Convergence of Double Fourier Series (*).

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Summary. - *Let a 2π-periodic function $f(x, y)$ be continuous in some neighbourhood of the point (x, y) except possibly along finitely many lines $l_1, l_2, \ldots, l_k$ terminating at (x, y). The problem of convergence of the Fourier series of $f(x, y)$ at the point (x, y) is examined in some detail. It is established that under certain restrictions on the variation of $f(x, y)$, and also on the lines $l_1, l_2, \ldots, l_k$, the fourier series converges to a value bounded above by the limit superior, and below by the limit inferior of $f(x+u, y+v)$, $u, v \to 0$, this value depending on the manner in which the series is summed.*

1. - Preliminary Remarks.

The problem of convergence of the double FOURIER series of a function $f(x, y)$ at a point has been investigated by a number of authors but at the present time it is still lacking a complete solution. For some of the results on this topic, the reader is referred to the works of G. H. HARDY [7], L. TONELLI [10] and [11], and E. W. HOBSON [8]. In this paper, we extend some of these known results. In particular, all of the results referred to require the existence of $f(x^+, y^+)$, $f(x^+, y^-)$, $f(x^-, y^+)$ and $f(x^-, y^-)$ for convergence at the point (x, y), and thus require that in some neighborhood of this point, the function is coninuous except possibly along lines through this point, parallel to the coordinate axes. We remove this restriction and relax somewhat the definition of bounded variation in a neighborhood of this point.

This same problem is pretty well solved in the one dimensional case, and many of the results may be found in any standard treatise on the subject. Among these may be mentioned the works of N.K. BARY [1], and A. ZYGMUND [13]. Some of the results carry over from the one dimensional to the two dimensional case in a very obvious manner. Others, however, are much more difficult to establish because of the greater amount of work involved, and also because, for instance, in going to two dimensions, the definition of bounded variation loses its uniqueness. Some of the ways in which this concept may be defined can be found in a paper by C. R. ADAMS and J. A. CLARKSON [5].

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2. - Some Preliminary Definitions.

In the sequel, a rectangle with vertices at (a, c), (b, c), (a, d) and (b, d), $a \leq b$, $c \leq d$, is denoted by $[b, d; a, c]$, and in particular, for a fixed point (x, y), we let $I = I(x, y) = [x + \pi, y + \pi; x - \pi, y - \pi]$, $I(x; \delta) = [x + \delta, y + \pi; x - \delta, y - \pi]$, $I(y; \delta) = [x + \pi, y + \delta; x - \pi, y - \delta]$, and $N = N((x, y); \delta) = I(x; \delta) \cup I(y: \delta)$, so that N is a cross-neighborhood of the point (x, y). Also, we let $f(b, d; a, c) = f(b, d) - f(a, d) - f(b, c) + f(a, c)$.

Let $R = [b, d; a, c]$. For fixed y, $c \leq y \leq d$, the total variation of $f(x\ y)$, considered as a function of x on $[a, b]$, is denoted by $V(f(x); y, R)$, with $V(f(y); x, R)$ defined in a similar manner. Occasionally it will be convenient to consider these total variation functions over a point set consisting of a single line, say the line joining the points (a, y) and (x', y) in the case of $V(f(x); y, R)$. In such a case we use the notation $V(f(x); y, [a, x'])$ or $V(f(x); y, (a, x'))$, depending on whether or not the end points are included.

A function $f(x, y)$ is said to be of bounded variation in the TONELLI sense on a rectangle $R = [b, d; a, c]$ if $V(f(x); y, R)$ and $V(f(y); x, R)$ exist for almost all x, respectively for almost all y, and if the integrals

$$\int_a^b V(f(y); x, R)dx, \qquad \int_c^d V(f(x); y, R)dy$$

exist in the LEBESQUE sense. This definition is extended to any open or closed irregularly shaped region in an obvious manner. First, let R be any closed region bounded by a simple, closed path. For each real constant c, the line $y = c$ intersects R on a set of at most countably many closed intervals I_i, and for each y, let

$$V(f(x); y, R) = \Sigma_i V(f(x); y, I_i).$$

In a similar way, let

$$V(f(y); x, R) = \Sigma_j V(f(y); x, I_j).$$

If $V(f(x); y, R)$ and $V(f(y); x, R)$ exist for almost all y, respectively for almost all x, and if the integrals

$$\int_\alpha V(f(y); x, R)dx, \qquad \int_\beta V(f(x); y, R)dy,$$

where $\alpha = \{x : \exists y$ and $(x, y) \in R\}$ and $\beta = \{y : \exists x$ and $(x, y) \in R\}$, exist in the LEBESGUE sense, then $f(x, y)$ is said to be of bounded variation in the TONELLI sense on R.

This definition is extended to the case where R is not closed by defining $V(f(x); y, I_i)$ as the supremum over I'_i of $V(f(x); y, I'_i)$, where I is any closed interval in I_i. $V(f(y); x, I_j)$ is defined in a similar manner. We use this general form of definition of bounded variation in the TONELLI sense in the sequel. Furthermore, we will say that $f(x, y)$ is of bounded variation in the restricted TONELLI sense, or more simply of bounded variation T on R if, in addition, the families $\{V(f(x); y, R)\}$ and $\{V(f(y); x, R)\}$ are bounded uniformly for almost all y, respectively for almost all x, on R. Of course, we are mainly concerned with the behavior of the FOURIER coefficients, and since these do not change with a change of the function on a set of measure zero, we may equally well assume that the families $\{V(f(x); y, R)\}$ and $\{V(f(y); x, R)\}$ are bounded uniformly whenever $f(x, y)$ is of bounded variation T on R.

For an example of a function which is of bounded variation in the TONELLI sense but not of bounded variation T on a region R, let $R = [1,1; 0,0]$ and let $f(x, y) = 1/(x^{1/2} + y^{1/2})$ if $x + y \neq 0$, and let $f(0, 0) = 0$. It is easily seen that $f(x, y)$ is such a function.

Now suppose that $V(f(y); x, R)$ exists for almost all x and the integral

$$\int_{\alpha} V(f(y); x, R)dx$$

exists in the LEBESGUE sense, where α is defined as before. If nothing is assumed about the set $\{V(f(x); y, R)\}$, $f(x, y)$ will be said to be of bounded variation in the TONELLI sense with respect to y on R. If in addition the family $\{V(f(y); x, R)\}$ is bounded uniformly for almost all x, then $f(x, y)$ is said to be of bounded variation in the restricted TONELLI sense with respect to y on R, or more simply of bounded variation T with respect to y on R. Bounded variation with respect to x on R is defined in a similar manner.

3. - Convergence of Double Fourier Series at a Point.

Before stating the main result of this paper, we define an integral which we express as a function $\varphi(\theta)$.

(3.1) DEFINITION. - Let $R(\theta)$ be the region bounded by the positive u-axis, a ray from the origin making an angle θ with the positive u-axis, and the boundary of the rectangle $[a, b; -a, -b]$.

Then

$$\varphi(\theta) = \lim \int_{R(\theta)} \frac{\sin u}{u} \frac{\sin v}{v} du\, dv, \ a, b \to \infty.$$

That $\varphi(\theta)$ exists for every real value of θ and that it is a continuous function of θ follows at once from Lemma 1 of Section 5. We use it in proving our main result:

(3.2) THEOREM. - Let $R = [x+\delta, y+\delta; x-\delta, y-\delta]$, $\delta > 0$, and let the k non-intersecting paths to the point (x, y), $l_1, l_2, \ldots, l_k$, $l_{k+1} = l_1$ divide R into k open regions R_i, $i = 1, 2, \ldots, k$, all enumerated in a counter-clockwise sequence with R_i bounded by l_i, l_{i+1} and the boundary of R. Suppose that for each i, l_i has a limiting angle of approach to the point (x, y), say θ_i, $0 \leq \theta_i < 2\pi$, and suppose there exists an integer k' such that an arbitrary horizontal or vertical line crosses l_i, $i = 1, 2, \ldots, k$, at most k' times. If a 2π-periodic function $f(x, y)$ is absolutely integrable on I, if in the cross-neighborhood of (x, y), $N = N((x, y); \delta) = I(x; \delta) \cup I(y; \delta)$, $f(x, y)$ is of bounded variation T with respect to y on $I(x; \delta)$ and with respect to x on $I(y; \delta)$, and if $f(x, y)$ is continuous at every point of each open region R_i with $\sup \{ V(f(x); y, R_i) \} \to 0$ and $\sup \{ V(f(y); x, R_i) \} \to 0$ as $\delta \to 0$, then, holding p/q constant, the $mn-th$ partial sums $s_{mn}(x, y)$ of the FOURIER series of $f(x, y)$ converge to

$$u_1 f_1 + u_2 f_2 + \ldots + u_k f_k,$$

where

$$u_i = (1/\pi^2) \{ \varphi(\tan^{-1}(q/p \tan \theta_{i+1})) - \varphi(\tan^{-1}(q/p \tan \theta_i)) \}$$

$$f_i = \lim f(x+u, y+v), \qquad u, v \to 0, \qquad (x+u, y+v) \in R_i$$

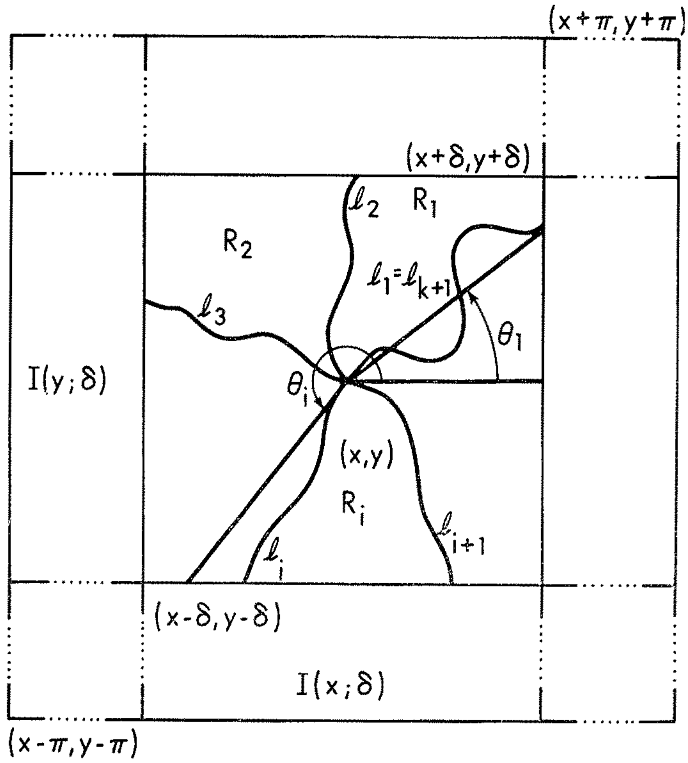
$$u_1 + u_2 + \ldots + u_k = 1$$

$$2p = 2m+1, \ 2q = 2n+1,$$

$\tan^{-1}(q/p \tan \theta)$ being taken in the same quadrant as θ. In particular, if $2m+1 = 2n+1$ as $m, n \to \infty$, then

$$u_i = (1/\pi^2) \{ \varphi(\theta_{i+1}) - \varphi(\theta_i) \}, \qquad i = 1, 2, \ldots, k.$$

Furthermore, u_i is independent of p/q if for some integer k'', $k'' = 0, 1, 2, 3$, $\theta_i = k''\pi/2$ and $\theta_{i+1} = (k''+1)\pi/2$, provided that for some fixed but otherwise arbitrary ε, $0 < \varepsilon < 1$, we have $1/\varepsilon \geq p/q \geq \varepsilon$ and in this case $u_i = 1/4$. If, in addition, l_i and l_{i+1} are rays from the point (x, y), each of which is parallel to one of the coordinate axes, and if $f(x, y)$ is also of bounded variation T with respect to y on $I(y;\ \delta)$, then the right hand restriction on p/q is removed. If it is also of bounded variation T with respect to x on $I(x;\ \delta)$, then the left hand restriction on p/q is removed also.



4. - Examples and Remarks.

In general, under the hypothesis of the theorem, the restriction on p/q cannot be removed. For let $g(x) = 0,\ -\pi < x < 0,\ g(x) = 1,\ 0 < x < \pi$, and let $h(x)$ be a continuous function with period 2π whose FOURIER series diverges to infinity at a point $x',\ -\pi < x' < 0$.
The FOURIER series of the function

$$f(x,\ y) = g(x)h(y) + g(y)h(x)$$

diverges at every point $(x',\ y)$ and $(x,\ x')$, and in particular, at the point $(x',\ x')$ in the square $S = (0, 0;\ -\pi,\ -\pi)$, over which the function is identically zero. On the other hand, for every point $(x,\ y) \in S$, all the conditions of the theorem are satisfied, so that we have convergence to the function value zero at each point in S provided that for some arbitrary, fixed $\varepsilon,\ 0 < \varepsilon \leq 1$, $1/\varepsilon \geq p/q \geq \varepsilon,\ m,\ n \to \infty$.

If we let $f(x, y) = g(x)h(y)$, then the FOURIER series of $f(x, y)$ converges at every point of the rectangle $(0, \pi; -\pi, -\pi)$ provided that $p/q \geq \varepsilon$. If $f(x, y) = h(x)g(y)$, then we have convergence at every point of the rectangle $(\pi, 0; -\pi, -\pi)$ provided that $1/\varepsilon \geq p/q$. Finally, if $f(x, y) = g(x)$, then $f(x, y)$ is of bounded variation T with respect to both variables in I, and the FOURIER series converges with the restriction on p/q removed entirely.

Since the coefficients u_i, $i = 1, 2, \ldots, k$, are functions of q/p, it follows that the particular value to which the FOURIER series converges depends on what value of p/q we choose, or to put in another way, it depends on the ray along which we sum the FOURIER series. This dependence can be shown directly in the case of some simple functions. We give two examples to illustrate the point.

EXAMPLE 1. - Let $f(x, y) = 1$, $(x, y) \in R = \{(x, y): 0 < y < x \text{ and } 0 < x < \pi\}$, let $f(x, y) = 0$, $(x, y) \notin R$, and let $f(x, y)$ be periodic with period 2π in each variable. The *mn-th* partial sum of the FOURIER series of $f(x, y)$, $s_{m,n}(x, y)$, evaluated at $(0, 0)$ is given by

$$s_{m,n}(0, 0) = 1/\pi^2 \int_{-\pi, -\pi}^{\pi, \pi} f(u, v) \frac{\sin pu}{u} \frac{\sin qv}{v} \, dudv + o(1)$$

$$= 1/\pi^2 \int_R \frac{\sin pu}{u} \frac{\sin qv}{v} \, dudv + o(1)$$

$$\to (1/\pi^2)\varphi(\tan^{-1}(q/p)), \qquad m, n \to \infty, \qquad p/q \text{ fixed},$$

since the transformation $u' = pu$, $v' = qv$ carries the line $v = u$ into the line $v' = (q/p)u'$. If we let $p/q = 1$, then $s_{mn}(0, 0) \to 1/\pi^2\varphi(\pi/4) = 1/8$, $m, n \to \infty$. On the other hand, if $p/q = \varepsilon$, then $s_{mn}(0, 0) \to 0$, $\varepsilon \to \infty$, and $s_{mn}(0, 0) \to 1/4$, $\varepsilon \to 0$, as $m, n \to \infty$. The geometric interpretation in this case is obvious.

EXAMPLE 2. - Suppose ε, $0 < \varepsilon < 1$, is given and suppose that in $I = [\pi, \pi; -\pi, -\pi]$, $f(x, y) = 1$ on the region $R = \{(x, y) : \varepsilon < |y/x| < 1/\varepsilon\}$, and $f(x, y) = 0$ otherwise. Geometrically, the region R is swept out by a line through the origin as this line swings from an angle of $\tan^{-1}\varepsilon$ to an angle of $\tan^{-1} 1/\varepsilon$, and again as it swings from an angle of $\pi/2 + \tan^{-1}\varepsilon$ to an angle of $\pi/2 + \tan^{-1} 1/\varepsilon$, all relative to the positive x-axis. As before, we have

$$s_{mn}(0,\ 0) = 1/\pi^2 \int_{-\pi,\ -\pi}^{\pi,\ \pi} f(u,\ v) \frac{\sin pu}{u} \frac{\sin qv}{v}\, dudv + o(1)$$

$$= 1/\pi^2 \int_R \frac{\sin pu}{u} \frac{\sin qv}{v}\, dudv + o(1)$$

$$\rightarrow 4/\pi^2 \{ \varphi(\tan^{-1} q/p\varepsilon) - (\tan^{-1} q\varepsilon/p) \}, \qquad p,\ q \rightarrow \infty,$$

since $\varphi(\theta'' + k\pi/2) - \varphi(\theta' + k\pi/2) = \varphi(\theta'') - \varphi(\theta')$, $k = 0,\ \pm 1,\ \pm 2,\ \ldots$. Now let $p,\ q \rightarrow \infty$. If at the same time, $p/q \rightarrow 0$ or $q/p \rightarrow 0$, then $s_{mn}(0,\ 0) \rightarrow 0$, and this is independent of the value ε, $0 < \varepsilon < 1$, that we choose. On the other hand, if $p/q = 1$ as $p,\ q \rightarrow \infty$ then $s_{mn}(0,\ 0) \rightarrow 4/\pi^2 \{ \varphi(\tan^{-1} 1/\varepsilon) - \varphi(\tan^{-1}\varepsilon) \neq 0$, and in this case we also have that $s_{mn}(0,\ 0) \rightarrow 1$, $p/q = 1$, $p,\ q \rightarrow \infty$, $\varepsilon \rightarrow 0$. In this case we note that the area of R, the area on which $f(x,\ y) = 1$, tends to the area of I as $\varepsilon \rightarrow 0$.

