A characteristic initial value problem for the Euler-Poisson-Darboux equation.

EUTQUIO C. YOUNG (Florida U.S.A.) (*) (**)

Summary. - *An explicit solution is obtained for a characteristic initial value problem for the Euler-Poisson-Darboux equation by Riesz method of analytic continuation of a generalized Riemann-Liouville integral.*

1. - Introduction.

This paper is concerned with an interior characteristic initial value problem for the nonhomogeneous EULER-POISSON-DARBOUX equation

$$(1) \quad L_x u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{k}{x_1} \frac{\partial u}{\partial x_1} - \sum_{i=1}^m \frac{\partial^2 u}{\partial x_i^2} = f(x_1, \dots, x_m)$$

in which the solution u is required to satisfy a vanishing initial condition

$$(2) \quad u(x_1, \dots, x_m) = 0$$

on the surface of the initial characteristic cone

$$(3) \quad S: x_1^2 - x_2^2 - \dots - x_m^2 = 0, \quad x_1 \geq 0.$$

The parameter k is assumed real and satisfies the condition $k > m - 2$ if m is even and $k > m - 3$ if m is odd. This restriction on k is made in order that the integral solution may converge. However, the uniqueness of the solution of the problem (1), (2) can be proved for $k \geq 0$ [1, pp. 645-47].

When $k = 0$, equation (1) reduces to the wave equation for which the solution of the present problem was given by RIESZ [2, pp. 107-125] (see also [3]). In the cases $m = 2$ and $m = 3$ for which $k > 0$, it will be evident that by letting k approach zero, the solution of (1), (2) reduces to the solution of the corresponding problem for the wave equation. However, in view of the restriction on k , it is not at all evident that for general m one can arrive at the problem for the wave equation by such a limiting process.

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The procedure used here is similar to that employed in [4] and [5] which is based on analytic continuation of a generalized RIEMANN-LIOUVILLE integral developed by RIESZ [2].

Let x and y denote two points in m -dimensional space-time or LORENTZ-space with coordinates $x_1, \dots, x_m$ and $y_1, \dots, y_m$, respectively. The square of the (LORENTZ) distance between these points is given by

$$(4) \quad R = (y_1 - x_1)^2 - (y_2 - x_2)^2 - \dots - (y_m - x_m)^2.$$

If x is fixed and y a variable point, then the equation $R=0$ with $x_1 - y_1 > 0$ defines the retrograde characteristic cone of equation (1) with vertex x . On the other hand, $R=0$ with $x_1 - y_1 < 0$ defines the direct or upright characteristic cone with the same vertex.

In the sequel, we shall also use x to denote the position vector of the point $(x_1, \dots, x_m)$. Let β and γ be two vectors with respective components $\beta_1, \dots, \beta_m$ and $\gamma_1, \dots, \gamma_m$. The scalar product (β, γ) of β and γ is defined by

$$(\beta, \gamma) = \beta_1 \gamma_1 - \beta_2 \gamma_2 - \dots - \beta_m \gamma_m.$$

In analogy with ordinary Euclidean space, the vectors β and γ are said to be orthogonal if $(\beta, \gamma) = 0$. A vector β is said to be timelike if $(\beta, \beta) > 0$ and spacelike if $(\beta, \beta) < 0$. In particular, a vector β is called a timelike unit vector or a spacelike unit vector according as $(\beta, \beta) = 1$ or $(\beta, \beta) = -1$.

2. - The Solution Formula.

Let x be a point in the interior of the direct initial cone S and let C^x denote the retrograde characteristic cone with vertex x , that is,

$$C^x: R = (y_1 - x_1)^2 - \dots - (y_m - x_m)^2 = 0, \quad y_m - x_m < 0.$$

Denote by D the domain enclosed by C^x and the initial cone S . The line joining the vertices of C^x and S will be called the axis of D . If u and v are two functions with continuous second derivatives in D , then by GREEN'S theorem,

$$(5) \quad \int_D [u M_y v - v L_y u] dy = \int_{S+C^x} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} - \frac{kuv}{y_1} \right) dS$$

where $dy = dy_1 \dots dy_m$ denotes the volume element in D and dS denotes the surface element on S and C^x . The operator

$$M_y = \frac{\partial^2}{\partial y_1^2} - \frac{k}{y_1} \frac{\partial}{\partial y_1} + \frac{k}{y_1^2} - \sum_{i=2}^m \frac{\partial^2}{\partial y_i^2}$$

is the adjoint of L .

