Some properties of pseudo-Abelian varieties.

Memoria di LEONARD ROTH (a Londra).

Summary. - *This work is a study of the algebraic varieties which are invariant under continuous groups of automorphisms the trajectories of which are* PICARD *varieties.*

The present work continues and amplifies previous studies (**18**, **19**) of pseudo-Abelian varieties, i. e. the varieties which admit continuous groups of automorphisms whose trajectories are PICARD varieties, forming *congruences* or systems of index unity. Since the previous papers on this subject were written, a number of important results (which are described in § 1) have appeared; these have made it possible to remove the restriction (imposed in **18**) that the trajectories of the group should have general moduli.

We begin by recapitulating or establishing some properties of algebraic varieties which are required in the sequel. In the first place, from a new definition, due to B. SEGRE (**23**), of the canonical varieties $X_k(U_p)$ of a variety U_p, we deduce relations between the canonical systems of any two varieties U_p, U_p^* in biregular (n, 1) correspondence; these include the case $k=p-1$, which is classical, and also the case $k=0$, which has been dealt with in (**18**). We next (§ 3) survey briefly the PICARD varieties, and the Abelian varieties which are mapped on them by involutions of various kinds; among such varieties, those for which the associated involutions are free from coincidences are specially important in our work. We call these Abelian varieties of the *first species*; it follows from § 2 that their canonical varieties, whether effective or virtual, are all of order zero.

We then proceed (§ 4 *et seq.*) to consider the pseudo-Abelian varieties. The essential fact about such a variety W_p is that, in addition to the congruence $\{V_q\}$ $(1 \leq q \leq p-1)$ of trajectories, it contains a complementary congruence of varieties V_{p-q}; by means of these two congruences W_p can be mapped on a multiple variety W_p^* which may or may not be pseudo-Abelian. From this mapping we deduce inequalities for the numbers g_i of linearly independent i-ple integrals of the first kind; we also prove that the canonical varieties of W_p are either of order zero or else belong to the congruence of trajectories, and we obtain equivalences for the varieties of the latter type. It follows from this that the canonical invariants of W_p are all zero, and that the arithmetic genus of W_p is equal to $(-1)^{p-1}$.

A particularly simple class of pseudo-Abelian variety is that of the *quasi-Abelian* varieties, which have been studied in detail by Severi (§ 6). A more interesting class, which we examine in §§ 7, 8, consists of the *improperly Abelian* varieties which are mapped by superficially irregular involutions on Picard varieties. Another class (§ 9) comprises those types which can be mapped on multiple quasi-Abelian varieties; this includes the elliptic surfaces of geometric genus zero, which were first investigated systematically by Enriques (7).

The problem of classifying the improperly Abelian varieties of higher species — i. e. those types for which the associated involutions on the Picard varieties possess coincidences of various kinds — depends in part on the determination of their canonical systems; as will appear, such an investigation would require results in the theory of dilatations (24) which are not yet available. Again, the question (the importance of which was recently pointed out to the writer by Prof. Severi) of classifying the pseudo-Abelian varieties of given dimension p whose canonical varieties are all (effective or virtual) of order zero depends for its solution on that of the analogous problem for varieties (not necessarily pseudo-Abelian) of dimension less than p; and as yet this is known only for $p=2$. From this limited result, however, we obtain (§ 10) a new characterization of the Picard threefolds as *the only threefolds which admit a finite continuous group of automorphisms and for which the canonical surface and canonical curve are both effective of order zero.*

1. **Preliminary results. The canonical systems.** - We begin by recalling a number of results, some classical, others recent, which will be required in our work; and, among the latter, a remarkably simple definition, due to B. Segre (23), of the canonical systems of an algebraic variety. Let U_p be a non-singular algebraic variety, and let S_i be a *general* hypersurface of U_p, i. e. one capable of varying in an ∞^p linear system the generic member of which is non-singular, and which is such that any ∞^{k+1} subsystem possesses a pure Jacobian variety $J_k(S_i)$ of dimension k $(k=0, 1, ..., p-1)$: then, if S_i $(i=1, 2, ..., r;\ r>p-k)$ are any r general hypersurfaces of U_p, it may be shown that the canonical varieties $X_k(U_p)$ $(k=0, 1, ..., p-1)$ are given by the formula

$$(1)\quad X_k(U_p)=\Sigma J_k(S_i)-\Sigma J_k(S_i+S_j)+...+(-1)^{r-1}J_k(S_1+S_2+...+S_r).$$

This result may be extended to the case where S_i is *any* non-singular hypersurface of U_p, not necessarily general. We then introduce the *adjoint* varieties $A_k(S)$ of any such hypersurface S by means of the equivalences

$$(2)\qquad X_k(U_p)\equiv A_k(S)-X_k(S)\qquad (k=1, 2, ..., p-1),$$

where $X_{p-1}(S) = S$. In the case $k = 0$, which merits separate consideration, we may instead proceed as follows. Let S be a hypersurface belonging to a rational pencil $|S|$, of general character, with Jacobian set δ; then X_0 is defined inductively, for $p = 2, 3, \ldots$, by means of the equivalence

(3) $$X_0(U_p) \equiv \delta - 2X_0(S) - X_0(S^2).$$

If S belongs to an irrational pencil, of genus $\rho > 0$, and of general character, then it may be proved that

(4) $$X_0(U_p) \equiv \delta + 2(\rho - 1)X_0(S).$$

The order of the series $\{X_0(U_p)\}$, called the *Severi series* of U_p, is equal to $I + (-1)^p 2p$, where I is the ZEUTHEN–SEGRE invariant of U_p.

Corresponding to the value $k = p - 1$, we have the virtual impure canonical system $|X_{p-1}|$, with effective freedom $P_g - 1$ and virtual freedom $P_a - 1$, where P_g and P_a are respectively the geometric genus and arithmetic genus ([1]) of U_p; the system $|iX_{p-1}|$ $(i > 1)$, if effective, has freedom $P_i - 1$, where P_i is the ith plurigenus of U_p. We denote by Ω_0 the virtual grade of the system $|X_{p-1}|$, and by Ω_i $(i = 1, 2, \ldots, p-1)$ the virtual arithmetic genera of the varieties (X^{p-i}); in the case where $|X_{p-1}|$ is free from multiple base elements, these satisfy the relation. conjectured by SEVERI (**29**) and established by HODGE (**13**),

(5) $$(1 - (-1)^p)P_a = \Omega_0 - \Omega_1 + \ldots + (-1)^{p-1}\Omega_{p-1} + p - (-1)^p.$$

The character Ω_0 is one of the set of *canonical invariants* which are defined as the intersection numbers $[X_{i_1}X_{i_2}\ldots X_{i_r}]$ of the various canonical systems; here the suffixes $i_1, i_2, \ldots$ may take any values such that $i_1 + i_2 + \ldots + i_r = (r-1)p$, repetitions of any variety X_i being allowed. It was shown by TODD (**31**) that, subject to the hypothesis that P_a is an enumerative character of U_p, the number $P_a + (-1)^p$ is expressible as a homogeneous linear function, with constant positive coefficients, of the canonical invariants. TODD's relation for P_a has since been obtained by HIRZEBRUCH (**12**) without making use of any unproved hypothesis.