Finally, we remark that the function $\varphi(\theta)$ is monotonically increasing with θ for $\theta > 0$. Thus, the constants u_i, $i = 1,\ 2,\ \ldots,\ k$ are all positive, with the obvious implication that under the conditions of the theorem, the FOURIER series converges to some value in the interval $[m,\ M]$, where M and m are, respectively, the maximum and the minimum of the set of values $\{ f : i = 1,\ 2,\ \ldots,\ k \}$.

5. - Some Preliminary Lemmas.

To expedite the proofs of our results, we prove a few lemmas. The point of departure for our work is the function $\varphi(\theta)$ defined in § 3. We content ourselves here by proving the continuity of $\varphi(\theta)$. That it exists for every choice of θ, $-\infty < \theta < \infty$, can be shown by a modification of the proof of continuity, if such a proof should be required, but on the other hand, it is an immediate corollary of Lemma 1.

(5.1) LEMMA 1. - Let $R(\theta)$ be a region in the uv-plane bounded by the positive u-axis, a ray from the origin making an angle θ with the positive u-axis, and the boundary of the rectangle $[a,\ b;\ -a,\ -b]$, $a > 0$, $b > 0$. Let

$$\varphi(\theta) = \lim \int_{R(\theta)} \frac{\sin u}{u} \frac{\sin v}{v}\, dudv, \qquad a,\ b \rightarrow \infty.$$

The function $\varphi(\theta)$ is a continuous function of θ with $\varphi(k\pi/4) = |k|\,\pi^2/8$, $k = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots$.

PROOF. - The numerical values of $\varphi(\theta)$ follow immediately on applying the known result that $\int_0^\infty \frac{\sin u}{u}\,du = \pi/2$, and noting that the integrand is an even function in each variable, $\varphi(\pi/4) = 1/2\varphi(\pi/2)$, and for $\theta > 0$, $\varphi(\theta + \pi/2) = \varphi(\theta) + \pi^2/4$. It remains to prove the continuity of $\varphi(\theta)$, and it is sufficient to prove it for $0 \leq \theta \leq \pi/2$.

First consider the interval $(0, \pi/4]$ and let θ', θ'' be such that $0 < \theta'$, $\theta'' \leq \pi/4$. Let G be the region in the uv-plane bounded by the rays $v = u \tan\theta'$ and $v = u \tan\theta''$ from the origin, and let G be otherwise unbounded. Then

$$|\varphi(\theta') - \varphi(\theta'')| = \left| \int_G \frac{\sin u}{u}\frac{\sin v}{v}\,dudv \right|.$$

Let

$$\begin{aligned} \psi(u, v) &= \sin u \sin v, \qquad && (u, v) \in G \\ &= 0, && (u, v) \notin G. \end{aligned}$$

Then

$$|\varphi(\theta') - \varphi(\theta'')| = \left| \sum_{0,0}^{\infty,\infty} (-1)^{i+j} u(i, j) \right| \tag{5.2}$$

where

$$u(i, j) = \int_{0,0}^{\pi,\pi} \frac{\psi(u, v)}{(u + i\pi)(v + j\pi)}\,dudv.$$

Summing first by rows, it is clear that the sum $\sum_{i=0}^{\infty} (-1)^{i+j} u(i, j)$ does not exceed in absolute value the integral

$$\int_{0,0}^{\pi,\pi} \frac{\sin u}{u + i\pi}\,\frac{\sin v}{v + j\pi}\,dudv,$$

where the first square $[(i + 1)\pi, (j + 1)\pi;\ i\pi, j\pi]$ in the j-th row, any part of which belongs to G, is in the i-th column. For $i > 0$, $j > 0$, this is bounded by $4/(ij\pi^2)$, and since $i \geq j$, this in turn is bounded by $1/j^2$. If R is the region bounded by the lines $v = u \tan\theta'$, $v = u \tan\theta''$ and $v = n\pi$, where n is some positive integer, and if $A(R)$ is the area of R, then

$$\begin{aligned} |\varphi(\theta') - \varphi(\theta'')| &\leq \left| \int_R \frac{\sin u}{u}\,\frac{\sin v}{v}\,dudv \right| + \sum_{j=n}^{\infty} 1/j^2 \\ &\lessdot A(R) + \sum_{j=n}^{\infty} 1/j^2. \end{aligned} \tag{5.3}$$

Given any $\varepsilon > 0$, n many be chosen large enough so that the contribution of the sum on the right does not exceed $\varepsilon/2$. On the other hand, $A(R) \to 0$ as $\theta' \to \theta''$ or as $\theta'' \to \theta'$. Thus, for θ' and θ'' near enough, $A(R) < \varepsilon/2$, so that for some $\delta > 0$,

$$| \varphi(\theta') - \varphi(\theta'') | < \varepsilon, \qquad | \theta' - \theta'' | < \delta \qquad 0 < \theta',\ \theta'' \leq \pi/4.$$

This proves continuity of $\varphi(\theta)$ over the interval $(0, \pi/4]$.

Continuity of $\varphi(\theta)$ over the interval $[\pi/4, \pi/2)$ is proved in exactly the same manner except the region R is taken to be the region bounded by the lines $v = u \tan\theta'$, $v = u \tan\theta''$ and $u = n\pi$, and the summation on the right in (5.2) outside of R is performed first along the columns to obtain the analogue of (5.3),

$$| \varphi(\theta') - \varphi(\theta'') | < A(R) + \sum_{i=n}^{\infty} 1/i^2.$$

Since $\varphi(\theta) = \pi^2/4 - \varphi(\pi/2 - \theta)$, to complete the proof of the lemma, we need only consider the case where one of θ', θ'', say θ'', equals $\pi/2$, and $\theta' \to \theta''$, $\theta' < \theta''$. Here the region G is bounded by the positive v-axis, and the ray from the origin $v = u \tan\theta'$, and is otherwise unbounded, and R is bounded by these same two rays and the line $v = n\pi$.

With $u(i, j)$ defined as in (5.2), let θ' be fixed and let $k = [n\pi \tan\theta']$. As before, we get

$$(5.4) \qquad | \varphi(\pi/2) - \varphi(\theta') | = | \sum_{0,0}^{\infty,\infty} (-1)^{i+j} u(i, j) |$$
$$< A(R) + | \sum_{0,n}^{\infty,\infty} (-1)^{i+j} u(i, j) |,$$

where the summation with respect to j now runs from n to ∞.

To estimate the last sum, perform the summation by columns and obtain

$$| \sum_{n}^{\infty} (-1)^j u(0, j) | \leq \int_{0,0}^{\pi,\pi} \frac{\sin u}{u} \frac{\sin v}{v + n\pi} \, du dv$$
$$< 2/n.$$

$$| \sum_{n}^{\infty} (-1)^{i+j} u(i, j) | \leq \int_{0,0}^{\pi,\pi} \frac{\sin u}{u + i\pi} \frac{\sin v}{v + n\pi} \, du dv$$
$$< 1/in, \qquad i = 1, 2, 3, \ldots, k$$
$$< 1/in' \leq 1/i^2, \qquad i > k,$$

where for $i > k$, the first square $[(i+1)\pi, (j+1)\pi; i\pi, j\pi]$ in the i-th column, any part of which belongs to G, is in the n'-th row. Since we may clearly assume that $\theta' > \pi/4$, it follows that $n' \geq i$, from which we get the last inequality. Then from (5.4), we get

$$|\varphi(\pi/2) - \varphi(\theta')| < A(R) + 1/n \{ 2 + \sum_1^k 1/i \} + \sum_k^\infty 1/i^2.$$

Note that the right hand side above is also a uniform bound for the left hand side of the inequalities for all angles θ such that $\theta' \leq \theta \leq \pi/2$, for by increasing θ' in this range, $A(R)$ is diminished, and the contribution of the sum $\sum_{j=n}^\infty (-1)^{i+j} u(i, j)$, $i = 1, 2, 3, \ldots$, does not increase in absolute value.