We now set $v = V^{\alpha+2}(x; y)$ in (5) where

$$(6) \quad V^{\alpha+2}(x; y) = \frac{2^k y_1^k R^{(\alpha+2-m)/2}}{H_m(\alpha+2) R'^{k/2}} F\left(\frac{k}{2}, \frac{\alpha+2+k-m}{2}; \frac{\alpha+4-m}{2}; \frac{R}{R'}\right)$$

with

$$R' = (y_1 + x_1)^2 - (y_2 - x_2)^2 - \dots - (y_m - x_m)^2 = R + 4x_1 y_1$$

and

$$H_m(\alpha) = 2^{\alpha-1} \pi^{(m-2)/2} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha+2-m}{2}\right).$$

The function (6) was determined in [4] and also in [6]. For α sufficient large, $V^{\alpha+2}$ satisfies the conditions

$$(7) \quad (i) \quad L_x V^{\alpha+2}(x; y) = V^{\alpha}(x; y)$$

$$(ii) \quad M_y V^{\alpha+2}(x; y) = V^{\alpha}(x; y)$$

and vanishes together with its derivatives on the surface of C^x . Writing

$$(8) \quad I^{\alpha} u(x) = \int_D u(y) V^{\alpha}(x; y) dy$$

and using (7ii) together with (1) and (2), we obtain from (5) the formula

$$(9) \quad I^{\alpha} u(x) = \int_D f(y) V^{\alpha+2}(x; y) dy.$$

It is clear that the integral (9) converges for $\alpha > m$ and $k \geq 0$. Under this condition it follows from (7i) that

$$(10) \quad L_x [I^{\alpha} u(x)] = \int_D f(y) L_x V^{\alpha+2}(x; y) dy = \int_D f(y) V^{\alpha}(x; y) dy = I^{\alpha} f(x).$$

As will be shown in the next section, the integral (8) can be continued analytically with respect to α for $k > m - 2$ and gives

$$(11) \quad I^0 u(x) = u(x)$$

verifying equation (1). Further, the analytic continuation of (9) to $\alpha = 0$ will yield the explicit solution

$$(13) \quad u(x) = I^{0+2} f(y).$$

This will be shown to vanish as x approaches the surface of the initial cone (3), thus verifying (2). It is in this sense that formula (9) provides the solution of the problem (1), (2).

3. - The Analytic Continuation of $I^2u(x)$.

In order to be able to perform the analytic continuation of the integral (8) with respect to x , we introduce a suitable coordinate system in D and assume that u is at least $[m/2]$ times continuously differentiable. We adapt here a coordinate system that was devised by RIESZ [7]. Let a be a fixed timelike unit vector issuing from x and pointing toward the origin, and let β be a variable spacelike unit vector orthogonal to a . The vector β actually lies in an $(m-1)$ -dimensional plane which passes through x and is parallel to the plane where the cones C^x and S meet. Thus we have

$$(14) \quad (a, a) = 1, \quad (\beta, \beta) = -1, \quad (a, \beta) = 0.$$

If the vector β issues from the origin, its endpoint describes a « unit sphere » ω_{m-1} defined by $\beta_1^2 - \beta_2^2 - \dots - \beta_m^2 = -1$ and lying in an $(m-1)$ -plane orthogonal to the vector x . Let y be the position vector of an arbitrary point y in D . Then we can write

$$(15) \quad y - x = ta + \rho\beta$$

where t and ρ are nonnegative real numbers. This expression can be put in the form

$$(16) \quad y - x = r(\gamma + s\tau)$$

where γ and τ are vectors defined by

$$\gamma = \frac{1}{2}(a + \beta), \quad \tau = \frac{1}{2}(a - \beta)$$

and r and s are scalars related to t and ρ by the equations

$$(17) \quad r = t + \rho, \quad s = \frac{t - \rho}{t + \rho}.$$

The inverse formulas of (17) are given by

$$(18) \quad t = \frac{1}{2}r(1 + s), \quad \rho = \frac{1}{2}r(1 - s).$$

The variables r , s and the angular variable β will now be used as the new coordinates in D . In this coordinate system, the square of the LORENTZ distance (4) becomes