Among the transcendental characters of U_p we may note the numbers g_i $(i = 1, 2, \ldots, p)$ of linearly independent i-ple integrals of the first kind; of these, the *superficial irregularity* g_1 and the geometric genus g_p $(= P_g)$ are specially important. The numbers g_i satisfy the relation, due to KODAIRA (**14**),

(6) $$P_a = g_p - g_{p-1} + \ldots + (-1)^{p-1}g_1$$

Finally, we note the applications of these results to the case of a product variety. Supposing that $U_p = U_q \times U_{p-q}$, where U_q and U_{p-q} are both non-

([1]) It is now known that the various alternative definitions of P_a are all equivalent (See **12**, **15**).

singular, we may show (Todd, 31) that

$$X_k(U_p) = \sum_{h=0} X_h(U_q) \times X_{k-h}(U_{p-q}) \qquad (k = 0, 1, ..., p-1), \tag{7}$$

where it is to be understood that meaningless symbols are replaced by zeros. We also have the obvious formula

$$g_i = \sum_{j=0}^{i} g_j(U_q) g_{i-j}(U_{p-q}) \qquad (i = 1, 2, ..., p), \tag{8}$$

where meaningless symbols are replaced by zeros, and where we write $g_0(U_q) = g_0(U_{p-q}) = 1$.

Evidently both (7) and (8) may at once be extended to the case where U_p is the product of any number of non-singular varieties.

2. **Correspondence formulae.** — Suppose now that U_p is in $(n, 1)$ correspondence with an irreducible non-singular variety U_p^*; i. e. that U_p^* is mapped by an involution I_n of order n on U_p and, again, that U_p^* is a rational transform of U_p. An immediate consequence of this last remark is that if g_i^* denote the characters of U_p^* corresponding to g_i, then we have

$$g_i^* \leq g_i \qquad (i = 1, 2, ..., p). \tag{9}$$

In the cases $i = 1$, $i = p$, this result is familiar; for $i = 1$, it seems to have been stated explicitly for the first time by Bagnera and De Franchis (2).

We now impose the condition that the correspondence between U_p and U_p^* is *biregular*; i. e. that there are no fundamental elements, and that the coincidence locus on U_p and the branch locus on U_p^* are both pure, consisting of one or more irreducible non-singular hypersurfaces such that no two components of either locus intersect. We may represent these loci by the symbols

$$\Sigma (s-1) B_{p-1}^{(s)}, \qquad \Sigma (s-1) B_{p-1}^{*(s)},$$

respectively; here each number s must be a divisor of n, and the characters s may possibly assume different values.

We then have *the following equivalences for the canonical varieties of* U_p:

$$X_k(U_p) \equiv \overline{X}_k(U_p^*) + \Sigma (s-1) X_k(B_{p-1}^{(s)}), \tag{10}$$

where the bar over a symbol denotes the transform of the variety in question. This result follows readily from (1): for if S_i^* is any general hypersurface of U_p^* (in the sense of § 1), with corresponding hypersurface S_i, the Jacobian $J_k(S_i)$ evidently consists of the transform of the Jacobian $J_k(S_i^*)$, together with the composite variety $\Sigma (s-1) J_k(S_i B_{p-1}^{(s)})$. Thus S_i is likewise general; hence, taking r general hypersurfaces S_i^* on U_p^*, and applying (1), we obtain (10).

In particular, *if the correspondence between U_p and U_p^* is without coincidences, it follows that the systems* $\{X_k(U_p)\}$ *are the transforms of the corresponding systems* $\{X_k(U_p^*)\}$. Moreover, *each canonical invariant of U_p is equal to n times the corresponding invariant of U_p^**. Hence, also, by the TODD-HIRZEBRUCH relation (§ 1), the arithmetic genera P_a, P_a^* of U_p, U_p^* satisfy the equation

$$P_a + (-1)^p = n\{P_a^* + (-1)^p\}. \tag{11}$$

The above considerations cannot in general be applied directly in a case where either the branch locus of U_p^* or the coincidence locus of U_p or both possess components of dimension less than $p-1$. In this case we must first perform one or more dilatations (24) so as to convert such components into hypersurfaces, after which we can apply (10) to the transforms of U_p and U_p^* so obtained. The latter will of course contain exceptional subvarieties in addition to any that the original models may have possessed; and in the present state of the theory it is not possible to say precisely how the canonical systems of U_p and U_p^* will be affected by transformations of this kind.

2. **Picard varieties and Abelian varieties.** - An Abelian variety W_p is an irreducible variety the coordinates of whose generic point are expressible as Abelian functions, of genus p, of p independent variables $u_1, u_2, \ldots, u_p$. The *rank* r of W_p is defined as the number of points in the primitive period parallelepiped which correspond to the generic point of W_p. In the case $r=1$, W_p is called a PICARD variety.

It has been shown by SIEGEL and others [2] (3) that any PICARD variety can be cleared of singularities; and it can then be shown (4) that it is possible to remove also any exceptional subvarieties, thereby rendering the correspondence between the points of the variety and the incongruent sets (u_i) (1,1) and unexceptional. We thus obtain a model which we denote by V_p and which will be used in all that follows; the symbol V_q will be used with a similar meaning.

It is known (16) that a *general* PICARD variety V_p (i. e. one with general moduli) contains no PICARD subvarieties, but that, if V_p contains one such variety V_q, it must contain a congruence $\{V_q\}$ of varieties V_q, the congruence itself being *Picardian*, i. e. representable by the points of a PICARD variety; and, further, that V_p must contain a second Picardian congruence $\{V_{p-q}\}$ of PICARD varieties V_{p-q}. We shall call such a variety V_p *special of type* q (or $p-q$). This specialization process may be repeatedly applied to both V_q and V_{p-q} so as to yield, for example, a variety V_p containing p congruences of elliptic curves; in particular, V_p may be the product of p elliptic curves.

[2] Including Prof. SEGRE, in an unpublished work which the writer has been privileged to consult.

It was first remarked by SEVERI (30), from transcendental considerations, that *the canonical varieties of V_p are all effective of order zero.* A geometrical proof of this result would be desirable.

Next, we observe that *the characters g_i of V_p are given by the formula* $g_i = \binom{p}{i}$ $(i = 1, 2, \dots, p)$. In the case where V_p is the product of p elliptic curves this follows from the extension of (8) already alluded to: the result, for any V_p, is a consequence of the fact that the local topological structure of the latter variety is identical with that of the former. Hence, by (6), V_p has arithmetic genus $(-1)^{p-1}$ - a result first obtained by LEFSCHETZ (16).