Let $\varepsilon > 0$ be chosen and fix k so that $\sum_k^\infty 1/i^2 < \varepsilon/3$. Now choose n so that $1/n \{ 2 + \sum_1^k 1/i \} < \varepsilon/3$. Then for all θ' such that $[\tan \theta'] \geq k$, we have

$$|\varphi(\pi/2) - \varphi(\theta')| < A(R) + 2\varepsilon/3.$$

Let $\theta' \to \pi/2$. Then $A(R) \to 0$, and so for all θ' large enough, $A(R) < \varepsilon/3$, and for all such θ',

$$|\varphi(\pi/2) - \varphi(\theta')| < \varepsilon.$$

This completes the proof of the lemma.

(5.5) COROLLARY. - Let R be a region in the uv-plane bounded by the rays $v = u \tan \theta'$ and $v = u \tan \theta''$, $\theta'' > \theta'$, and let R be unbounded otherwise. Then

$$u(R) = \int_R \frac{\sin u}{u} \frac{\sin v}{v} \, dudv$$

exists as a finite, real number.

PROOF.

$$u(R) = \{ \varphi(\theta'') - \varphi(\theta') \}.$$

(5.6) REMARKS. - It is clear that $\varphi(\theta) = 0(\theta)$. Also $\varphi(\theta)$ is monotonically increasing for $\theta > 0$ as already indicated.

(5.7) LEMMA 2. - Let l' be a continuous path in the uv-plane, terminating at the origin, and let θ' be the limiting angle of approach of l' to the origin. Suppose also that there exists an integer k' such that any line l parallel to either coordinate axis will cross l' at most k' times. If r' is the ray $v = u \tan \theta'$, and R' is the region lying on one side of r' and bounded by l', r' and the boundary of the square $S = [\delta, \delta; -\delta, -\delta]$, then

$$\int_{R'} \frac{\sin pu}{u} \frac{\sin qv}{v} \, du dv = o(1), \qquad \delta \to 0, \qquad 1/\varepsilon \geq p/q \geq \varepsilon,$$

where ε is arbitrary but fixed, $0 < \varepsilon \leq 1$, and the $o(1)$ term tends to zero with δ, uniformly in p/q.

(5.8) REMARKS. - In general, there will be two such regions, one on each side of the ray r'.

If R' is contained in the interior of two adjacent quadrants, then clearly r' is in one of these quadrants. Since l' has a limiting angle of approach to the origin, δ can be chosen small enough so that l', and so also R', must eventually be contained in one quadrant. Thus we assume that R' is already contained in some one quadrant.

If R' is in some quadrant other than the first, then we can reflect it in a suitable manner about the appropriate coordinate axes, and reduce the proof of the lemma to a region R' in the first quadrant. This can be done without penalty since the integrand is an even function in each variable. Hence we assume that R' is already in the first quadrant.

PROOF. - Suppose first that $\theta' = \pi/2$ so that r' coincides with the positive v-axis, and R' is in the first quadrant. Let r be the ray $v = u \tan \theta$, terminating at the origin, touching l' at least at one point distinct from the origin in the interior of $S = [\delta, \delta; -\delta, -\delta]$, and such that l', restricted to S, lies entirely on one side of r. Since l' has a limiting direction of approach to the origin, it follows that $\theta \to \pi/2$ as $\delta \to 0$.

Since $1/\varepsilon \geq p/q \geq \varepsilon$, the transformations $u' = pu$, $v' = qv$ map the coordinate axes into the coordinate axes, the path l' into some new path, and the ray r into a new ray $v' = u'(q/p) \tan \theta = u' \tan \psi$. Because of the restriction imposed on p/q, the minimum value that q/p can take on is ε, so that the minimum slope the transformed ray will have is $\varepsilon \tan \theta$. Since $\theta \to \pi/2$ as $\delta \to 0$, it follows that the lower bound for the slope of the transformed ray may be taken arbitrarily large by choosing δ small enough.

Proceeding as in the proof of Lemma 1, let G be the region bounded by the positive v-axes, the transform of the path l', and the boundary of the transform of $[\delta, \delta; 0, 0]$. That is, G is the transform of the region R' under the transformation $u' = up$, $v' = qv$. Let R be that part of G below the line $v = n\pi$, where n is some integer. Then

$$(5.9)\qquad \left|\int\limits_{R'} \frac{\sin pu}{u}\,\frac{\sin qv}{v}\,dudv\right| = \left|\int\limits_{G} \frac{\sin u}{u}\,\frac{\sin v}{v}\,dudv\right|$$

$$\leq A(R) + k'/n\,\{\,2 + \sum_{1}^{k} 1/i\,\} + k' \sum_{k}^{\infty} 1/i^2,$$

where $k = [\tan\psi]$ and the factor k' appears because of the assumption that a vertical or horizontal line will cross l' at most k' times. Since k' is fixed, choose k large enough so that for a given $\varepsilon' > 0$, the contribution of the last term on the right in (5.9) does not exceed $\varepsilon'/3$. Now choose n so that the middle term does exceed $\varepsilon'/3$, and choose ψ so that $[\tan\psi] \geq k$. This can always be done since $\tan\psi = q/p \tan\theta \geq \varepsilon \tan\theta$, so that $\psi \to \pi/2$ as $\theta \to \pi/2$, and so as $\delta \to 0$.

Now choose δ so small so that $A(R) < \varepsilon'/3$. The left side of (5.9) then does not exceed $\varepsilon'/3 + \varepsilon'/3 + \varepsilon'/3 = \varepsilon'$, proving the lemma for the case considered.

The case where $\theta' = 0$ and R' is along the upper side of the positive u-axis is proved in a similar manner.

To complete the proof, suppose that the ray r, is oriented at some angle θ', $0 < \theta' < \pi/2$. With the ray r, $v = u\tan\theta$, as defined above, the transformation $u' = pu$, $v' = qv$ carries r' into the ray $v' = u'(q/p)\tan\theta'$, and r into the ray $v' = u'(q/p)\tan\theta$. The maximum difference in the slopes of these rays is $1/\varepsilon(\tan\theta' - \tan\theta)$, and since $\theta \to \theta'$ as $\delta \to 0$, this difference tends to zero with δ.

Given $\varepsilon' > 0$, choose k so that $k' \sum_{k}^{\infty} 1/i^2 < \varepsilon'/4$. Let G be the transform of the region R' and let R be the part of G below the line $v = k\pi$ or to the left of the line $u = k\pi$, or both. Then

$$(5.10)\qquad \left|\int\limits_{R'} \frac{\sin pu}{u}\,\frac{\sin qv}{v}\,dudv\right| = \int\limits_{G} \frac{\sin u}{u}\,\frac{\sin v}{v}\,dudv\,\Bigg|$$

$$\leq A(R) + 2k' \sum_{k}^{\infty} 1/i^2,$$

where the factor k' again appears because of the assumption that a horizontal or vertical line will cross l' at most k' times, and the factor 2 appears because a part of G may fall above and a part below the main diagonal in the transformed plane. The integral over the part of G in the complement of R is taken over individual squares of the type $[(i+1)\pi,\ (j+1)\pi;\ i\pi,\ j\pi]$, and the contributions over such squares are summed by columns above the main diagonal and by rows below the main diagonal.

The second term on the right in (5.10) does not exceed $\varepsilon'/2$. Since $A(R) \to 0$ as $\theta \to \theta'$, and so as $\delta \to 0$, uniformly in p/q for $1/\varepsilon \geq p/q \geq \varepsilon$, it follows that under this restriction, the left hand side does not exceed ε' if δ is taken small enough, uniformly in p/q. This completes the proof of Lemma 2.