$$(19) \quad R = (y - x, y - x) = t^2 - \rho^2 = (t + \rho)^2 \frac{t - \rho}{t + \rho} = r^2 s$$

by virtue of (14) and (15). When y lies on the cone C^* , $R = r^2 s = 0$. Since the vertex x is given by the single equation $r = 0$, it follows that the cone C^* apart from the vertex is given by $s = 0$. Points on the axis $y = x + ta$ (or $\rho = 0$) correspond to $s = 1$. Moreover, for $r = c = \text{constant}$, the equation $t + \rho = r = c$ defines a direct characteristic cone with vertex $y = x + ca$. Thus the initial cone S is defined by the equation $r = r_0$, where $r_0 = (x_1^2 - x_2^2 - \dots - x_m^2)^{1/2}$ is distance of x from the origin. Hence the domain D including its boundary is characterized by the inequalities

$$0 \leq s \leq 1, \quad 0 \leq r \leq r_0.$$

If we denote as usual the surface element on the «unit sphere» ω_{m-1} by $d\omega_{m-1}$, then the element of volume in D is given by

$$dy = r^{m-2} d\rho dt d\omega_{m-1}.$$

On changing from the variables ρ , t to the variables r , s using (18), this becomes

$$(20) \quad dy = 2^{1-m} r^{m-1} (1-s)^{m-2} dr ds d\omega_{m-1}.$$

We now proceed to prove (11). We distinguish between the cases m even and m odd, concerning ourselves here only with the case m even, $m \geq 2$. We expand in infinite series the hypergeometric function appearing in V^2 of the integral (8) and consider the general term of the first $\frac{m}{2}$ terms, namely

$$(21) \quad I_j^2 u(x) = \frac{C_j(\alpha)}{\Gamma\left(\frac{\alpha+2-m}{2} + j\right)} \int \frac{y_1^k u(y) R^{(\alpha-m+2j)/2}}{R^{k/2+j}} dy$$

$0 \leq j \leq \frac{m-2}{2}$, where

$$C_j(\alpha) = \frac{\Gamma\left(\frac{\alpha+2-m}{2}\right) \Gamma\left(\frac{k}{2} + j\right) \Gamma\left(\frac{\alpha+k-m}{2} + j\right) 2^k}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{\alpha+k-m}{2}\right) H_m(\alpha) j!}.$$

The leading term corresponding to $j = 0$ is given by

$$\begin{aligned} I_0^\alpha u(x) &= \frac{2^k}{H_m(\alpha)} \int_D \frac{y_1^k u(y) R^{(\alpha-m)/2}}{R^{k/2}} dy \\ &= \frac{1}{H_m(\alpha)} \int_D g(y) R^{(\alpha-m)/2} \\ &= J^\alpha g(x) \end{aligned}$$

where we have set

$$g(y) = \frac{2^k y_1^k u(y)}{R^{k/2}}.$$

The last integral $J^\alpha g(x)$, which converges for $\alpha > m - 2$ and $k > m - 2$, is identical with that considered in [2, p. 111] for which it is true that α can be continued analytically to $\alpha = 0$, giving the result

$$J^0 g(x) = g(x).$$

Hence,

$$I_0^\alpha u(x) = g(x) = \frac{2^k x_1^k u(x)}{[(2x_1)^2]^{k/2}} = u(x).$$

It remains to be shown that all the remaining terms in the expansion of $I^\alpha u(x)$ vanish as α is continued analytically to $\alpha = 0$.