Every variety V_p admits a completely transitive permutable continuous group of ∞^p automorphisms, represented by the equations

$$u_i' = u_i + a_i \qquad (i = 1, 2, \dots, p), \tag{12}$$

where the a_i are arbitrary constants; for a given set of these constants, equations (12) represent *a transformation of the first kind.* Conversely, every variety of dimension p which admits such a group is a PICARD variety V_p (4).

It is convenient to remark here that, if the group of automorphisms of the variety in question is only *generally* transitive, we obtain instead what SEVERI (27) has called a *quasi-Abelian variety* W_p. SEVERI has proved (28) that W_p is either birational or is the product of a PICARD variety V_q and a linear space S_{p-q}; thus W_p is a simple special case of the pseudo-Abelian varieties considered in § 4.

Turning now to the Abelian varieties W_p of rank $r > 1$, we first observe that, from the definition, it is clear that W_p can be mapped by a simple involution [3] of order r on a PICARD variety V_p; hence a classification of W_p can be obtained from a study of the various kinds of involution which V_p, whether general or special, can carry. This study is based on transcendental and group-theoretic methods which we shall now briefly describe.

In practice it proves necessary to consider involutions I_n whose orders n are multiples of r. The primary characteristic of any such involution I_n is the nature of the coincidence locus, which may have any dimension from 0 to $p-1$ inclusive; and, in our work, particular importance attaches to those involutions which are without coincidences. In the case where V_p is general, it may be shown that any simple involution, without coincidences, on V_p, is necessarily Picardian, and generable by a finite group of transformations of the first kind. ENRIQUES (7) has proved that *any variety which contains a Picard involution, without coincidences, is itself Picardian.* The remaining types of involution without coincidences which V_p can carry emerge from the general considerations which follow.

(3) If, however, degenerate Abelian functions are allowed in the definition of W_p, this may not be true (see § 8); but we shall not consider this case.

To begin with, we deduce from the correspondence between V_p and W_p that *the pure canonical and pluricanonical hypersurfaces of W_p, if effective, are all of order zero.* Thus the geometric genus and plurigenera of W_p satisfy the inequalities $P_g \leq 1$, $P_i \leq 1$. Next, it follows from § 2 that the characters g_i of W_p satisfy the inequalities $g_i \leq \binom{p}{i}$ $(i = 1, 2, \ldots, p)$; in particular, *the superficial irregularity q $(= g_1)$ is given by $q \leq p$.* It may be shown (**25**) that $q = p$ if, and only if, W_p is a PICARD variety. In the third place, it may be proved (**19**) that, if the involution I_n possesses ∞^{p-1} coincidences, then $P_i = 0$ (all i).

The group-theoretic method of classification, which was first applied systematically to the case $p = 2$ by BAGNERA and DE FRANCHIS (**2**), and also by ENRIQUES and SEVERI (**8**), is based on the theorem: *If W_p has some plurigenus greater than zero, then I_n can be generated by a finite group $\mathcal{G}_n$ of automorphisms of V_p.* This result, for the case $p = 2$, was established in two stages ([4]) by BAGNERA - DE FRANCHIS (**2**) and DE FRANCHIS (**6**) respectively; the extension, for $p > 2$, is due to ANDREOTTI (**1**). It does not follow that, if all the plurigenera of W_p are zero, I_n cannot be generated by a group $\mathcal{G}_n$ - in fact, examples of such groups are easily constructed; but it means that the systematic classification of the Abelian varieties has to be restricted to those types which have some plurigenus greater than zero.

Assuming, then, that I_n is generable by a group $\mathcal{G}_n$, we may show (**16, 17**), that $\mathcal{G}_n$ itself can be generated by a finite set of linear substitutions, each of which is of the form

$$u_i' = \sum_{j=1}^{p} a_{ij} u_j + b_i \qquad (i = 1, 2, \ldots, p), \tag{13}$$

where a_{ij} and b_i are constants. In the case where W_p has superficial irregularity $q > 0$, we may show further (**16, 17**) that q of the above relations may be taken to be

$$u_i' = u_i + b_i \qquad (i = 1, 2, \ldots, q). \tag{14}$$

LEFSCHETZ (**17**) has remarked that, by modifying suitably the period matrix of V_p, the remaining transformations of the set (13) may be reduced to the canonical form

$$u_j' = \varepsilon_j u_j + b_j \qquad (j = q + 1, q + 2, \ldots, p). \tag{15}$$

The constants ε_j, called the *multipliers* of the substitution, are all roots of unity other than unity itself.

([4]) Separated by an interval of nearly 30 years.

In the case where $\mathcal{G}_n$ is cyclic, all the transformations are represented by powers of (14) and (15); when $\mathcal{G}_n$ is non-cyclic, the generating transformations all have equations (14) in common.

Supposing then that W_p has superficial irregularity q $(0 < q < p)$ and virtual canonical hypersurface of order zero, we see from (14) that I_n cannot possess coincidences unless $b_i = 0$ (all i), in which case I_n must have ∞^q coincidences at least.

It is important to note that, *if I_n is free from coincidences, the canonical varieties of W_p are all (effective or virtual) of order zero; and W_p is superficially irregular*. The first statement is an immediate consequence of § 2; and it means incidentally that the transcendental considerations described above will apply to W_p. To prove the second statement we observe that, when $q = 0$, the equations (15), in which u_j is substituted for u_j', always admit solutions.

We shall call such a variety W_p an Abelian variety of *the first species*, it being understood that $q < p$, i. e. that W_p is not a PICARD variety.

4. **Pseudo-Abelian varieties.** - We consider a non-singular variety W_p which admits a permutable continuous group $\mathcal{G}$ of ∞^q automorphisms $(1 \leq q \leq p - 1)$. The trajectories of $\mathcal{G}$ constitute a congruence $\{ V_q \}$ of varieties V_q, the generic member of which is irreducible; each variety V_q is invariant under $\mathcal{G}$, and no two trajectories intersect.

Now when a group possesses trajectories, its transformations may be reflected in the trajectories in various ways: here we shall make the assumption that $\mathcal{G}$ acts transitively, without exceptions, on the generic V_q and, further, that V_q represents biunivocally and unexceptionally the transformations of $\mathcal{G}$; it follows therefore that V_q is a PICARD variety in the reduced form described in § 3; moreover, the operations of $\mathcal{G}$ are transformations of the first kind ([5]) on V_q.

We call W_p *a pseudo-Abelian variety of type* q; as particular cases we may mention the elliptic surfaces $(p = 2,\ q = 1)$, which have been studied in some detail, and the pseudo-Abelian threefolds $(p = 3,\ q = 1,\ 2)$, which have been considered in previous work (**20**, **21**, **22**).

It will appear shortly that, as a hypothesis of generality, we may assume that every irreducible trajectory of $\mathcal{G}$ is non-singular; but it will also appear that we may expect $\{ V_q \}$ to contain a certain aggregate of reducible members, each consisting of an irreducible non-singular PICARD variety counted with a certain multiplicity.