(5.11) LEMMA 3. - Let R' be a region in the xy-plane, bounded by two non-intersecting paths l' and l'' terminating at a point (x, y), and the boundary of the neighbourhood S of (x, y), $S = [x + \delta, y + \delta; x - \delta, y - \delta]$. Suppose that l' and l'' have limiting angles of approach to (x, y), say θ' and θ'' respectively, $\theta'' > \theta'$, and there exists an integer k' such that a horizontal or vertical line will cross l' or l'' at most k' times. If $f(x, y)$ is defined, continuous and of bounded variation T on R', and if $\sup \{ V(f(x); y, R') \} \to 0$ and $\sup \{ V(f(y); x, R') \} \to 0$ as $\delta \to 0$, then

$$\lim \int_{R'} f(x + u, y + v) \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

$$= u'f' + o(1),$$

where the limit is taken as p, $q \to \infty$, p/q fixed, the $o(1)$ term tends to zero with δ, and

$$f' = \lim f(x + u, y + v), \qquad u, v \to 0, \qquad (x + u, y + v) \in R',$$

$$u' = u'(p/q, R) = \lim \int_R \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

$$= \varphi(\tan^{-1}(q/p \tan \theta'')) - \varphi(\tan^{-1}(q/p \tan \theta')),$$

$$p, q \to \infty,$$

the region R being the region bounded by the rays $v = u \tan \theta'$, $v = u \tan \theta''$, and the boundary of the square $[\delta, \delta; -\delta, -\delta]$.

PROOF. - We identify the point (x, y) with the origin in the uv-plane, making it convenient to identify a region R, having a prescribed configuration relative to the point (x, y), with the region R in the uv-plane, having the same configuration relative to the origin.

Assume that R' is contained in the first quadrant. Then

$$(5.12) \qquad \int_{R'} f(x+u,\ y+v)\frac{\sin pu}{u}\frac{\sin qv}{v}\,dudv$$

$$= f'\int_{R'}\frac{\sin pu}{u}\frac{\sin qv}{v}\,dudv$$

$$+\int_{R'}\{f(x+u,\ y+v)-f'\}\frac{\sin pu}{u}\frac{\sin qv}{v}\,dudv.$$

To estimate the first term, we draw the rays r', given by $v = u\tan\theta'$, and r'', given by $v = u\tan\theta''$, in the uv-plane, and get at most four regions, say R_1', R_2', R_1'' and R_2'', the first two being bounded by r' and l', and the last two by r'' and l'', with R_1' and R_1'' on the clockwise side of r' and r'' respectively, and R_2' and R_2'' on the counter-clockwise side of r' and r''. These regions are of the type described in Lemma 2, and we have

$$\int_{R'}\frac{\sin pu}{u}\frac{\sin qv}{v}\,dudv$$

$$=\int_{R}\frac{\sin pu}{u}\frac{\sin qv}{v}\,dudv$$

$$+\left\{\int_{R_1'}-\int_{R_2'}-\int_{R_1''}+\int_{R_2''}\right\}\frac{\sin pu}{u}\frac{\sin qv}{v}\,dudv.$$

By Lemma 2, each of the last four integrals is $o(1)$, $\delta\to 0$. Now let

$$u' = u'(p/q,\ R) = \lim\int_{R}\frac{\sin pu}{u}\frac{\sin qv}{v}\,dudv, \qquad p,\ q\to\infty;$$

where p/q is held constant, and get

$$(5.13) \qquad \lim f'\int_{R'}\frac{\sin pu}{u}\frac{\sin qu}{v}\,dudv$$

$$= u'f' + o(1), \qquad p,\ q\to\infty, \qquad p/q \text{ constant},$$

where the $o(1)$ term tends to zero with δ.

To estimate the contribution of the second term, set $g(u, v) = f(x + u, y + v) - f'$, and extend $g(u, v)$ to the square $S' = S'(\delta) = [\delta, \delta; 0, 0]$ by setting $g(u, v) = 0$, $(u, v) \notin R'$. If r is any ray in the first quadrant with end point at the origin, set

$$g(u(r), v(r)) = g^+(u(r), v(r)) - g^-(u(r), v(r)),$$

where g^+ and g^- have the usual meaning and we use the notation $(u(r), v(r))$ to indicate that u and v are restricted to vary along r: if θ is the angle between r and the positive u-axis, then $v = u \tan \theta$.

Above r, along lines parallel to the v-axis terminating on r, let $V^+(g(v); u(r), v)$ and $V^-(g(v); u(r), v)$ denote the positive and negative variation functions of $g(u, v)$, considered as functions of v alone for each fixed u, and set $V^+(g(v); u(r), v) = V^-(g(v); u(r), v) = 0$ below r. Below r, let $V^+(g(u); v(r), u)$ and $V^-(g(u); v(r), u)$ denote the positive and negative variation functions of $g(u, v)$, considered as functions of u alone along lines parallel to the u-axis, and set $V^+(g(u); v(r), u) = V^-(g(u); v(r), u) = 0)$ above r. Then

$$g(u, v) = g_1'(u, v) - g_2'(u, v)$$

where

$$g_1'(u, v) = g^+(u(r), v(r)) + V^+(g(v); u(r), v) + V^+(g(u); v(r), u)$$

$$g_2'(u, v) = g^-(u(r), v(r)) + V^-(g(v); u(r), v) + V^-(g(v); v(r), u)$$

is a decomposition of $g(u, v)$ into two non-negative functions with the property that above r, these functions are mononically increasing in v, and below r they are monotonically increasing in u.

For a fixed ray r, let $M(r) = \sup \{ g_1', g_2' \colon (u, v) \in S' \}$. Let $M(\delta) = \sup \{ M(r) \}$, where now the supremum is taken over all rays in the first quadrant with end point at the origin. Since f' exists, $\sup \{ |g(u, v)| : (u, v) \in S' \} \to 0$, $\delta \to 0$, and also $\sup \{ g^+(u(r), v(r)) \} \to 0$ and $\sup \{ g^-(u(r), v(r)) \} \to 0$, $\delta \to 0$. Since $\sup \{ V(f(x); y, R') \}$ and $\sup \{ V(f(y); x, R') \}$ tend to zero with δ, it follows that $\sup \{ V^+(g(v); u(r), v) \}$, $\sup \{ V^-(g(v); u(r), v) \}$, $\sup \{ V^+(g(u); v(r), u) \}$ and $\sup \{ V^-(g(u); v(r), u) \}$ all tend to zero with δ, uniformly in r. Thus also $M(\delta) \to 0$, $\delta \to 0$, uniformly in r.

Let r be fixed and let

$$\begin{aligned} g(u, v) &= \{ M(\delta) - g_2'(u, v) \} - \{ M(\delta) - g_1'(u, v) \} \\ &= g_1(u, v) - g_2(u, v). \end{aligned}$$

This is a decomposition of $g(u, v)$ into two functions, and from the foregoing we conclude that these functions have the following properties. Each function is non-negative, and for fixed r, each is monotonically decreasing in v above r, and monotonically decreasing in u below r. These functions are uniformly bounded in r, and the uniform bound, $M(\delta)$, tends to zero with δ.

Let $S'' = [p\delta, q\delta; 0, 0]$ be the transform of S' under the transformation $u' = pu$, $v' = qv$. Proceeding as in the proof of Lemma 1, we get

$$\int_{S'} g_1(u, v) \frac{\sin pu}{u} \frac{\sin qv}{v} dudv \tag{5.14}$$

$$= \int_{S''} g_1\left(\frac{u}{p}, \frac{v}{q}\right) \frac{\sin u}{u} \frac{\sin v}{v} dudv$$

$$= \sum_{0,0}^{\infty,\infty} (-1)^{i+j} u(i, j)$$

where

$$u(i, j) = \int_{0,0}^{\pi,\pi} g_1\left(\frac{i\pi + u}{p}, \frac{j\pi + v}{q}\right) \frac{\sin u}{i\pi + u} \frac{\sin v}{j\pi + v} dudv.$$

Summing by rows below the transform of the ray $v = (p/q)u$, and by columns above this transform, it is easily verified that the above sum does not exceed the quantity $2M(\delta)\{\pi^2 + \sum_1^{\infty} 1/i^2\}$ in absolute value. Since $M(\delta)$ tends to zero with δ, it follows that the left hand side in (5.14) tends to zero with δ uniformly in p/q.