For $1 \leq j \leq \frac{m-2}{2}$, we write (21) in the new coordinates r, s , and β using (16), (19), and (20), obtaining

$$(22) \quad I_j^\alpha u(x) = \frac{C_j(\alpha) 2^{1-m}}{\Gamma\left(\frac{\alpha+2-m}{2} + j\right)} \int_0^{r_0} r^{x+2j-1} dr \int_0^1 M(h) (1-s)^{m-2} s^{(\alpha-m+2j)/2} ds.$$

Here

$$(23) \quad M(h) = \int_{\omega_{m-1}} h(r, s) d\omega_{m-1}$$

with

$$h(r, s) = \frac{[x_1 + r(\gamma_1 + s\tau_1)]^k u[x + r(\gamma + s\tau)]}{\{r^2 s + 4x_1[x_1 + r(\gamma_1 + s\tau_1)]\}^{k/2}}.$$

With the differentiability condition required of u and with $k > m - 2$, inte-

gration by parts with respect to s in the inner integral of (22) can be performed $\frac{m-2j}{2}$ times, yielding the result

$$(24) \quad I^\alpha u(x) = \frac{C_j(\alpha) 2^{1-m}}{\Gamma\left(\frac{\alpha+2}{2}\right)} (-1)^{(m-2j)/2} \int_0^{r_0} r^{\alpha+2j-1} dr \\ \times \int_0^1 s^{\alpha/2} \left(\frac{\partial}{\partial s}\right)^{(m-2j)/2} [M(h)(1-s)^{m-2}] ds.$$

The integral (24) is now convergent for $\alpha > -2$ and $k > m-2$. Hence letting $\alpha \rightarrow 0$, it follows that $I_j^\alpha u(x) = 0$, $1 \leq j \leq \frac{m-2}{2}$, since $C_j(0) = 0$ in view of the factor $\Gamma(\alpha/2)$ in $H_m(\alpha)$.

The remaining terms in the series expansion of the hypergeometric function can be expressed as

$$A \left(\frac{R}{R'}\right)^{m/2} {}_3F_2\left(\frac{R}{R'}\right)$$

where A is some constant and ${}_3F_2$ is a generalized hypergeometric function. Substituting this for the hypergeometric function in (8), we see that the resulting integral

$$I_*^\alpha u(x) = \frac{2^k A}{H_m(\alpha)} \int_D \frac{y_1^k u(y) R^{k/2}}{R'^{(k+m)/2}} {}_3F_2\left(\frac{R}{R'}\right) dy$$

converges for $\alpha > -2$ and $k > m-2$. For the same reason as above, it follows that this integral vanishes as α is allowed to approach zero. This establishes equation (11).

In view of what has just been proved, we can now conclude from (10) and (12) that (9) indeed satisfies equation (1).

4. - The Explicit Solution.

We now perform the analytic continuation with respect to α in the integral (9) to arrive at an explicit solution of the problem (1), (2). For this purpose we require that the function f has continuous derivatives up to the order $\left[\frac{m+2}{2}\right]$. In the cases $m=2$ and $m=3$, we readily obtain the solution of

(1), (2) by simply setting $\alpha = 0$ in (9). In fact, when $m = 2$ we have

$$(25) \quad u(t, x) = 2^{k-1} \int_D \frac{\tau^k f(\tau, \xi)}{R'^{k/2}} F\left(\frac{k}{2}, \frac{k}{2}; 1; \frac{R}{R'}\right) d\xi d\tau$$

where $R = (\tau - t)^2 - (\xi - x)^2$, $R' = (\tau + t)^2 - (\xi - x)^2$, and D is the rectangle bounded by the characteristic lines $\xi = \tau$, $\xi = -\tau$, $\xi + \tau = x + t$, and $\xi - \tau = x - t$. The solution (25) is valid for $k \geq 0$ and yields the solution of the corresponding problem for the wave equation $u_{tt} - u_{xx} = 0$ where $k = 0$. In characteristic coordinates $s = \xi + \tau$, $r = \xi - \tau$ with $s_0 = x + t$, $r_0 = x - t$, (25) assumes a more familiar form (see [8])

$$(26) \quad u(s_0, r_0) = \frac{1}{4} \int_0^{s_0} \int_0^{r_0} \frac{(s-r)^k}{[(r-s_0)(r_0-s)]^{k/2}} f\left(\frac{s+r}{2}, \frac{s-r}{2}\right) \times F\left(\frac{k}{2}, \frac{k}{2}; 1; z\right) dr ds$$

where $z = (r - r_0)(s - s_0)/[(r - s_0)(s - r_0)]$.

