# Eigenvectors and Surjectivity for $\alpha$-Lipschitz Mappings in Banach Spaces.

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**Summary.** – *The main purpose of this paper is to find the conditions under which an lipschitz mapping, defined in a Banach space, admits eigenvectors. We then apply the obtained results to some surjectivity problems proving theorems which contain, as particular cases, the well known theorems of Rothe, Krasnoselskij and Schaefer.*

## 1. – Introduction.

In this paper we consider surjectivity and eigenvector problems involving $\alpha$-Lipschitz mappings, defined in a real Banach space $X$.

Such mappings will be defined below (see Section 2).

We recall here some of the simplest examples of $\alpha$-Lipschitz mappings: completely continuous mappings (i.e. those that are continuous and map bounded sets into precompact ones); Lipschitz mappings (i.e. $L: X \to X$ such that $\|L(x) - L(y)\| \leqslant K\|x - y\|$ for any pair $x, y \in X$, and $0 < K < \infty$); sums of these two types of mappings; mappings of semicontractive type (see [7]); $\alpha$-contractions (see [2]).

The main purpose of this paper is to find conditions under which an $\alpha$-Lipschitz mapping admits eigenvectors. At the same time we obtain some surjectivity results for those mappings.

Our results contain, as particular cases, some Theorems proved by W. V. PETRYSHYN (see [7] and [8], see also [14]) and well known theorems of E. ROTHE (see [10]), M. KRASNOSELSKIJ (see [11]) and H. SCHAEFER (see [12]).

## 2. – Notations and definitions.

The following list contains our basic notations and definitions.

1) $X$ indicates a Banach space;

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2) $B(0, R) = \{x \in X \,|\, \|x\| \leq R\}$;

3) $\partial B(0, R) = \{x \in X \,|\, \|x\| = R\}$;

4) given any $A \subset X$, $\overline{\text{co}}(A)$ indicates the convex closure of $A$;

5) the mappings considered are always continuous;

6) given $T: X \to X$ we say that $x \in X$ is an eigenvector of $T$ if $T(x) = \lambda x$ for some real number $\lambda$.

By $\alpha(A)$, where $A$ is any bounded set of $X$, we denote the infimum of all $\varepsilon > 0$, such that $A$ can be covered by a finite family of subsets with diameter less than $\varepsilon$ (see C. KURATOWSKI [1]).

We will use the following properties of the number $\alpha$ (frequently called measure of noncompactness)

*a*) $\alpha(A) = 0$ iff $A$ is precompact;

*b*) $\alpha(\overline{\text{co}}(A)) = \alpha(A)$ (see G. DARBO [2]);

*c*) $\alpha(L(A)) \leq \|L\| \alpha(A)$ for any linear mapping $L: X \to X$, where by $\|L\|$ we indicate the norm of the mapping $L$.

A mapping $T: X \to X$ is said to be *$\alpha$-Lipschitz* with constant $k$, $0 < k < \infty$, if for any bounded and non precompact $A \subset X$

$$\alpha(T(A)) < k\alpha(A) .$$

If $0 < k < 1$, then it is called *$\alpha$-contractive* with constant $k$.

Some examples of $\alpha$-contractions can be found in a recent paper by A. VIGNOLI [3]. If $k = 1$, then $T$ is called *densifying* [4]. If $\alpha(T(A)) \leq \alpha(A)$, then $T$ is called *$\alpha$-nonexpansive*; moreover $T$ is completely continuous if $\alpha(T(A)) = 0$ for any bounded subset $A$ of $X$.

We will use the following result proved by M. FURI and A. VIGNOLI (see [6]).

PROPOSITION 1. – *Let $T: Q \to Q$ be densifying mapping of a bounded, closed and convex subset of a Banach space $X$. Then $T$ has a fixed point in $Q$.*

The idea of densifying mapping was first introduced by B. N. SADOVSKIJ (see [13]) though he used a different measure of noncompactness. B. N. SADOVSKIJ called those mappings *condensing* and proved a theorem analogous to Proposition 1. Here we use Proposition 1, which is a consequence of more general results obtained by M. FURI and A. VIGNOLI [6], since all of our results will be related to the Kuratowski measure of noncompactness.