The first step in the discussion is to prove that W_p *contains a second congruence* $\{ V_{p-q} \}$, *Picardian or Abelian of the first species, of* ∞^q *birationally*

([5]) This follows from the fact (**7**) that $\mathcal{G}$ must be *simply* transitive on V_q.

equivalent varieties V_{p-q}. This is achieved exactly as in (18), where it was assumed that V_q was general; in any case, however, the method there employed yields a congruence $\{ V_{p-q} \}$ of varieties V_{p-q} which are transforms of one another under $\mathfrak{G}$, and which cut the generic V_q in sets of an involution i_d, of order $d = [V_q V_{p-q}]$, free from coincidences. If V_q is general, i_d is necessarily Picardian, otherwise i_d may be Abelian of the first species (§ 3). The number d, called the *determinant* of W_p, is an important character of the variety.

We shall assume that the generic V_{p-q} is non-singular and thus, as will be seen, we are led to suppose that every V_{p-q} is non-singular.

Conversely, it may be shown that *any variety* W_p *which contains a congruence* $\{ V_q \}$ *as defined above is pseudo-Abelian of type* q; the proof of this result, which is similar to that of the analogous theorem for elliptic surfaces (7), depends on the fact that transformations of the first kind are transferred from one variety V_q to another, and rationally determinable, by means of the involutions i_d, which can always be constructed.

We now proceed to obtain a mapping of W_p on a multiple variety W_p^* which is fundamental in what follows. First, in the case $d = 1$, the congruences $\{ V_q \}$, $\{ V_{p-q} \}$ are birationally equivalent to V_{p-q} and V_q respectively, so that we may map W_p on the product $V_q \times V_{p-q}$; and, by the assumptions already made, the representation is free from exceptional elements.

To obtain a mapping in the case $d > 1$, we first construct the variety $W_p^* = V_q^* \times V_{p-q}^*$, where V_q^* and V_{p-q}^* are birationally and unexceptionally equivalent to $\{ V_{p-q} \}$ and $\{ V_q \}$ respectively; such a variety contains two congruences which, without risk of confusion, we may denote by $\{ V_q^* \}$ and $\{ V_{p-q}^* \}$, the varieties V_q^* being either Picardian or Abelian of the first species, according to the nature of $\{ V_{p-q} \}$; in the former case W_p^* is a pseudo-Abelian variety of determinant unity. We now make correspond to the generic point of W_p^* the set of d points $(V_q V_{p-q})$, thereby obtaining a representation of W_p on the d-ple variety W_p^*.

Is this mapping each trajectory V_q corresponds to a d-ple variety V_q^* in a representation which is without branch points, since the involution i_d is free from coincidences. Hence the branch locus on W_p^* is either lacking altogether or else consists of a number of irreducible varieties belonging to the congruence $\{ V_q^* \}$, i. e. generated by varieties V_q^*; such varieties, which we shall suppose to be non-singular, may have any dimension varying from q to $p - 1$ inclusive, those of dimension q consisting of isolated varieties V_q^*. To each generator $V_{q,s}^*$, say, of an $(s-1)$-fold component of the branch locus, there corresponds a variety $V_{q,s}$ which is an $(s-1)$-fold element of the coincidence locus of the involution I_d defined by the sets $(V_q V_{p-q})$, and which is such that $s V_{q,s} \equiv V_q$; the numbers s $(2 \leq s \leq d)$ may *a priori* be any divisors of d.

Evidently each variety $V_{q,s}$ is itself a trajectory of the group $\mathcal{G}$, and is therefore Picardian; and it is mapped on the d/s-ple variety $V^*_{q,s}$ without the intervention of branch points. As a hypothesis of generality we shall assume that each $V_{q,s}$ is non-singular.

We shall denote by $(s-1)B^{*(s)}_h$ $(q \leq h \leq p-1)$ a typical $(s-1)$-fold component of the branch locus on W^*_p; to it there corresponds an $(s-1)$-fold component — $(s-1)B^{(s)}_h$, say — on the coincidence locus on W_p. The variety $B^{(s)}_h$, which we shall suppose to be non-singular, belongs to the congruence $\{V_q\}$; any two varieties $B^{(s)}_h$, $B^{(s)}_k$ which correspond to the same value of s may intersect, in which case their common part consists entirely of trajectories. Each variety $B^{(s)}_h$ is pseudo-Abelian of type q, and is mapped on the d/s-ple variety $B^{*(s)}_h$ without the intervention of branch points.

In the case where all the numbers h are equal to $p-1$, the correspondence between W_p and W^*_p is biregular if — as we shall suppose — W^*_p is non-singular. In any other case it follows from a result of SEVERI (26) that, if we require W^*_p to be non-singular, each variety $B^{(s)}_h$ for which $h < p-1$ must be mapped by a hypersurface on W^*_p, so that the correspondence possesses fundamental elements. However, with the hypotheses of generality we have made above, the representation of W_p on W^*_p has no other exceptional features than those already described.

Two immediate consequences of the mapping may be noted here. In the first place, the characters g_i, g^*_i of W_p, W^*_p satisfy the inequalities $g_i \geq g^*_i$ (§ 2), where the numbers g^*_i are given by (8); in particular, we have

$$g_1 \geq g_1(V^*_q) + g_1(V^*_{p-q}). \tag{16}$$

And since V^*_q is either Picardian or Abelian of the first species, it follows that *in all cases W_p is superficially irregular.*

In the second place, while the varieties $B^{(s)}_h$ are in general algebraically isolated, it may happen that there exist two varieties $B^{(s)}_h$, $B'^{(s)}_h$, corresponding to the same value of s, such that $sB^{(s)}_h \equiv sB'^{(s)}_h$; in that case W_p has *divisor* σ_h *greater than unity*, i. e. is endowed with torsion.

We conclude this section with some remarks concerning the construction and classification of W_p. In order to construct a projective model of W_p, it is first necessary to obtain a mapping of V_q on the d-ple variety V^*_q, which is Picardian or Abelian of the first species, as the case may be. The general problem involved here has not so far been studied except for $q=1$; however, the case where the representation on V^*_q is cyclic presents no difficulty, and is dealt with exactly as in (18), to which we may refer for further details.

The process of classifying the variety W_p, for given values of p and q, is based on the study of the involution j_d, say, which the congruence $\{V_q\}$ cuts on any V_{p-q}. The coincidences in j_d arise solely from the varieties $B^{(s)}_h$; thus each $B^{(s)}_h$ cuts V_{p-q} in a variety which is an $(s-1)$-fold coincidence locus of j_d. Next, we observe that V_{p-q} *is invariant under a permutable group* $\mathcal{G}_d$ *of automorphisms,* namely the transformations of $\mathcal{G}$ which leave each V_{p-q} invariant. It follows that our first task is to determine those varieties V_{p-q} which admit such groups of automorphisms, and to obtain the characters of the corresponding involutions j_d; for each involution j_d we shall have a determinate congruence $\{V_q\}$, since j_d and $\{V_q\}$ are birationally equivalent.