The estimate for the integral

$$\int_{S'} g_2(u, v) \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

is obtained in a similar manner. Then

$$\int_{R'} \{ f(x + u, y + v) - f' \} \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

$$= \int_{S'} \{ g_1(u, v) - g_2(u, v) \} \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

$$= o(1) + o(1) = o(1), \ \delta \to 0.$$

Combining this with the estimate (5.13) of the contribution of the first term on the right in (5.12) completes the proof of the main part of the lemma, To complete the proof, it remains to show that the limit u' of the integral

$$\int_R \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

as $p, q \to \infty$, p/q constant, is given by

$$\varphi(\tan^{-1}(q/p \tan \theta'')) - \varphi(\tan^{-1}(q/p \tan^{-1}\theta')).$$

The transformation $u' = pu$, $v' = qv$ carries the region R, bounded by the rays $v = u \tan \theta''$ and $v = u \tan \theta'$, and the boundary of S' into the region R'', bounded by the rays $v = u(q/p) \tan \theta''$ and $v = u(q/p) \tan \theta'$, and the boundary of $S'' = [p\delta, q\delta; 0, 0]$. Since $p\delta, q\delta \to \infty$ as $p, q \to \infty$, the integral

$$\int_{R''} \frac{\sin u}{u} \frac{\sin v}{v} dudv$$

tends to

$$\varphi(\tan^{-1}(q/p \tan \theta'')) - \varphi(\tan^{-1}(q/p \tan \theta'))$$

by the corollary (5.5) to Lemma 1.

(5.15) COROLLARY. - Let $S = [x + \delta, y + \delta; x - \delta, y - \delta]$ be a neighborhood of the point (x, y), and suppose that k non-intersecting paths $l_1, l_2, \ldots, l_k$, $l_{k+1} = l_1$, terminating at (x, y), divide S into k open regions $R_1', R_2', \ldots, R_k'$, so that the boundary of R_i' is traced out by l_i, l_{i+1}, and the boundary of S, the paths l_i, $i = 1, 2, \ldots, k$ being enumerated in a counter-clockwise sequence. Suppose that for each i, l_i has a limiting angle of approach to (x, y), say θ_i, and there exists an integer k' such that a horizontal or vertical line will cross l_i at most k' times. If $f(x, y)$ is continuous in each open region R_i', and if for each i, $\sup\{V(f(x); y, R_i')\} \to 0$ and $\sup\{V(f(y); x, R_i')\} \to 0$ as $\delta \to 0$, then

$$\lim \int_S f(x + u, y + v) \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$
$$= u_1 f_1 + u_2 f_2 + \ldots + u_k f_k + o(1),$$

where the limit is taken as $p, q \to \infty$, p/q fixed, the $o(1)$ term tends to zero with δ, and for $i = 1, 2, \ldots, k$,

$$u_i = \varphi(\tan^{-1}(q/p \tan \theta_{i+1})) - \varphi(\tan^{-1}(q/p \tan \theta_i))$$

$$f_i = \lim f(x + u, y + v), \qquad u, v \to 0, \qquad (x + u, x + v) \in R_i',$$

$\tan^{-1}(q/p \tan \theta)$ being taken in the same quadrant as θ.

PROOF. - We have

$$\int_S f(x+u,\ y+v)\frac{\sin pu}{u}\,\frac{\sin qv}{v}\,dudv$$

$$=\left\{\int_{R_2'}+\int_{R_2'}+\ldots+\int_{R_k'}\right\}f(x+u,\ y+v)\frac{\sin pu}{u}\,\frac{\sin qv}{v}\,dudv.$$

Applying Lemma 3 to each of the integrals on the right completes the proof of the corollary. That $\tan^{-1}(q/p\tan\theta)$ must be taken in the same quadrant as θ follows immediately from the geometry of the transformation $u'=pu$, $v'=qv$.

(5.16) REMARKS. - We also have the trivial result that for each i, u_i exists as a real number, and for every choice of p/q,

$$u_1+u_2+\ldots+u_k=\pi^2.$$

(5.17) LEMMA 4. - Suppose that $f(x,\ y)$ is of bounded variation T with respect to x on the rectangle $R'=[x+\pi,\ y+\delta;\ x+\delta,\ y]$, $\delta>0$, and for some $\varepsilon>0$, $f(x,\ y)$ is of bounded variation T with respect to y on the part of R' above the line l' through the point $(x+\delta,\ y)$, with slope ε. Then

$$\int_{R'} f(x+u,\ y+v)\frac{\sin pu}{u}\,\frac{\sin qv}{v}\,dudv$$

$$=o(1),\qquad p,\ q\to\infty,\qquad p/q\geq\varepsilon.$$

PROOF. - Let l be any line through the point $(x+\delta,\ y)$ with slope not less than ε. As in the proof of Lemma 3, $f(x,\ y)$ can be expressed on R' as the difference of two functions,

$$f(x+u,\ y+v)=g_1(u,\ v)-g_2(u,\ v),$$

where the functions $g_1(u,\ v)$ and $g_2(u,\ v)$ are non-negative, monotonically decreasing in x below l and monotonically decreasing in y above l. Since $f(x,\ y)$ is bounded, of bounded variation T with respect to x on R' and with respect to y on the part of R' above l', these functions may be assumed to be uniformly bounded for every choice of line l through $(x+\delta,\ y)$ with slope not less than ε. Let $M(\varepsilon)$ be such a uniform bound.

Then

$$\int_{R'} f(x+u,\ y+v)\frac{\sin pu}{u}\,\frac{\sin qv}{v}\,dudv \tag{5.18}$$

$$= \int_{R'} g_1(u,\ v)\frac{\sin pu}{u}\,\frac{\sin qv}{v}\,dudv$$

$$- \int_{R'} g_2(u,\ v)\frac{\sin pu}{u}\,\frac{\sin qv}{v}\,dudv.$$

To estimate the contribution of the first integral on the right in (5.18), assume that p, q are fixed, $p/q \geq \varepsilon$, and $g_1(u,\ v)$ is decreasing monotonically in u below the line l'' defined by $v = (p/q)(u - \delta)$, and that it is decreasing monotonically in v above this line, and is non-negative. The transformation $u' = pu$, $v' = qv$ allows us to write

$$\int_{R'} g_1(u,\ v)\frac{\sin pu}{u}\,\frac{\sin qv}{v}\,dudv$$

$$= \int_{R} g_1\left(\frac{u}{p},\ \frac{v}{q}\right)\frac{\sin u}{u}\,\frac{\sin v}{v}\,dudv$$

where $R = [p\pi,\ q\delta;\ p\delta,\ 0]$ is the tranform of the region R'. This transformation carries the line l'' into the line l, defined by $v = u - p\delta$, with unit slope. The function $g_1\left(\frac{u}{p},\ \frac{v}{q}\right)$ remains bounded by $M(\varepsilon)$ and is monotonically decreasing in u below l and in v above l

Now choose integers a, b, c, such that

$$a\pi \leq p\delta < (a+1)\pi$$

$$b\pi \leq p\pi < (b+1)\pi$$

$$c\pi \leq q\delta < (c+1)\pi,$$

and let

$$\psi(u,\ v) = \sin u \sin v, \qquad (u,\ v) \in R$$

$$= 0, \qquad (u,\ v) \notin R$$

$$u(i,\ j) = \int_{0,\,0}^{\pi,\,\pi} g_1\left(\frac{i\pi}{p} + \frac{u}{p},\ \frac{j\pi}{q} + \frac{v}{q}\right)\frac{\psi(u,\ v)}{(i\pi + u)(j\pi + v)}\,dudv. \tag{5.19}$$