The mapping $\pi\colon X \to B(0, R)$ defined by

$$\pi(x) = \begin{cases} x & \text{if} \quad \|x\| \leqslant R\,, \\ Rx/\|x\| & \text{if} \quad \|x\| > R\,, \end{cases}$$

is called the *radial retraction* of $X$ onto $B(0, R)$.

## 3. - Some results.

We begin by stating a Lemma due to R. D. NUSSBAUM [5]. In view of its importance and for completeness sake we will prove it. Our proof is simpler than that one given by Nussbaum.

LEMMA. - *Let $X$ be a Banach space and $B$ the unit ball of $X$ about the origin. Then the radial retraction $\pi\colon X \to B$ is $\alpha$-nonexpansive*

PROOF. - Let $A \subset X$ be a bounded set. Clearly $\pi(A) \subset \overline{\text{co}}\,(\{0\} \cup A)$. Therefore $\alpha(\pi(A)) \leqslant \alpha\big(\overline{\text{co}}\,(\{0\} \cup A)\big) = \alpha(\{0\} \cup A) = \alpha(A)$.

The sharpest result of this section is represented by Theorem 2, but Theorem 1 is more useful for applications.

THEOREM 1. - *Let $T\colon B(0, R) \to X$ be an $\alpha$-Lipschitz mapping with constant $K$ and let $L\colon X \to X$ be an isomorphism.*

*Assume that*

i) $\|L^{-1}\|K \leqslant 1$;

ii) $T(x) = \lambda L(x)$ *for some* $x \in \partial B(0, R)$ *implies* $0 \leqslant \lambda \leqslant 1$.

*Then the set* $M = \{x | T(x) = L(x)\}$ *is nonempty and compact.*

PROOF. - Consider the mapping $F\colon B(0, R) \to X$ defined by $F(x) = L^{-1} \circ T(x)$. The mapping $F$ is densifying. Indeed, let $A$ be any non precompact subset of $B(0, R)$. We have

$$\alpha(F(A)) = \alpha(L^{-1} \circ T(A)) \leqslant \|L^{-1}\|\alpha(T(A)) < \|L^{-1}\|K\alpha(A)\,.$$

Since $\|L^{-1}\|K \leqslant 1$, it follows that $\alpha(F(A)) < \alpha(A)$. Let $\pi$ be the radial retraction of $X$ onto $B(0, R)$. Since $\pi$ is $\alpha$-nonexpansive, the composite mapping $G = \pi \circ F$: $B(0, R) \to B(0, R)$ is evidently densifying.

Thus, by Proposition 1 there exists an element $x \in B(0, R)$ such that

$$F(x) = x, \text{ i.e. } \pi \circ L^{-1} \circ T(x) = x$$

We must show that $L^{-1}\circ T(x)=x$. This is the case if

$$\|L^{-1}\circ T(x)\|\leqslant R$$

Assume that $\|L^{-1}\circ T(x)\|>R$. Then $x\in\partial B(0,R)$, therefore $\|x\|=R$ and $L^{-1}\circ T(x)=\lambda x$, i.e. $T(x)=\lambda L(x)$. Hence

$$\|L^{-1}\circ T(x)\|=\|\lambda x\|=|\lambda|\,\|x\|=|\lambda|R>R$$

But this is impossible since $0\leqslant\lambda\leqslant 1$. Thus $L^{-1}\circ T(x)=x$ and, consequently, $L(x)=T(x)$ and $M=\{x|T(x)=L(x)\}$ is nonempty. Clearly $M$ is closed and compact. Indeed since $G(M)=M$ we have to have that $\alpha(M)=0$ otherwise the assumption $\alpha(M)>0$ would lead to the contradictory inequality $\alpha(M)=\alpha(G(M))<\alpha(M)$, which follows from the densifying property of $G$.

COROLLARY. 1. – *Let $T\colon B(0,R)\to X$ be as in Theorem 1.*

*Assume that*

i) *if $T(x)=\lambda x$ for some $x\in\partial B(0,R)$ then $|\lambda|\leqslant h$ where $h\geqslant K$.*

*Then the set $M=\{x|T(x)=hx\}$ is nonempty and compact.*

PROOF. – The assertion follows immediately from Theorem 1 by putting $L(x)=hx$ for any $x\in X$.