5. **The canonical systems of W_p.** - Since the canonical systems of an algebraic variety are invariant under any regular automorphisms, we should expect that those systems $\{X_k(W_p)\}$ which are not of order zero would belong to the congruence of trajectories. More precisely, we prove that: *The canonical systems* $\{X_k(W_p)\}$ $(k=0, 1, \ldots, p-1)$ *of any pseudo-Abelian variety* W_p *of type* q *are of order zero for all* $k<q$, *while for* $k \geq q$, *they satisfy the equivalences*

$$X_k(W_p) \equiv \overline{X}_k(W_p^*) + \Sigma(s-1)X_k(B_h^{(s)}) \quad (k=q, q+1, \ldots, p-1), \tag{17}$$

where each meaningless symbol is replaced by zero, and where $X_k(W_p)$ passes $(s-1)$-ply through each $B_h^{(s)}$ for which $h<k$.

The first part of this theorem has already been established in (18), so that we need only outline the proof here. This is by induction on p and k; we consider first a general pencil $|S|$ of hypersurfaces belonging to $\{V_q\}$ or, when $q=p-1$, the pencil $\{V_{p-1}\}$ itself. Since the varieties S and S^2 are both pseudo-Abelian of type q, it follows from the inductive hypothesis that $X_0(S)$ and $X_0(S^2)$ have order zero; we then show that the virtual number δ in (3) is likewise zero, whence, by (3), $X_0(W_p)$ has order zero. The result, for $k=1, 2, \ldots, q-1$, then follows from the inductive hypothesis and equation (2).

Suppose now that $k \geq q$; then, if the correspondence between W_p and W_p^* is biregular, (17) is merely a restatement of (10), the term $X_k(W_p^*)$ being evaluated by using (7). Since $W_p^* = V_q^* \times V_{p-q}^*$, where V_q^* is either Picardian or Abelian of the first species — so that $X_h(V_q^*)$ is effective or virtual of order zero for all $h<q$ — it follows that *the only (possibly) non-zero term in the expression for* $X_k(W_p^*)$ *is* $V_q^* \times X_{k-q}(V_{p-q}^*)$.

In the case where the correspondence is not biregular, the first term in the equivalence for $X_h(W_p)$ is still the transform $\overline{X}_h(W_p^*)$ of the corresponding canonical variety of W_p^*, but the rest of the former reasoning cannot now be employed; however, we can obtain the required result by applying (1) directly to W_p.

To this end consider, for $q<p-1$, a general hypersurface S belonging to the congruence $\{V_q\}$, it being understood that the definition of generality given in § 1 is now modified so as to apply only for $k \geq q$. We first note

that any Jacobian $J_k(S)$ $(k \geq q)$ necessarily belongs to $\{V_q\}$, and also that, if S contains a variety $sV_{q,s}$ (§ 4), the latter counts $(s-1)$ times as part of the Jacobian $J_q(S)$. Now any linear system $|S|$ of freedom $k+1$ cuts a variety $B_h^{(s)}$ in a linear system of the same freedom; then, provided that $k \leq h$ (all h), taking a sufficiently large number of general hypersurfaces belonging to $\{V_q\}$ and applying (1), we obtain (17); if instead there exists any variety $B_h^{(s)}$ such that $h < k$, this will be an $(s-1)$-fold component of $X_k(W_p)$.

We now use the inductive hypothesis that the result holds for all pseudo-Abelian varieties of dimension less than p; since each variety $B_h^{(s)}$ is pseudo-Abelian, the variety $X_k(B_h^{(s)})$ is either of order zero or belongs to the congruence $\{V_q\}$; and since $\overline{X}_h(W_p^*)$ obviously belongs to the congruence, the theorem is established for all values of k.

It now follows that *the canonical invariants of W_p are all zero.* For these invariants (§ 1) are the respective intersection numbers of appropriate sets of canonical varieties, and the latter either belong to the congruence $\{V_q\}$ or else have order zero. Hence also, *the arithmetic genus of W_p is equal to* $(-1)^{p-1}$; this is an immediate consequence of the last result and the TODD-HIRZEBRUCH relation (§ 1).

Consider in particular the system $|X_{p-1}|$; by (17), this contains each variety $B_h^{(s)}$ as $(s-1)$-fold component and passes $(s-1)$-ply through each $B_h^{(s)}$ for which $h < p-1$. Since the intersection of any number of varieties X_{p-1} cannot have dimension less than q, it is clear that $\Omega_0 = 0$, $\Omega_i = (-1)^{i-1}$ $(i = 1, 2, \dots, q)$. In the special case where the varieties $B_h^{(s)}$ are all absent, we obviously have $\Omega_i = (-i)^{i-1}$ $(i = 1, 2, \dots, p-1)$, a result which agrees with the SEVERI-HODGE relation (5). But in a case where $|X_{p-1}|$ possesses multiple base elements the proof of this result — assuming that it still holds — does not appear easy.

6. **Examples: the quasi-Abelian varieties** - We illustrate the preceding results by considering first a *quasi-Abelian variety* W_p (§ 3), i. e. a variety of the form $V_q \times S_{p-q}$ $(q > 0)$. It follows from § 3 that, in this case, $g_i = \binom{q}{i}$ (where it is understood that meaningless symbols are replaced by zeros); hence, in particular, W_p has superficial irregularity q. And since all the canonical varieties of V_q are effective of order zero, we have $X_k(W_p) = V_k \times X_{p-q}(S_{p-q})$ $(k = 0, 1, \dots, p-1)$. Now B. SEGRE (23) has shown that, for any linear space S_r, $X_h(S_r) \equiv (-1)^{r-h}\binom{r+1}{h+1}S_h$ $(h = 0, 1, \dots, r-1)$. From this result we deduce that

$$X_k(W_p) \equiv (-1)^{p-k}\binom{p-q+1}{k-q+1} V_q \times S_{k-q} \quad (k = 0, 1, \dots, p-1),$$

where meaningless symbols are replaced by zeros. Hence, for $k < q$, $X_k(W_p)$ has order zero; $X_q(W_p) \equiv (-1)^{p-q}(p-q+1)V_q$; while, for $k > q$, $X_k(W_p)$ is the sum of a number of quasi-Abelian varieties of type q, but is never of order zero. This means that, for all $k \geq q$, the variety $X_k(W_p)$, whether effective or virtual, is never of order zero, and always belongs to the congruence $\{V_q\}$.

In particular, putting $k = p-1$, we have $X_{p-1} \equiv -(p-q+1)V_q \times S_{p-q-1}$. Thus $P_g = P_i = 0$ (all i); and obviously $\Omega_0 = 0$, $\Omega_i = (-1)^{i-1}$ $(i = 1, 2, \ldots, p-1)$. And, as for all pseudo-Abelian varieties, we have $P_a = (-1)^{p-1}$.

