almost all hyperplanes $a_1 X_1 + \dots + a_n X_n = 0$, the ideal $(\mathfrak{p}, a_1 X_1 + \dots + a_n X_n)$ is an $(r-1)$ -dimensional prime ideal.

PROOF. — Let (ξ) be a generic point of V/k . Let k' be the algebraic closure of k in $k(\xi)$. Let k'' be the least normal extension of k containing k' . By [10; Theorem 12a, p. 40], $\mathfrak{p} \cdot k'[X]$, $\mathfrak{p} k''[X]$ are intersection of prime ideals. Let $\mathfrak{p} \cdot k'[X] = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s$, and let $\mathfrak{p} \cdot k''[X] = \mathfrak{p}'_1 \cap \dots \cap \mathfrak{p}'_t$. Since k'' is a normal extension of k , $\mathfrak{p}'_1, \dots, \mathfrak{p}'_t$ are conjugate [1; theorem 8, p. 260]. $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are the only prime ideals in $k'[X]$ lying over $\mathfrak{p}$, and one of them must be the prime ideal determined by (ξ) over k' , say $\mathfrak{p}_1$. Thus $\mathfrak{p}_1$ is absolutely irreducible. Hence $\mathfrak{p}_1 \cdot k''[X]$ is a prime ideal, which is therefore appearing among $\mathfrak{p}'_1, \dots, \mathfrak{p}'_t$, say $\mathfrak{p}_1 \cdot k''[X] = \mathfrak{p}'_1$. By Corollary 1 to Theorem 1, for almost all $a_1 X_1 + \dots + a_n X_n$, $a_1, \dots, a_n \in k$, $(\mathfrak{p}, a_1 X_1 + \dots + a_n X_n) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_l$, where $\mathfrak{q}_1, \dots, \mathfrak{q}_l$ are prime ideals of dimension $r-1$ in $k[X]$. Claim $\mathfrak{q}_1 = \mathfrak{q}_2 = \dots = \mathfrak{q}_l$. Indeed there is a prime ideal $\mathfrak{n}$ in $k''[X]$ lying over $\mathfrak{q}_1$ and by [1; Theorem 2, p. 253] and [3; (10, 14), p. 32], $\mathfrak{n}$ is necessarily of dimension $r-1$. $\mathfrak{n}$ contains $\mathfrak{p} \cdot k''[X]$ hence $\mathfrak{n}$ contains a $\mathfrak{p}'_i$. $\mathfrak{n}$ contains $(\mathfrak{p}'_i, a_1 X_1 + \dots + a_n X_n)$. But $(\mathfrak{p}'_i, a_1 X_1 + \dots + a_n X_n)$ is conjugate to $(\mathfrak{p}_1, a_1 X_1 + \dots + a_n X_n)$ which, by Theorem 4, is a prime ideal almost always. Therefore $(\mathfrak{p}'_i, a_1 X_1 + \dots + a_n X_n)$ is a prime of dimension $r-1$ almost always and $\mathfrak{n} = (\mathfrak{p}'_i, a_1 X_1 + \dots + a_n X_n)$. Applying automorphisms of k''/k to $(\mathfrak{p}', a_1 X_1 + \dots + a_n X_n)$, we have that $(\mathfrak{p}'_i, a_1 X_1 + \dots + a_n X_n)$, for $i = 1, 2, \dots, t$, are

are prime ideals and lying over $\mathfrak{q}_1$. Applying the same argument to each of $\mathfrak{q}_2, \dots, \mathfrak{q}_l$, we conclude that $\mathfrak{q}_1 = \dots = \mathfrak{q}_l$. Thus for almost all $a_1 X_1 + \dots + a_n X_n$, $a_1, \dots, a_n \in k$, $(\mathfrak{p}, a_1 X_1 + \dots + a_n X_n)$ is a prime ideal.

5. - Normality of the algebraic variety of $(\mathfrak{p}, a_1 X_1 + \dots + a_n X_n)$ at (0) .

THEOREM 6. - Let H_τ be the generic hyperplane through (0) defined by $\tau_1 X_1 + \dots + \tau_n X_n = 0$, and let V/k be of dimension $r \geq 3$ and normal at (0) . If $V \cap H_\tau$ is $k(\tau)$ -normal at (0) , then for almost all hyperplanes $a_1 X_1 + \dots + a_n X_n = 0$, $a_1, \dots, a_n \in k$, the sections of V by $a_1 X_1 + \dots + a_n X_n = 0$ is k -normal at (0) .

PROOF. - By [2; Corollary 2, p. 398], $(\mathfrak{p}, \tau_1 X_1 + \dots + \tau_n X_n) = \mathfrak{p}_\tau$ is a prime ideal. Let $E(Z_1, \dots, Z_r; \tau; u)$ be the ground of $\mathfrak{p}_\tau$. By [2; proposition 3, p. 395], $(\mathfrak{p}_\tau, \partial E / \partial \tau_r) \cdot k(\tau, u)[X]_{(0)}$ is unmixed. By Theorem 3 $(\mathfrak{p}_a, \partial E(Z_1, \dots, Z_r; a; u) / \partial Z_r) \cdot k(u)[X]_{(0)}$ is unmixed almost always. Where $\mathfrak{p}_\tau$ and $E(Z_1, \dots, Z_r; \tau; u)$ are specializations of $\mathfrak{p}_\tau$ and $E(Z_1, \dots, Z_r; \tau; u)$ respectively over the specialization $(\tau) \rightarrow (a)$ and $(a) = (a_1, \dots, a_n) \in k^n$. Since V is k -normal at (0) , we have that V is locally free of $(r-1)$ -dimensional singularities at (0) . By Corollary 2 to Theorem 1, the algebraic variety of $\mathfrak{p}_a$ is locally free of $(n-2)$ -dimensional singularities at (0) almost always. By [Theorem 6, p. 367], $E(Z_1, \dots, Z_r; a; u)$ is almost always the ground form of $\mathfrak{p}_a$. Therefore, by [2; proposition 3, p. 395], that $(\mathfrak{p}_a, \partial E(Z; a; u) / \partial Z_r)$ is unmixed implies that the variety of $\mathfrak{p}_a$ is k -normal at (0) almost always.

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