On the existence of periodic solutions of a certain non-autonomous differential equation

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Summary. - *Applying the Leray-Schauder fixed point theorem we prove the existence of a periodic solution for a non-autonomous differential equation with a bounded nonlinear term.*

Let us consider the differential equation

$$x^{(n+1)} + a_1 x^{(n)} + \dots + a_n x' + f(x) = p(t) \tag{1}$$

$$[n \geqq 1,\ p(t+\omega) \equiv p(t)]$$

where the functions f and p are continuous for all values x respectively t and the constant coefficients $a_i > 0$ $(1 \leqq i \leqq n)$ satisfy the Hurwitz conditions

$$D_1 > 0, \dots, D_n > 0$$

for the n-th order equation

$$\lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0.$$

Herein the following definition is used:

$$D_1 = a_1,\ D_k = \begin{vmatrix} a_1 & a_3 & & a_{2k-1} \\ 1 & a_2 & & a_{2k-2} \\ 0 & a_1 & \dots & a_{2k-3} \\ 0 & 1 & & a_{2k-4} \\ & & \vdots & \\ 0 & 0 & \dots & a_k \end{vmatrix} \quad (k > 1,\ a_m = 0 \text{ for } m > n).$$

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Later on we have to consider the $(n+1)-th$ order equation

$$\lambda^{n+1} + a_1 \lambda^n + \dots + a_n \lambda + a = 0$$

taking a sufficiently small value $a > 0$ and the same values a_k $(1 \leqq k \leqq n)$ as above. It is evident that the corresponding Hurwitz conditions

$$D_k(a) > 0 \, (1 \leqq k \leqq n), \;\; D_{n+1}(a) = a \;\; D_n(a) > 0$$

can be fulfilled too because $D_k(a)$, $1 \leqq k \leqq n$, is a polynomial with respect to a having the initial value $D_k(0) = D_k > 0$.

The purpose of this paper is to prove

THEOREM 1. - Equation (1) admits at least one periodic solution of period ω if

(i) $|f(x)| \leqq F$ for all x

(ii) $f(x)$ sgn $x > 0$ for $|x| \geqq h > 0$

(iii) $|p(t)| \leqq m$, $|P(t)| = \left| \int_0^t p(\tau) d\tau \right| \leqq M$.

[Condition (iii) is equivalent to $P(\omega) = \int_0^\omega p(\tau)\, d\tau = 0$.]

The main tool of our proof will be the Leray-Schauder fixed point theorem (for example, see [4]).

a) Let $x(t)$, $0 \leqq t < T$, be a solution of equation (1); its derivative $x' = y$ may be considered as a solution of the linear $n-th$ order differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = p(t) - f(x(t)). \tag{2}$$

This equation is of the asymptotic stable type and a wellknown fact is that the following estimation holds for $0 \leqq i \leqq n-1$:

$$|y^{(i)}(t)| \leqq C\left[m + F + e^{-\varkappa t} \sum_{j=0}^{n-1} |y^{(j)}(0)| \right], \;\; t \in [0, T)$$

(C and $\varkappa$ positive constants only depending on the coefficients $a_1, \dots, a_n$).

A similar estimation may be written for $y^{(n)}(t)$.

Now, by a simple conclusion, the solution considered by us can be defined for all $t \geqq 0$ (i.e. $T = \infty$).

Furthermore we obtain (with a constant $C_0 > C$)

$$|y^{(i)}(t)| \leqq C_0(m+F) \text{ for } t \geqq T_0 \tag{3}$$

[T_0 sufficiently large and depending on the initial values $y^{(j)}(0)$].

b) Integrating equation (1) from $t' \geqq T_0$ to $t > t'$ we find

$$x(t) = x(t') - \frac{1}{a_n}[x^{(n)}(t) + a_1 x^{(n-1)}(t) + \dots + a_{n-1}x'(t)]$$

$$+ \frac{1}{a_n}[x^{(n)}(t') + a_1 x^{(n-1)}(t') + \dots + a_{n-1}x'(t')]$$

$$- \frac{1}{a_n}\int_{t'}^{t} f(x(\tau))\,d\tau + \frac{1}{a_n}[P(t) - P(t')]$$

$$\leqq x(t') - \frac{1}{a_n}\int_{t'}^{t} f(x(\tau))d\tau + \rho,$$

$$\rho = \frac{2}{a_n}[n\alpha\, C_0(m+F) + M], \ \alpha = \text{Max}(1, a_1, \dots, a_{n-1}).$$

If $x(t) \geqq h$ [and consequently $f(x(t)) > 0$] for this interval $t \geqq t'$ we obtain

$$x(t) \leqq x(t') + \rho$$

and furthermore

$$x(t) \leqq x(t') + \rho - \delta'(t - t'),$$

$$a_n \delta' = \{\inf f(x);\ x(t') \leqq x \leqq x(t') + \rho\}.$$

There must be a $t'' > t'$ such that

$$x(t'') = h,$$

and for $t \geqq t''$ the estimation

$$x(t) \leqq h + \rho$$

is valid.