As a second example we consider the case — of great importance in the theory of Abelian varieties — where W_p has *an effective canonical hypersurface of order zero*. In order to discuss this question we must assume that the model W_p which we have constructed is free from exceptional hypersurfaces (all that we actually know is that each trajectory is similarly free). In this case, then, there can be no varieties $B_{p-1}^{(s)}$ on W_p, since these would be fixed parts of $|X_{p-1}|$. And since, therefore, $X_{p-1}(W_p)$ is the transform of $X_{p-1}(W_p^*)$, of which — as we have already remarked — the only non-zero part (if any) is the variety $V_q^* \times X_{p-q-1}(V_{p-q}^*)$, it follows that V_{p-q}^* must have a canonical hypersurface of order zero; though this need not be effective.

The condition imposed on X_{p-1} does not however exclude the presence of coincidence loci $B_h^{(s)}$ $(h < p-1)$. For if these are dilated into hypersurfaces on a model $\overline{W}_p$, say, it can still happen that the sum of such hypersurfaces, each counted with the proper multiplicity, and the variety $\overline{X}_{p-1}(W_p^*)$, constitutes an exceptional hypersurface on $\overline{W}_p$ which, on removal, leaves an effective canonical hypersurface of order zero.

Suppose, however, that the involution I_d is free from coincidences; in this case, by § 2, we have

$$X_k(V_{p-q}^*) \equiv X_k(V_{p-q}) \qquad (k = 0, 1, \ldots, p-q-1).$$

It then follows that V_{p-q} likewise has a canonical hypersurface (effective or virtual) of order zero. It follows also that each canonical invariant of V_{p-q} is equal to d times the corresponding invariant of V_{p-q}^*.

If we suppose further that each variety $X_k(W_p)$ $(k = p-2, p-3, \ldots, q)$ has order zero, we see that *all the canonical varieties of V_{p-q}^*, and also those of V_{p-q}, have order zero.* These may be either effective or virtual.

So far we have assumed that $X_{p-1}(W_p)$ is effective; if instead we are merely given that this variety is virtual (of order zero) the discussion is more complicated, for in this case hypersurfaces $B_{p-1}^{(s)}$ will in general be present. (See §§ 10, 11).

7. Improperly Abelian varieties. - Let V_p be a Picard variety which is special of type q (§ 3), with complementary congruences $\{\overline{V}_q\}$, $\{\overline{V}_{p-q}\}$, say, of Picard varieties; and consider, on V_p, a (simple) involution I_n which is either free from coincidences or is such that the coincidence locus belongs to the congruence $\{\overline{V}_q\}$; thus no variety $\overline{V}_q$ which does not form part of the coincidence locus can intersect it. And suppose further that the generic $\overline{V}_q$ is not united in I_n.

It then follows that the variety W_p which maps I_n contains a congruence $\{V_q\}$ of varieties V_q, image of the congruence $\{\overline{V}_q\}$; the generic member V_q maps a set of n varieties $\overline{V}_q$, in general distinct from one another, so that V_q is itself a Picard variety. Also since the Picard congruence $\{\overline{V}_q\}$ is mapped on the n-fold congruence $\{V_q\}$, it follows that the latter is either Picardian or Abelian; the aggregate of branch elements (if any) of the correspondence arises from the varieties $\overline{V}_{q,s}$, say, which are loci of sets of n/s $(2 \leq s \leq n)$ points homologous to each other in I_n: corresponding to a variety $\overline{V}_{q,s}$ we have a reducible variety $sV_{q,s}$ $(\equiv V_q)$ such that $V_{q,s}$ is in $(1, n/s)$ correspondence with $\overline{V}_{q,s}$. Each variety $V_{q,s}$ is an $(s-1)$-fold element of the branch locus; and, by § 3, $V_{q,s}$ is itself a Picard variety.

Corresponding to the congruence $\{\overline{V}_{p-q}\}$ we have on W_p a congruence $\{V_{p-q}\}$ which maps the former in a correspondence without branch elements; hence $\{V_{p-q}\}$ is either Picardian or Abelian of the first species - and if $\{\overline{V}_{p-q}\}$ is general, the first alternative must hold. The varieties V_{p-q} are all birationally equivalent; in the case where I_n is without coincidences V_{p-q} is either Picardian or Abelian of the first species, while, if I_n possesses coincidences, V_{p-q} is Abelian (but not of the first species), being mapped by an involution on $\overline{V}_{p-q}$ whose coincidence locus is the intersection of $\overline{V}_{p-q}$ with the coincidence variety of I_n.

In any case we see that W_p is either special Picardian or pseudo-Abelian of type q: and that, except when V_q, V_{p-q}, $\{V_q\}$ and $\{V_{p-q}\}$ are all Picardian, the second alternative must hold.

In conclusion, then, *the variety W_p representing a simple involution I_n on a special Picard variety V_p of type q which is such that the coincidence locus (if any) belongs to the congruence $\{\overline{V}_q\}$ while $\{\overline{V}_q\}$ is not united in I_n, is either special Picardian or pseudo-Abelian of type q. In the latter case W_p contains an Abelian congruence $\{V_q\}$ of trajectories, image of $\{\overline{V}_q\}$, and a complementary congruence $\{V_{p-q}\}$, Picardian or Abelian of the first species, of Abelian varieties V_{p-q} which are Picardian or Abelian of the first species if, and only if, I_n is without coincidences.*

8. We now obtain a condition sufficient to ensure that *an Abelian variety W_p, which is mapped by an involution I_n on a Picard variety V_p, should be pseudo-Abelian of type q.* In § 7 we made no hypothesis concerning

the plurigenera of W_p: but we shall now require to assume that W_p *possesses an effective or virtual canonical hypersurface of order zero.* This is necessary in order that the group-theoretic considerations of § 3 should apply; for it is known that there exist Abelian varieties of plurigenera zero, representable by involutions I_n on PICARD varieties, which cannot be generated by groups $\mathcal{G}_n$ of automorphisms. And there exist also Abelian varieties (in the wider sense) of plurigenera zero which are not so representable. Consider, for example, a quasi-Abelian variety of type q, $W_p = V_q \times S_{p-q}$ $(q > 0)$: if this could be mapped by an involution on a PICARD variety V_p, the PICARD congruence $\{V_q\}$ corresponding to the birational congruence $\{V_q\}$ on W_p would be mapped on the latter without branch elements (since there are no trajectories $V_{q,s}$) - and this is impossible.

Suppose then that W_p is an Abelian variety, mapped on V_p by an involution I_n; and suppose further that W_p has some plurigenus greater than zero. Let W_p have superficial irregularity q $(0 < q < p)$: then it follows from § 3 that I_n must be generable by a finite group of automorphisms of V_p, the group being representable by a finite set of equations such as (14) and (15). Now it is obvious that equations (14) are invariant under a continuous group $\mathcal{G}$ of ∞^q transformations of the first kind, the trajectories of which are PICARD varieties; hence W_p is a pseudo-Abelian variety of type q. Moreover, the congruence $\{V_q\}$ of trajectories of $\mathcal{G}$ can arise only from a congruence $\{\overline{V}_q\}$ of varieties on V_p: and from the general theory of PICARD varieties we know that this congruence must be Picardian and that its members must be PICARD varieties: that is, V_p must be special of type q. We know also that V_p must contain a second PICARD congruence $\{\overline{V}_{p-q}\}$ of PICARD varieties $\overline{V}_{p-q}$: and this congruence gives rise to the complementary congruence $\{V_{p-q}\}$ on W_p which we have described in § 7.