As a particular case of Corollary 1 we have

COROLLARY 2 (see A. VIGNOLI [3], see also W. V. PETRYSHYN [14]). – *Let $T\colon B(0,R)\to X$ ba $\alpha$-contractive with constant $k(0<k<1)$ and let $T$ satisfy the following condition on $\partial B(0,R)$*

i) *if $T(x)=\beta x$ for some $x\in\partial B(0,R)$, then $|\beta|<\mu$ where $\mu$ is any real number such that $k\leqslant\mu\leqslant 2-k$.*

*Then there exists $x\in B(0,R)$, such that $T(x)=\mu x$.*

COROLLARY 3 (see W. V. PETRYSHYN [7]). – *Let $T\colon B(0,R)\to X$ be a densifying mapping which satisfies the boundary condition*

i) *if $T(x)=\lambda x$ for some $x\in\partial B(0,R)$ then $0\leqslant\lambda\leqslant 1$.*

*Then $M$, the set of fixed points of $T$ in $B(0,R)$ is nonempty and compact.*

PROOF. – Put $L=I$, the identity mapping of $X$, and apply Theorem 1.

COROLLARY 4. – *Let $T\colon B(0,R)\to X$ be a densifying mapping such that for any $x\in\partial B(0,R)$*

i) $\|x-T(x)\|^n\geqslant\|T(x)\|^n-\|x\|^n$, $n\geqslant 2$.

*Then $M$, the set of fixed points of $T$, is nonempty and compact.*

PROOF. – Assume that $T(x) = \lambda x$ for some $x \in \partial B(0, R)$, $\lambda \geqslant 0$. By assumption i) we have

$$\|x - \lambda x\|^n = |1 - \lambda|^n \|x\|^n \geqslant \lambda^n \|x\|^n - \|x\|^n .$$

Then $|1 - \lambda|^n \geqslant \lambda^n - 1$. This inequality implies that $\lambda \leqslant 1$ and $T$ satisfies the boundary condition of Corollary 3.

REMARK. – As a special case, for $n = 2$, we obtain a result of V. and A. ISTRATESCU (see [9]) and W. V. PETRYSHYN (see [7]).

COROLLARY 5 (see E. ROTHE [10]). – *Let $T\colon B(0, R) \to X$ be a compact mapping. if for every $x \in \partial B(0, R)$, $\|T(x)\| \leqslant \|x\|$ then $T$ has at least one fixed point in $B(0, R)$.*

COROLLARY 6 (see M. KRASNOSELSKIJ [11]). – *Let $T\colon B(0, R) \to H$ be a compact mapping, where $H$ is a Hilbert space. If for any $x \in \partial B(0, R)$*

$$\langle T(x), x\rangle \leqslant \|x\|^2$$

*then $T$ has at least one fixed point.*

PROOF. – Assume $T(x) = \lambda x$, $\lambda \geqslant 0$. Then $\langle \lambda x, x\rangle = \lambda \|x\|^2 \leqslant \|x\|^2$ Thus $\lambda \leqslant 1$, and $T$ satisfies the boundary condition of Corollary 3.

Let $F\colon X \to X$ be a mapping. We say that $F$ is *$\alpha$-expansive* if for any $A \subset X$ we have

$$\alpha(F(A)) \geqslant h\alpha(A) , \qquad h > 0$$

Perhaps, the simplest example of an $\alpha$-expansive mapping is the following. Let $F\colon X \to X$ be such that $\|F(x) - F(y)\| \geqslant k\|x - y\|$, $k > 0$, for all $x, y \in X$, then $F$ is $\alpha$-expansive.

In the proof of the following Theorem 2, which is the most general result of this paper, we shall use for shortness sake Corollary 3, but another direct proof can be given.