It is easily seen from (26) that u vanishes on the characteristic lines $s = 0$ and $r = 0$.

When $m = 3$, we obtain from (9)

$$(27) \quad u(t, x, y) = \frac{2^{k-1}}{\pi} \int_D \frac{\tau^k f(\tau, \xi, \eta)}{R^{1/2} R'^{k/2}} F\left(\frac{k-1}{2}, \frac{k}{2}; \frac{1}{2}; \frac{R}{R'}\right) d\xi d\eta d\tau$$

where $R = (\tau - t)^2 - (\xi - y)^2 - (\eta - y)^2$, $R' = (\tau + t)^2 - (\xi - x)^2 - (\eta - y)^2$ and D is the domain bounded by the retrograde characteristic cone $R = 0$, $t - \tau > 0$ and the direct characteristic cone $\tau^2 - \xi^2 - \eta^2 = 0$, $\tau > 0$. This solution is $k \geq 0$.

Thus we need continue (9) only in the case $m \geq 4$. We consider here the case when m is even and simply state the result for m odd, $m \geq 5$. Expanding the hypergeometric function appearing in $V^{\alpha+2}$ on the right side of (9), we valid for consider the general term of the first $(m-4)/2$ terms,

$$(28) \quad U_j(\alpha) = \frac{B_j(\alpha)}{\Gamma\left(\frac{\alpha+4-m}{2} + j\right)} \int_D \frac{y_1^k f(y) R^{(\alpha+2-m+2j)/2}}{R'^{k/2+j}} dy$$

$0 \leq j \leq \frac{m-4}{2}$, where

$$B_j(\alpha) = \frac{2^k \Gamma\left(\frac{\alpha+4-m}{2}\right) \Gamma\left(\frac{k}{2} + j\right) \Gamma\left(\frac{\alpha+k+2-m}{2} + j\right)}{H_m(\alpha+2) \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{\alpha+k+2-m}{2}\right) j!}$$

In the new coordinates r , s , and β , (28) becomes

$$U_j(\alpha) = \frac{2^{1-m} B_j(\alpha)}{\Gamma\left(\frac{\alpha+4-m}{2} + j\right)} \int_0^{r_0} r^{\alpha+2j+1} dr \int_0^1 M(h)(1-s)^{m-2} s^{(\alpha+2-m+2j)/2} ds$$

where M is as defined in (23) with f replacing u in the expression for h . Under the differentiability condition imposed on f , we can perform integration by parts with respect to $s^{\frac{m-2-2j}{2}}$ times. The resulting integral (see equation (24)) converges for $\alpha > -2$ and $k > m-2$. Thus letting $\alpha \rightarrow 0$, we obtain the result

$$(29) \quad U_j(0) = (-1)^{(m-2j)/2} 2^{1-m} B_j(0) \int_0^{r_0} r^{2j+1} \times \left\{ \left(\frac{\partial}{\partial s} \right)^{(m-4-2j)/2} [M(h)(1-s)^{m-2}] \right\}_{s=0} dr$$

with

$$B_j(0) = \frac{2^{k-1} \Gamma\left(\frac{k}{2} + j\right) \Gamma\left(\frac{k+2-m}{2} + j\right)}{\Pi^{(m-2)/2} \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k+2-m}{2}\right) j!}, \quad 0 \leq j \leq \frac{m-4}{2}.$$

Let $b = \frac{k}{2}$, $c = \frac{k+\alpha+2-m}{2}$, $d = \frac{\alpha+4-m}{2}$, $\nu = \frac{m-2}{2}$ and put $z = \frac{R}{R'}$. The remaining term in the expansion of the of the hypergeometric function of $V^{\alpha+2}$ can be expressed as

$$\begin{aligned} F^*(z) &= \frac{(b)_\nu (c)_\nu}{(d)_\nu} \frac{z^\nu}{\nu!} \left[1 + \frac{(b+\nu)(c+\nu)}{(d+\nu)(\nu+1)} z + \dots \right] \\ &= \frac{(b)_\nu (c)_\nu}{(d)_\nu} \frac{z^\nu}{\nu!} {}_3F_2(b+\nu, c+\nu, 1; d+\nu; \nu+1; z) \end{aligned}$$