Now it is clear that the involution (contained in or coincident with I_n) defined by (14) and (15) will admit coincidences if, and only if, $b_i = 0$ (all i): and in that case there are at least ∞^q coincidences, belonging to the congruence $\{\overline{V}_q\}$. Hence

Every variety W_p of superficial irregularity q $(0 < q < p)$ and with some plurigenus greater than zero, which is mapped by an involution on a Picard variety V_p, is pseudo-Abelian of type q: moreover, V_p must be special of type q, and the coincidence locus of the involution must belong to the congruence which maps the congruence of trajectories on W_p.

With the notation of § 4, we may map W_p on the d-ple variety $W_p^* = V_q^* \times V_{p-q}^*$, where, by what has been said, V_q^* and V_{p-q}^* are Abelian varieties of genera q and $p-q$ respectively: hence the coordinates of the generic point P^* of W_p^* are expressible as rational functions of the coordinates of two points, lying on Abelian varieties of genera q and $p-q$ respectively. It follows that the coordinates of the generic point P of W_p, which is mapped on the d-ple variety W_p^*, are expressible as algebraic functions of the coordinates of P^*, that is to say, as algebraic functions of Abelian functions of genus q and other Abelian functions of genus $p-q$. By virtue of this result (19) we may call W_p an ***improperly Abelian*** variety (6). In many cases, one or both of the congruences $\{V_q\}$, $\{V_{p-q}\}$ are themselves improperly Abelian, and then the genera of the Abelian functions required for the parametric representation can be lowered still further.

Conversely, if W_p is a pseudo-Abelian variety of type q which is also Abelian, and which is the image of an involution on a PICARD variety V_p, the latter must be special of type q; also the involution is either free from coincidences or else the coincidence locus belongs to the congruence which maps the trajectories on W_p. And W_p is necessarily improperly Abelian.

We may note here two interesting special cases:

(*i*) If, on the variety W_p, the varieties V_{p-q} are Picardian, then W_p is Abelian of the first species. For then the congruence $\{V_q\}$ cuts on V_{p-q} an involution which is necessarily Abelian, and so $\{V_q\}$ is itself Abelian. And in any case $\{V_{p-q}\}$ is either Picardian or Abelian of the first species (§ 4). We may thus map W_p on a PICARD variety V_p, which is special of type q, by an involution which is free from coincidences.

(*ii*) Suppose that W_p is an Abelian variety satisfying the conditions stated at the beginning of § 8, and that the corresponding congruence $\{V_q\}$ contains no trajectories $V_{q,s}$. This means that the involution cut by $\{V_q\}$ on any V_{p-q} is without coincidences, so that V_{p-q} is either Picardian or Abelian of the first species; moreover, if the associated congruence $\{\overline{V}_q\}$ is general, the first alternative must hold. If $\{V_q\}$ is Picardian, so also is V_{p-q}, since V_{p-q} then carries a PICARD involution free from coincidences (§ 3); and $\{V_{p-q}\}$ must be Abelian of the first species, since if this congruence were Picardian, W_p would also be Picardian. If instead $\{\overline{V}_q\}$ is special of some type, then $\{V_q\}$ may be Abelian of the first species.

9. **The case where W_p^* is quasi-Abelian.** - An important class of pseudo-Abelian varieties is obtained by supposing in § 4 that the congruence $\{V_q\}$ is *birational* and, more particularly, that it has *invariant order unity* (26); we may then take the variety V_{p-q}^* to be a linear space S_{p-q}^*, so that W_p^* is of the form $V_q^* \times S_{p-q}^*$. If we further assume that the congruence $\{V_{p-q}\}$ is Picardian, it follows that W_p^* is quasi-Abelian of type q (§ 3).

The variety W_p mapped in the usual way on W_p^* is of considerable interest; thus, for $p=2$, it includes the entire class of elliptic surfaces of geometric genus zero; more generally, for $q=p-1$, we have a variety W_p containing a rational pencil of trajectories V_{p-1}. The discussion of this case

(6) Thus, in particular, every Abelian variety of the first species is improperly Abelian.

is substantially the same as that of the particular case $p = 3$ (21), to which we may therefore refer for details. In what follows we shall suppose that $q < p - 1$.

To begin with, we remark that, since the involution cut by $\{V_q\}$ on any V_{p-q} is birational, it must possess coincidences. By the hypotheses made concerning the nature of $\{V_q\}$ and the correspondence between W_p and W_p^*, the coincidence locus on W_p must be a hypersurface to which corresponds, on W_p^*, the product of a Picard variety and a hypersurface in S_{p-q}^*. Since any two hypersurfaces of S_{p-q}^* intersect, and the base for hypersurfaces in S_{p-q}^* is a prime, there is no loss of generality in supposing that the coincidence locus on W_p is an irreducible variety $B_{p-1}^{(s)}$, corresponding to an irreducible variety $B_{p-1}^{*(s)}$ on W_p^*; $B_{p-1}^{(s)}$ is mapped on the d/s-ple variety $B_{p-1}^{*(s)}$ without branch points, so that, by § 2,

$$X_k(B_{p-1}^{(s)}) \equiv \overline{X}_k(B_{p-1}^{*(s)}) \qquad (k = 0,\ 1, \ldots,\ p-2).$$

Now $B_{p-1}^{*(s)}$ is of the form $V_q^* \times V_{p-q-1}^*$, where V_q^* is a Picard variety and V_{p-q-1}^* a non-singular primal of S_{p-q}^*, of some order n; so that $X_k(B_{p-1}^{*(s)}) = V_q^* \times X_k(V_{p-q-1}^*)$. All that we require, therefore, is to determine the canonical varieties of the primal V_{p-q-1}^*. These have been obtained by B. Segre (23); thus, let A, C denote respectively a prime and a non-singular primal, of order n, in S_r, so that $C \equiv nA$: then Segre has shown that $X_h(C) = \gamma(h)A^{r-h}$, where

$$\gamma(h) = \sum_{j=0}^{r-h-1} (-1)^{r-h-1-j}\binom{r+1}{r-h-1-j} n^{j+1}.$$

For the canonical varieties on W_p we then use the equivalences (§ 5)

$$X_k(W_p) \equiv \overline{X}_k(W_p^*) + (s-1)X_k(B_{p-1}^{(s)}) \qquad (k = 0,\ 1, \ldots,\ p-1),$$

where, by § 6,

$$X_k(W_p^*) \equiv (-1)^{p-k}\binom{p-q+1}{k-q+1} V_q^* \times S_{k-q}^*.$$

In particular, putting $k = q - 1$, we obtain

$$X_{p-1}(W_p) \equiv \overline{X}_{p-1}(W_p^*) + (s-1)B_{p-1}^{(s)},$$

where $X_{p-1}(W_p^*) \equiv -(p-q+1)V_q^* \times S_{p-q-1}^*$. Since $B_{p-1}^{(s)}$ is algebraically isolated on W_p, it follows that W_p has geometric genus zero. This result may be otherwise obtained by a simple induction argument. We consider, on W_p, the ∞^{p-1} system $|W_{p-1}|$ of varieties corresponding to the varieties $V_q^* \times S_{p-q-1}^*$; evidently these belong to the class of manifold at present under discussion. We know that, for $p = 2$, W_p has geometric genus zero; assuming, then, that a like property holds for W_{p-1}, we deduce that it holds for W_p. For if W_p had geometric genus greater than zero, the characteristic system of $| W_{p-1} |$ would necessarily be special, and this is impossible.