THEOREM 2. – *Let $F\colon X \to X$ be a homeomorphism (possibly nonlinear) $\alpha$-expansive will constant $h > 0$.*

*Let $T\colon B(0, R) \to X$ be an $\alpha$-Lipschitz mapping with constant $k$. Assume that*

i) $0 < k \leqslant h$,

ii) $T(x) = F(\beta x)$

*for some $x \in \partial B(0, R)$ implies $0 \leqslant \beta \leqslant 1$.*

*Then* $M = \{x \in X | T(x) = F(x)\}$ *is nonempty and compact.*

PROOF. – Let $A$ be a bounded and non precompact subset of $X$. Clearly $\alpha(F^{-1}(A)) \leqslant h^{-1}\alpha(A)$. Indeed, $\alpha(A) = \alpha(F \circ F^{-1}(A)) \geqslant h\alpha(F^{-1}(A))$ i.e. $\alpha(F^{-1}(A)) \leqslant \leqslant h^{-1}\alpha(A)$. Consider the composite mapping $G = F^{-1} \circ T \colon B(0, R) \to X$. We must show that $G$ is densifying. For any subset $A \subset B(0, R)$ we have

$$\alpha(G(A)) = \alpha(F^{-1} \circ T(A)) \leqslant h^{-1}\alpha(T(A)) < kh^{-1}\alpha(A)$$

Since $k \leqslant h$ it follows that $\alpha(G(A)) < \alpha(A)$ and $G$ is densifying.

Condition ii) implies that $G$ satisfies the boundary condition of Corollary 3, so the assertion of Theorem 2 is proved.

## 4. – Applications.

In this section we give some applications of Theorems 1 and 2 to surjectivity and other problems.

THEOREM 3. – *Let* $T \colon X \to X$ *be an* $\alpha$*-Lipschitz mapping with constant* $k$ *and let* $F \colon X \to X$ *be a homeomorphism (possibly nonlinear)* $\alpha$*-expansive with constant* $h > 0$. *Assume that*

i) $0 < k \leqslant h$;

ii) *there exists a sequence* $\{\partial B(0, \beta_n)\}$ *of spheres and a sequence* $\{\gamma_n\}$ *of positive real numbers* $\gamma_n \to \infty$ *as* $n \to \infty$ *such that for any* $\lambda > 1$ *and any* $x \in \partial B(0, \beta_n)$

$$\|T(x) - F(\lambda x)\| \geqslant \gamma_n .$$

*Then the mapping* $T - F$ *is surjective.*

PROOF. – We have to show that for any given $y \in X$ there exists an element $x \in X$ such that $T(x) - F(x) = y$. Since $\gamma_n \to \infty$ as $n \to \infty$, choose $n$ sufficiently large that $\|y\| < \gamma_n$. Evidently, the mapping $G = T - y$ is $\alpha$-Lipschitz with constant $k$. Therefore, if $G(x) = F(\lambda x)$ for some $x \in \partial B(0, \beta_n)$ implies that $\lambda \leqslant 1$, then Theorem 2 gives the existence of an $x \in B(0, \beta_n)$ such that $G(x) = F(x)$ and we are done. Take $\lambda > 1$. Then

$$0 = \|G(x) - F(\lambda x)\| = \|T(x) - F(\lambda x) - y\| \geqslant \|T(x) - F(\lambda x)\| - \|y\| \geqslant \gamma_n - \|y\| > 0 .$$

This contradiction shows that $\lambda \leqslant 1$ and the theorem is proved.

The following result, although being less general than Theorem 3, is more useful for applications. Theorem 4 below can be regarded as a Corollary of Theorem 3, but here we give a proof based on Theorem 1.

THEOREM 4. – *Let $T\colon X \to X$ be an $\alpha$-Lipschitz mapping with constant $k$ and let $L\colon X \to X$ be an isomorphism. Assume that*

i) $\|L^{-1}\|k \leqslant 1$;

ii) *there exists a sequence $\{\partial B(0, \beta_n)\}$ of spheres and a sequence $\{\gamma_n\}$ of positive real numbers $\gamma_n \to \infty$ as $n \to \infty$ such that for any $\lambda > 1$ and any $x \in \partial B(0, \beta_n)$*

$$\|T(x) - \lambda L(x)\| \geqslant \gamma_n$$

*Then the mapping $T - L$ is surjective.*

PROOF. – Let $y \in X$. We must show that there exists an $x \in X$ such that $T(x) - L(x) = y$. Choose $n$ so large that $\|y\| < \gamma_n$.