where $(\lambda)_\nu = \Gamma(\lambda+\nu)/\Gamma(\lambda)$. The generalized hypergeometric function ${}_3F_2(z)$ converges uniformly in z for $z \leq \sigma$ with any $\sigma < 1$. Hence for the remaining terms in the expansion of (9) we have

$$W(\alpha) = A(\alpha) \int \frac{y_1^k f(y) R^{\alpha/2}}{R^{(k+m-2)/2}} {}_3F_2\left(\frac{k+m-2}{2}, \frac{\alpha+k}{2}, 1; \frac{\alpha+2}{2}; \frac{m}{2}; \frac{R}{R'}\right) dy$$

with A denoting all constant factors. This integral converges uniformly for

$\alpha > -2$ and $k > m - 2$. So letting $\alpha \rightarrow 0$, we have

$$W(0) = A(0) \int \frac{y_1^k f(y)}{R^{(k+m-2)/2}} F\left(\frac{k}{2}, \frac{k+m-2}{2}; \frac{m}{2}; \frac{R}{R'}\right) dy$$

with

$$A(0) = \frac{2^{k-1} \Gamma\left(\frac{k+m-2}{2}\right)}{\Pi^{(m-2)/2} \Gamma\left(\frac{k+2-m}{2}\right) \Gamma\left(\frac{m}{2}\right)}.$$

Therefore for m even, $m \geq 4$, and $k > m - 2$, the solution of the problem (1), (2) is

$$\begin{aligned} u(x) &= \frac{2^{k-m}}{\Pi^{(m-2)/2}} \sum_{j=0}^{(m-4)/2} (-1)^{(m-2j)/2} \frac{\Gamma\left(\frac{k}{2} + j\right) \Gamma\left(\frac{k+2-m}{2} + j\right)}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k+2-m}{2}\right) j!} \\ &\quad \times \int_0^{r_0} r^{2j+1} \left\{ \left(\frac{\partial}{\partial s} \right)^{(m-4-2j)/2} [M(h)(1-\tau)^{m-2}] \right\}_{s=0} dr \\ &\quad + \frac{2^{k-1} \Gamma\left(\frac{k+m-2}{2}\right)}{\Pi^{(m-2)/2} \Gamma\left(\frac{k+2-m}{2}\right) \Gamma\left(\frac{m}{2}\right)} \int \frac{y_1^k f(y)}{R^{(k+m-2)/2}} F\left(\frac{k}{2}, \frac{k+m-2}{2}; \frac{m}{2}; \frac{R}{R'}\right) dy. \end{aligned}$$

The case m odd, $m \geq 5$, and $k > m - 3$ can be treated in the same way. We obtain the result

$$\begin{aligned} u(x) &= \frac{2^{k-m}}{\Pi^{(m-1)/2}} \sum_{j=0}^{(m-5)/2} (-1)^{(m-3-j)/2} \frac{\Gamma\left(\frac{k}{2} + j\right) \Gamma\left(\frac{k+2-m}{2} + j\right)}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k+2-m}{2}\right) j!} \\ &\quad \times \int_0^{r_0} r^{2j+1} \int_0^1 s^{-1/2} \left(\frac{\partial}{\partial s} \right)^{(m-3-2j)/2} [M(h)(1-s)^{m-2}] ds dr \\ &\quad + \frac{2^{k-1} \Gamma\left(\frac{k-1}{2}\right) \Gamma\left(\frac{m-3+k}{2}\right)}{\Pi^{(m-1)/2} \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{2-m+k}{2}\right) \Gamma\left(\frac{m-1}{2}\right)} \int \frac{y_1^k f(y) R^{-1/2}}{R^{(m+k-3)/2}} \\ &\quad \times {}_3F_2\left(\frac{k-1}{2}, \frac{k+m-3}{2}; 1; \frac{1}{2}, \frac{m-1}{2}; \frac{R}{R'}\right) dy. \end{aligned}$$

That the solution (27) for $m = 3$ or (30) for m even, $m \geq 4$, or (31) for m odd, $m \geq 5$, satisfies the initial condition (2) follows readily in the new coordinates r, s, β from the observation that as x approaches the initial cone (3), r_0 tends to zero and vice versa.

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