A similar result may be derived for $-x(t)$.
Let be $T_1 > T_0$ sufficiently large and

$$k = \text{Max}\ (C_0, \frac{2n\alpha C_0}{a_n}, \frac{2}{a});$$

then we can summarize:

$$(4)\qquad \begin{cases} |x(t)| \leqq h + k(m + M + F) \\ |x^{(i)}(t)| \leqq k(m + F), \ 1 \leqq i \leqq n \\ \text{for } t \geqq T_1 . \end{cases}$$

c) Choosing (like above) a sufficiently small constant $a \in (0, \frac{1}{k})$ we investigate the more general equation

$$(5)\qquad x^{(n+1)} + a_1 x^{(n)} + \dots + a_n x' + ax = \mu[ax - f(x) + p(t)]$$

$$(0 \leqq \mu \leqq 1).$$

For $\mu = 1$ we obtain the original equation (1).

If on the right hand side the variable x is replaced by a continuous ω-periodic function $u(t)$ equation (5) becomes a non-homogeneous linear equation with a uniquely determined ω-periodic solution

$$(6)\qquad v(t) = \mu\, T\{u(t)\},\ 0 \leqq \mu \leqq 1.$$

The following interpretation is possible:
May B be the Banach space of all continuous functions $u(t)$, $0 \leqq t \leqq \omega$ with the boundary condition $u(0) = u(\omega)$ and the norm

$$\|u(t)\| = \operatorname*{Max}_{[0,\omega]} |u(t)|.$$

Then the operator T maps B into itself.
After replacing equation (5) by an equivalent first order system it is easily be shown that the mapping T is continuous and compact.

Applying the Leray-Schauder fixed point theorem we can state:
If there exists an *a priori* estimation for the fixed points of the operator $\mu T (0 \leqq \mu \leqq 1)$ i.e. for the ω-periodic solutions of equation (5), this equation will admit at least one ω-periodic solution for each value $\mu \in [0, 1]$, especially for $\mu = 1$.

REMARK. - If all solutions of equation (5) are ultimately bounded and the bound can be chosen independently on μ, then the needed *a priori* estimation follows immediately.

In order to apply the preceding boundedness result (4) we write instead of (5)

$$x^{(n+1)} + a_1 x^{(n)} + \dots + a_n x' + f_\mu(x) = \mu p(t), \tag{7}$$

$$f_\mu(x) = (1-\mu)\, a x + \mu f(x).$$

But the term $f_\mu(x)$, $\mu < 1$ doesn't fulfil condition (i). For this reason an auxiliary consideration is necessary.
May equation (7) possess an ω-periodic solution $x(t)$, and be

$$S = \|x(t)\| = \operatorname*{Max}_{[0,\,\omega]} |x(t)|.$$

Replacing $f_\mu(x)$ by the function

$$f^*_\mu(x) = \begin{cases} (1-\mu)\, a x + \mu f(x), & 0 \leqq |x| \leqq S \\ (1-\mu)\, a S \operatorname{sgn} x + \mu f(x), & |x| \geqq S \end{cases}$$

we obtain an equation which admits the same ω-periodic solution $x(t)$. This equation belongs to the class (1) because conditions (i) and (ii) are satisfied:

$$|f^*_\mu(x)| \leqq (1-\mu)\, a S + \mu F < a S + F \text{ for all } x;$$

$$f^*_\mu(x) \operatorname{sgn} x > 0 \text{ for } |x| \geqq h.$$

Applying the estimation (4) we conclude

$$S \leqq h + k(m + M + aS + F),$$

$$S \leqq \frac{h + k(m + M + F)}{1 - ka}. \tag{8}$$

This *a priori* bound for the ω-periodic solutions of (7) is only dependent on the parameters $a_1, \dots, a_n$, m, M, F, h of the system (1) and on the arbitrarily chosen value a; it is evident that we can finally set $a = 0$ if $\mu = 1$.

REMARK. - If $n = 0$ we have the differential equation

$$x' + f(x) = p(t). \tag{9}$$

It is rather easy to prove

THEOREM 2. - Equation (9) admits at least one ω-periodic solution if the (weaker) conditions are valid:

(i) $f(x)$ sgn $x > 0$ for $|x| \geqq h$

(ii) $|P(t)| \leqq M$ for all t.

The proof is based on a boundedness result for $x(t)$ [derived by integration of equation (9)] and on the Leray-Schauder fixed point theorem.

If $n = 3$ Theorem 1. is a generalization of some former results obtained by EZEILO [1] and SEDZIWY [5].

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