The mapping $G = T - y$ is $\alpha$-Lipschitz with constant $k$. If $G(x) = \lambda L(x)$ for some $x \in \partial B(0, \beta_n)$ implies that $\lambda \leqslant 1$, then, by Theorem 1, there exists $x \in B(0, \beta_n)$ such that $G(x) = L(x)$ i.e. $T(x) - L(x) = y$. Assume $\lambda > 1$. Then

$$0 = \|G(x) - \lambda L(x)\| = \|T(x) - \lambda L(x) - y\| \geqslant \|T(x) - \lambda L(x)\| - \|y\| \geqslant \gamma_n - \|y\| > 0 .$$

But this is impossible, so $\lambda \leqslant 1$ and the assertion is proved.

COROLLARY 7. – *Let $T\colon X \to X$ be an $\alpha$-Lipschitz mapping with constant $k$ and let $h \geqslant k$. Assume that*

i) *there exists a sequence $\{\partial B(0, \beta_n)\}$ of spheres and a sequence $\{\gamma_n\}$ of positive real numbers, $\gamma_n \to \infty$ as $n \to \infty$, such that for any $|\lambda| > h$ and any $x \in \partial B(0, \beta_n)$*

$$\|T(x) - \lambda x\| \geqslant \gamma_n .$$

*Then the mapping $T - hI$ is surjective.*

PROOF. – Let $y \in X$. We must show that there exists an $x \in X$, such that $T(x) - hx = y$.

Choose $n$ so large that $\|y\| < \gamma_n$ and consider the mapping $G = T - y$. If $G(x) = \lambda x$ for some $x \in \partial B(0, \beta_n)$ implies that $|\lambda| \leqslant h$, then, by Corollary 1, there exists $x \in B(0, \beta_n)$, such that $G(x) = hx$, i.e. $T(x) - hx = y$. Assume $|\lambda| > h$. We have

$$0 = \|G(x) - \lambda x\| = \|T(x) - \lambda x - y\| \geqslant \|T(x) - \lambda x\| - \|y\| \geqslant \gamma_n - \|y\| > 0 .$$

But this is a contradiction, so $|\lambda| \leqslant h$.

REMARK. – Theorem 3 and Corollary 7 give extensions to $\alpha$-Lipschitz mappings of some results proved by W. V. PETRYSHYN (see [8]) for nonlinear $P$-compact operators.

COROLLARY 8 (see A. VIGNOLI [3], see also W. V. PETRYSHYN [14]). - *Let $T: X \to X$ be an $\alpha$-contractive mapping with constant $k$, $0 < k < 1$, and let $\mu$, $k \leqslant \mu \leqslant 2 - k$. If there exists a sequence $\{\partial B(0, \beta_n)\}$ of spheres and a sequence $\{\gamma_n\}$ of positive real numbers, $\gamma_n \to \infty$ as $n \to \infty$, such that for any $\lambda > \mu$ and any $x \in \partial B(0, \beta_n)$*

$$\|T(x) - \lambda x\| \geqslant \gamma_n .$$

*Then the mapping $T - \mu I$ is surjective.*

As another application of the results obtained in section 3 we will give a generalization of a very well known result of SCHAEFER (see [12]). We prove a general theorem (see Theorem 5 below) and we obtain Schafer's result as a Corollary to that theorem (see Corollary 10).

THEOREM 5. - *Let $T: X \to X$ be an $\alpha$-Lipschitz mapping with constant $k$ and let $F: X \to X$ be a (possibly nonlinear) homeomorphism $\alpha$-expansive with constant $h > 0$ such that $0 < k \leqslant h$. If there is no $x \in X$ such that $T(x) = F(x)$, then the set $M = \{x \in X | T(x) = F(\lambda x)$ for some $\lambda > 1\}$ is unbounded.*