Then

$$\int_R g_1\left(\frac{u}{p}, \frac{v}{q}\right) \frac{\sin u}{u} \frac{\sin v}{v} dudv$$

$$= \sum_{a,0}^{b,c} (-1)^{i+j} u(i, j).$$

Performing the summation by columns above l and by rows below l, we get

$$\left|\sum_a^b (-1)^i u(i, 0)'\right| \leq M(\varepsilon) \int_{0,0}^{\pi,\pi} \frac{\sin u}{u + a\pi} \frac{\sin v}{v} dudv$$

$$< 2M(\varepsilon)/a$$

$$\left|\sum_{a+j}^b (-1)^{i+j} u(i, j)'\right| \leq M(\varepsilon) \int_{0,0}^{\pi,\pi} \frac{\sin u}{u + (a+j)\pi} \frac{\sin v}{v + j\pi} dudv$$

$$< M(\varepsilon)/j(a+j), \qquad j = 1, 2, \ldots, c$$

$$\left|\sum_0^c (-1)^{a+j} u(a, j)'\right| \leq M(\varepsilon) \int_{0,0}^{\pi,\pi} \frac{\sin u}{u + a\pi} \frac{\sin v}{v} dudv$$

$$< 2M(\varepsilon)/a$$

$$\left|\sum_i^e (-1)^{a+i+j} u(a+i, j)'\right| \leq M(\varepsilon) \int_{0,0}^{\pi,\pi} \frac{\sin u}{u + (a+i)\pi} \frac{\sin v}{v + i\pi} dudv$$

$$< M(\varepsilon)/i(a+i), \qquad i = 1, 2, \ldots, c.$$

Here the prime indicates that only that part of the first term which results from performing the integration (5.19) below the line l enters the sum in the case of summation by rows, and only that part of the first term which results from performing the same integration above l enters the sum in the case of summation by columns. Collecting the results, we have

(5.20)
$$\left|\int_R g_1\left(\frac{u}{p}, \frac{v}{q}\right) \frac{\sin u}{u} \frac{\sin v}{v} dudv\right|$$

$$< M(\varepsilon)\left\{4/a + \sum_1^c 1/j(a+j) + \sum_1^b 1/i(a+i)\right\}$$

$$< 2M(\varepsilon)\left\{2/a + \sum_1^\infty 1/i(a+i)\right\}.$$

Since $a \to \infty$ as $p, q \to \infty$, and the term inside the brackets tends to zero as $a \to \infty$, the left hand side in (5.20), and so the first term on the right in (5.18), is $o(1)$, $p, q \to \infty$.

In a similar manner, it can be shown that the right side in (5.20) is an upper bound for the second term on the right in (5.18) as well. Then

$$\int_{R'} f(x+u,\ y+v) \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

$$= o(1) + o(1) = o(1), \qquad p,\ q \to \infty, \qquad p/q \gtrsim \varepsilon.$$

This completes the proof of the lemma.

(5.21) REMARK. - If $f(x, y)$ is also of bounded variation T with respect to y on R', then $M(\varepsilon)$ may be replaced by a uniform bound on the functions $g_1(u, v)$ and $g_2(u, v)$. In such a case, the restriction $p/q \gtrsim \varepsilon$ can be removed. Taking this into account, we have

(5.22) COROLLARY 1. - Let $\varepsilon > 0$ be given and let $R = [x+\pi,\ y+\delta;\ x+\delta,\ y-\delta] \cup [x-\delta,\ y+\delta;\ x-\pi,\ y-\delta] = R_1 \cup R_2$. If $f(x, y)$, defined on R, is of bounded variation T with respect to x on R, and if on R_1 above the line through $(x+\delta, y)$ with slope ε and below the line through the same point with slope $-\varepsilon$, and on R_2 above the line through $(x-\delta, y)$ with slope $-\varepsilon$ and below the line through the same point with slope ε, $f(x, y)$ is is also of bounded variation T with respect to y, then

$$\int_{R} f(x+u,\ y+v) \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

$$= o(1), \qquad p,\ q \to \infty, \qquad p/q \gtrsim \varepsilon.$$

If, in addition, $f(x, y)$ is of bounded variation T with respect to y on R, then the restriction $p/q \gtrsim \varepsilon$ is removed.

PROOF. - Let $R = [x+\pi,\ y+\delta;\ x+\delta,\ y] \cup [x+\pi,\ y;\ x+\delta,\ y-\delta] \cup [x-\delta,\ y+\delta;\ x-\pi,\ y] \cup [x-\delta,\ y;\ x-\pi,\ y-\delta] = R'_1 \cup R'_2 \cup R'_3 \cup R'_4$. Then

$$\int_{R} f(x+u,\ y+v) \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

$$= \left\{ \int_{R_1} + \int_{R'_2} + \int_{R'_3} + \int_{R'_4} \right\} f(x+u,\ y+v) \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

$$= \int_{R_1'} \{ f(x+u,\ y+v) + f(x+u,\ y-v) + f(x-u,\ y+v)$$

$$+ f(x-u,\ y-v) \} \frac{\sin pu}{u} \frac{\sin qv}{v}\, dudv.$$

By the hypothesis, the functions $f(x+u, y+v)$, $f(x+u, y-v)$, $f(x-u, y+v)$ and $f(x-u, y-v)$ are of bounded variation T with respect to x on R_1', and with respect to y on the part of R_1' above the line through the point $(x+\delta, y)$ with slope ε. The corollary now follows by an application of the lemma and the remark (5.21).

(5.23) COROLLARY 2. - Let $\varepsilon > 0$ be given and let $R = [x+\delta, y+\pi; x-\delta, y+\delta] \cup [x+\delta, y-\delta; x-\delta, y-\pi] = R_1 \cup R_2$. If $f(x, y)$, defined on R, is of bounded variation T with respect to y on R, and if on R_1 on the right of the line through the point $(x, y+\delta)$ with slope $1/\varepsilon$ and on the left of the line through the same point with slope $-1/\varepsilon$, and on R_2 on the right of the line through the point $(x, y-\delta)$ with slope $-1/\varepsilon$ and on the left of the line through the same point with slope $1/\varepsilon$, $f(x, y)$ is also of bounded variation T with respect to x, then

$$\int_R f(x+u,\ y+v) \frac{\sin pu}{u} \frac{\sin qv}{v}\, dudv$$

$$= o(1), \qquad p,\ q \to \infty, \qquad 1/\varepsilon \geq p/q.$$

If, in addition, $f(x, y)$ is of bounded variation T with respect to x on R, then the restriction $1/\varepsilon \geq p/q$ is removed.

PROOF. - The proof of Corollary 2 is immediate since it is only a restatement of Corollary 1 with the roles of p, and q, x and y, u and v, and ε and $1/\varepsilon$ interchanged.

(5.24) LEMMA 5. - Suppose that $f(x, y)$ is defined on the square $I = [x+\pi, y+\pi; x-\pi, y-\pi]$, and suppose that for some $\delta > 0$, $f(x, y)$ is of bounded variation T with respect to x on $I(y; \delta)$ and with respect to y on $I(x; \delta)$, where $N = N((x, y); \delta) = I(x; \delta) \cup I(y; \delta)$ is a cross-neighborhood of the point (x, y). Then, given $\varepsilon > 0$, there exists $\delta' > 0$ such that on the cross-neighborhood $N' = N((x, y); \delta')$, $f(x, y)$ has the following properties. (1) The function $f(x, y)$ is of bounded variation T with respect to both variables on the square $S = [x+\delta', y+\delta'; x-\delta', y-\delta']$. (2) On $R' = [x+\pi, y+\delta'; x+\delta', y-\delta'] \cup [x-\delta', y+\delta'; x-\pi, y-\delta'] = R_1' \cup R_2'$, $f(x, y)$ is of bounded variation T with respect to x; on the parts of R_1' above the line through the point

$(x+\delta', y)$ with slope ε and below the line through the same point with slope $-\varepsilon$, and on the parts of R_2' above the line through the point $(x-\delta', y)$ with slope $-\varepsilon$ and below the line through the same point with slope ε, $f(x, y)$ is also of bounded variation T with respect to y. (3) On $R''=[x+\delta', y+\pi; x-\delta', y+\delta'] \cup [x+\delta', y-\delta'; x-\delta', y-\pi]=R_1'' \cup R_2''$, $f(x, y)$ is of bounded variation T with respect to y; on the parts of R_1'' on the right of the line through $(x, y+\delta')$ with slope $1/\varepsilon$ and on the left of the line through the same point with slope $-1/\varepsilon$, and on the parts of R_2'' on the right of the line through $(x, y-\delta')$ with slope $-1/\varepsilon$ and on the left of the line through the same point with slope $1/\varepsilon$, $f(x, y)$ is also of bounded variation T with respect to x.

PROOF. - That δ' can be chosen so that $f(x, y)$ is of bounded variation T with respect to both variables on S, with respect to x on R' and with respect to y on R'' is obvious, since by assumption, this is already true for the case $\delta' \leq \delta$. To prove the remaining part of the lemma, choose δ' so that $0<\delta' \leq \varepsilon\delta/(1+\varepsilon)$. Then S, the parts of R' on which $f(x, y)$ is required to be of bounded variation T with respect to y, and the parts of R'' on which $f(x, y)$ is required to be of bounded variation T with respect to x are all contained in the square $[x+\delta, y+\delta; x-\delta, y-\delta]$, so that the conclusions of the lemma follow.