PROOF. - Let $B_n = \{x \in X | \|x\| \leqslant n\}$ and let $\pi_n$ be the radial retraction of $X$ onto $B_n$ (see section 2). Put $G_n = \pi_n \circ F^{-1} \circ T$. Clearly $G_n$ is a densifying mapping of $B_n$ into itself then, by Theorem $A$ (see Introduction), there exists $x_n \in B_n$, such that

$$x_n = G_n(x_n) = \pi_n \circ F^{-1} \circ T(x_n) .$$

Evidently $\|F^{-1} \circ T(x_n)\| > n$, otherwise $\pi_n \circ F^{-1} \circ T(x_n) = F^{-1} \circ T(x_n)$, i.e. $T(x_n) = F(x_n)$, contradicting the hypothesis. Now, from the definition of the radial retraction $\pi_n$ we get $\|x_n\| = n$ and $T(x_n) = F(\lambda x_n)$, where $\lambda = n/\|F^{-1} \circ T(x_n)\| > 1$ and the theorem is proved.

The following result is a linear version of Theorem 5 (in the sense that the mapping $L$ is an isomorphism). Theorem 6 below is not only much more useful in applications, but its formulation is very similar to the one of the above mentioned result of SCHAEFER (see Corollary 10).

THEOREM 6. - *Let $T: X \to X$ be an $\alpha$-Lipschitz mapping with constant $k$ and let $L: X \to X$ be an isomorphism such that $\|L^{-1}\| k \leqslant 1$. If there is no $x \in X$ such that $T(x) = L(x)$, then the set $M = \{x \in X | \lambda T(x) = L(x)$ for some $\lambda \in (0, 1)\}$ is unbounded.*

PROOF. - Let $B_n = \{x \in X | \|x\| \leqslant n\}$ and let $\pi_n$ be the radial retraction of $X$ onto $B_n$. Put $G_n = \pi_n \circ L^{-1} \circ T$. Since $G_n$ is a densifying mapping of $B_n$ into itself there exists $x_n \in B_n$, such that $x_n = G_n(x_n) = \pi_n \circ L^{-1} \circ T(x_n)$. Clearly $\|L^{-1} \circ T(x_n)\| > n$, otherwise $\pi_n \circ L^{-1} \circ T(x_n) = L^{-1} \circ T(x_n)$, i.e. $T(x_n) = L(x_n)$, contradicting the hypothesis. Then from the definition of radial retraction $\pi_n$, it follows that $\|x_n\| = n$ and $L^{-1} \circ T(x_n) = h x_n$, with $h > 1$. Thus $h^{-1}(T x_n) = L(x_n)$ and $h^{-1} \in (0, 1)$.

COROLLARY 9. – *Let $T: X \to X$ be a densifying mapping. If there is no $x \in X$, such that $T(x) = x$ then the set*

$$M = \{x \in X | \lambda T(x) = x \text{ for some } \lambda \in (0, 1)\} \text{ is unbounded.}$$

PROOF. – Put $L = I$, the identity mapping of $X$, and apply Theorem 6.

COROLLARY 10 (Schaefer's Theorem [12]). – *Let $T$ be a compact self-mapping of a Banach space $X$. If there exists $\lambda_0 \in [0, 1]$ such that the equation $x = \lambda_0 T(x)$ does not have any solution, then the set $M = \{x \in X | x = \lambda T(x),\ 0 < \lambda < \lambda_0\}$ is unbounded.*

PROOF. – Suppose there is a $\lambda_0 \in [0, 1]$, such that the equation $x = \lambda_0 T(x)$ does not have any solution. Let $G_n: X \to X$ defined by $G_n(x) = \pi_n(\lambda_0 T(x))$. As in the proof of Theorem 6, we can find $x_n \in B_n$, $B_n = \{x \in X | \|x\| \leqslant n\}$, such that $G(x_n) = \pi_n(\lambda_0 T(x_n)) = x_n$. Clearly $\|\lambda_0 T(x_n)\| > n$, otherwise $\lambda_0 T(x_n) = x_n$. Then $\|x_n\| = n$, and $\lambda_0 T(x_n) = \mu^{-1} x_n$ with $0 < \mu < 1$, so $\mu \lambda_0 T(x_n) = x_n$ and $0 < \mu\lambda_0 < \lambda_0$.

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