6. - Proof of the theorem.

The proof of the Theorem now reduces to interpreting the results of Section 5.

We identify the point (x, y) in the xy-plane with the origin in the uv-plane, and a region R in the xy-plane, having a fixed configuration relative to the point (x, y), with a region R, having the same configuration relative to the origin in the uv-plane. Thus the region R has a fixed meaning without ambiguity in either plane, and this allows us to transfer the discussion from one plane to another penalty. We do this in the sequel without specific mention.

The mn-th partial sum of the FOURIER series of $f(x, y)$, $s_{mn}(x, y)$, evaluated at the point (x, y), is given by

$$s_{mn}(x, y)=\frac{1}{\pi^2}\int_I g(u, v)\frac{\sin pu}{u}\frac{\sin qv}{v}\,dudv \tag{6.1}$$

where $2m+1=2p$, $2n+1=2q$, I is the period square and

$$g(u, v)=f(x+u, y+v)uv/(4\sin u/2 \sin v/2).$$

If we let $\sin t/t = 1$ when $t = 0$, then it is easily seen that the variation properties of f and g over any region in I are identical. If f is of bounded variation T with respect to either variable on any region in I, then so is g, and conversely. Also, f is bounded, continuous and absolutely integrable over any such region if and only if g is bounded, continuous and absolutely integrable. Finally, we note that if R' is the region described in the statement of Lemma 3, then

$$\lim g(u, v) = \lim f(x + u, y + v) = f', \quad u, v \to 0, \ (u, v) \in R'.$$

From the foregoing, it follows that if we replace $f(x + u, y + v)$ by $g(u, v)$ in the statement of the lemmas and corollaries of the preceding section, then in each case the conclusions are valid.

We use the notation of Lemma 5 and denote the complement of $N = I(x, \delta) \cup I(y; \delta) = S \cup R' \cup R''$, relative to the period square $I = [x + \pi, y + \pi; x - \pi, y - \pi]$ by C. Fix p/q, then choose $\varepsilon > 0$, $1/\varepsilon \geq p/q \geq \varepsilon$. Since otherwise we could choose a smaller cross-neighborhood, we assume that on S, R' and R'', the function $f(x, y)$ already has the properties (1), (2) and (3), so that $g(u, v)$ has these properties in the image of these regions in the uv-plane. Then

$$s_{mn}(x, y) = 1/\pi^2 \left\{ \int_S + \int_{R'} + \int_{R''} + \int_C \right\} g(u, v) \frac{\sin pu}{u} \frac{\sin qv}{v} \, du dv. \tag{6.2}$$

Since $p, q \to \infty$ as $m, n \to \infty$ and $g(u, v)$, and so also $g(u, v)/(uv)$, is absolutely integrable on C, by the RIEMANN-LEBESGUE lemma we have

$$1/\pi^2 \int_C g(u, v) \frac{\sin pu}{u} \frac{\sin qv}{v} \, du dv = o(1), \qquad p, q \to \infty. \tag{6.3}$$

Next, under the hypothesis of Theorem 1, $f(x, y)$, and so also $g(u, v)$, satisfies the hypotheses of Lemmas 3 and 4. By the corollary to Lemma 3, and the foregoing remarks,

$$1/\pi^2 \int_S g(u, v) \frac{\sin pu}{u} \frac{\sin qv}{v} \, du dv = u_1 f_1 + u_2 f_2 + \dots + u_k f_k + o(1)' + o(1)'', \tag{6.4}$$

where the $o(1)'$, term tends to zero with δ, and the $o(1)''$ term tends to zero as $p, q \to \infty$, p/q fixed. If we denote the coefficients in the corollary to

Lemma 3 by u_i', then because of the factor $1/\pi^2$, the coefficients here are defined by $u_i = u_i'/\pi^2$, $i = 1, 2, \ldots, k$.

Finally, by the corollaries 1 and 2 to Lemma 4, we have

$$(6.5) \qquad 1/\pi^2 \int_{R'} g(u, v) \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

$$= o(1), \qquad p, q \to \infty, \qquad p/q \geq \varepsilon$$

$$(6.6) \qquad 1/\pi^2 \int_{R''} g(u, v) \frac{\sin pu}{u} \frac{\sin qv}{v} dudv$$

$$= o(1), \qquad p, q \to \infty, \qquad 1/\varepsilon \geq p/q.$$

Using the estimates (6.3) to (6.6) in (6.2) we have

$$(6.7) \qquad s_{mn}(x, y) = u_1 f_1 + u_2 f_2 + \ldots + u_k f_k + o(1)' + o(1)'',$$

$$p, q \to \infty, \qquad p/q \ \text{ fixed},$$

where the $o(1)'$ term tends to zero with δ and the $o(1)''$ term tends to zero as $p, q \to \infty$.

Let $\varepsilon' > 0$ be given. Choose δ so small that the $o(1)'$ term does not exceed $\varepsilon'/2$. Choose an integer k such that for $p > k$, $q > k$, the $o(1)''$ term does not exceed $\varepsilon'/2$. Then for all such p, q,

$$| s_{mn}(x, y) - u_1 f_1 - u_2 f_2 - \ldots - u_k f_k | \leq \varepsilon'.$$

Since $\varepsilon' > 0$ is arbitrary, we conclude the proof of the main result of the theorem remarking that by Section 5,

$$f_i = \lim f(x + u, y + v), \qquad u, v \to 0 \qquad (u, v) \in R_i,$$

$$u_i = 1/\pi^2 \{ \varphi(\tan^{-1}(q/p \tan \theta_{i+1})) - \varphi(\tan^{-1}(q/p \tan \theta_i)) \},$$

where $\tan^{-1}(q/p \tan \theta)$ is taken in the same quadrant as θ.

The remaning results of the theorem follow easily. We have $u_1 + u_2 + \ldots + u_k = 1/\pi^2 \varphi(2\pi) = 1$. That $u_i = 1/\pi^2 \{ \varphi(\theta_{i+1}) - \varphi(\theta_i) \}$ when $p/q = 1$ is obvious. Furthermore, u_i is independent of p/q if for some integer k'', $k'' = 0$, 1, 2, 3, $\theta_i = k''\pi/2$ and $\theta_{i+1} = (k'' + 1)\pi/2$, for then one of $\tan \theta_i$, $\tan \theta_{i+1}$ is zero and the other is infinite, so that $\varphi(\tan^{-1}(q/p \tan \theta_{i+1})) - \varphi(\tan^{-1}(q/p \tan \theta_i)) = \pi^2/4$ independently of what value we choose for p/q, $qp \neq 0$. However, in this case the $o(1)$ term in Lemma 3, and so the $o(1)'$ term above might not go to zero with δ uniformly in p/q unless the restriction

$1/\varepsilon \geq p/q \geq \varepsilon$ is retained. But then this error term is zero indentically in p/q if the paths l_i and l_{i+1} are rays from the point (x, y).

The restriction on p/q, $1/\varepsilon \geq p/q \geq \varepsilon$, arises in yet another instance. It arises in obtaining the estimate for the integral

$$\left\{ \int_{R'} + \int_{R''} \right\} g(u, v) \frac{\sin pu}{u} \frac{\sin qv}{v} \, dudv$$

by way of Lemma 4. In this case, this restriction is removed on the right only if $f(x, y)$ is also of bounded variation T with respect to y on R', and it is removed on the left only if $f(x, y)$ is of bounded variation T with respect to x on R''. For then in the first instance the uniform upper bound $M(\varepsilon)$ may be chosen independent of ε as regards Corollary 1, and in the second instance in may be chose independent of $1/\varepsilon$ as regard Corollary 2. In fact, it is sufficient to replace it by $M = \sup \{ V(f(x); y, I(y; \delta)) \} + \sup \{ V(f(y); x, I(y; \delta)) \} + \sup \{ | f(x', y') - f(x'', y'') | : (x', y'), (x'', y'') \in I(y; \delta) \}$ and $M = \sup \{ V(f(y); x, I(x; \delta)) \} + \sup \{ V(f(x); y, I(x; \delta)) \} + \sup \{ | f(x', y') - f(x'', y'') | : (x', y'), (x'', y'') \in I(x; \delta) \}$ in the respective cases. This completes the proof of the theorem.

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