As regards the plurigenera of W_p, these may or may not be positive; it is a simple matter to construct examples for which $P_i = 0$ (all i) (C. f. 20).

We have so far assumed that $\{ V_q \}$ has invariant order unity; in a case where $\{ V_q \}$ is birational but of higher invariant order, it is possible to have a number of non-intersecting hypersurfaces $B_{p-1}^{(s)}$ corresponding to the same or different values of s. For V_{p-q}^{*} can then contain a pencil of varieties V_{p-p-1}^{*} free from base points, and we can take $B_{p-1}^{*(s)}$ to be of the form $V_q^{*} \times V_{p-q-1}^{*}$; the images of such varieties on W_p will be non-intersecting. Conversely, if the congruence $\{ V_q \}$ is birational and it is known that there exists a coincidence locus with non-intersecting components, or with components corresponding to different values of s, then $\{ V_q \}$ must have invariant order greater than unity.

10. **Applications to threefolds.** - We begin by recalling the properties of surfaces with canonical system or canonical series of order zero which we shall require below. Let W_2 be a surface, free from exceptional curves of the first kind, with effective or virtual canonical varieties X_1 and X_0: then a first result, due to ENRIQUES (7), states that, if W_2 has arithmetic genus -1, the surface is either elliptic (including elliptic scrollar) or Picardian. In either case W_2 has (absolute) linear genus 1; moreover, W_2 is improperly Abelian or Picardian according as X_1 is virtual or effective of order zero.

From this DANTONI (5) has deduced the striking theorem: *if X_0 has order zero, then W_2 is elliptic (including elliptic scrollar) or Picardian.* DANTONI has further shown that the SEVERI series of a surface, if of order zero, must be effective. As before, W_2 will be improperly Abelian or Picardian according as X_1 is virtual or effective of order zero.

While it seems unlikely that such precise results as the above could be established for threefolds, it is possible to restate some of them in a weaker form which admits of extension. Thus, if we recall (7) that any surface which admits a finite continuous group [7] of automorphisms is either scrollar, elliptic or Picardian, we may assert that

(*i*) any non-scrollar surface which admits a finite continuous group of automorphisms has arithmetic genus -1 and linear genus 1;

(*ii*) the only surfaces which admit a finite continuous group of automorphisms and which possess an effective canonical curve of order zero are the PICARD surfaces.

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(7) There is no loss of generality in assuming that the group is *algebraic* (C. f. 7).

Both these results, as we shall see, extend to threefolds. In order to state the corresponding theorems as concisely as possible, it will be convenient to introduce the following definitions. We shall say that a threefold is *scrollar* or *planar* according as it is birationally equivalent to a threefold generated by a congruence of lines or a pencil of planes. We shall now prove that

Any threefold which admits a finite continuous group of automorphisms and which is neither scrollar nor planar has arithmetic genus and linear genus unity.

The Picard threefolds are the only threefolds which admit a finite continuous group of automorphisms and which are endowed with effective canonical surface and canonical curve of order zero.

Here it is of course to be understood that the threefolds in question have been cleared of exceptional surfaces; at present it is not known whether such an assumption is restrictive.

For the proof of the above results we require the classification, due to HALL (11), of the threefolds which admit finite continuous groups of automorphisms; in the case where the group in question has dimension 1 or 2, it may be shown that the threefold is scrollar or planar or pseudo-Abelian of type 1 or 2; if the group has dimension 3, and is completely or generally transitive, the threefold is Picardian or quasi-Abelian, as the case may be, while if the group is intransitive one of the former possibilties must hold. And since every PICARD or pseudo-Abelian threefold has arithmetic genus 1 and (absolute) linear genus 1, the first of our theorems is established.

Turning now to the second, we remark that, if the threefold W_3 has an effective canonical surface of order zero, then it must be Picardian or pseudo-Abelian, since in all the other cases listed above we have $P_g = 0$; we have thus to examine the pseudo-Abelian threefolds, for which, in the notation of § 4, $q = 1, 2$.

Consider first the case $q = 2$; this gives the hyperelliptic threefold, examined in detail in (21); from the equivalences there given for the canonical systems, we see that, if the surface X_2 is effective of order zero, the pencil $\{V_2\}$ of trajectories must be elliptic and free from surfaces $V_{2,s}$. Hence $\{V_2\}$ cuts a curve V_1 of the complementary congruence $\{V_1\}$ in sets of an elliptic involution which has no coincidences, from which it follows that V_1 is elliptic. Also the congruence $\{V_1\}$ must be Picardian; for if it were Abelian of the first species, X_2 would be virtual, instead of effective, of order zero (21). Thus W_3 is a PICARD threefold.

Next, let $q = 1$; then, as in § 6, we observe that, if X_2 is effective of order zero, then there can be no surfaces $B_2^{(s)}$ in the congruence $\{V_1\}$ of trajectories; also, if X_1 is effective of order zero, there can be no curves $B_1^{(s)}$ either. And since X_2 cuts each surface of the elliptic pencil $\{V_2\}$ on W_3 in a canonical curve, it follows that V_2 must possess an *effective* canonical curve of order zero; hence, from the known classification (7) of surfaces, we can say that V_2 is either regular or Picardian.

Now the congruence $\{V_1\}$ cuts on V_2 an involution free from coincidences; if, then, V_2 is Picardian, $\{V_1\}$ is either Picardian or Abelian of the first species. Thus W_3 can be regarded as a pseudo-Abelian threefold of type 2, with $\{V_2\}$ as the pencil of trajectories; hence $\{V_1\}$ must be Picardian since, as remarked above, X_2 would otherwise be virtual of order zero. Thus W_3 is a PICARD threefold.

Suppose, in the second place, that V_2 is regular; then the involution cut by $\{V_1\}$ on V_2 must be regular of genus zero (7). Thus the image V_2^* of $\{V_1\}$ must be regular of genus zero; and, by § 5, in order that X_2 and X_1 should both have order zero, it is necessary that V_2^* should have linear genus unity and SEVERI series of order zero; which, by the above result of DANTONI, is impossible. Hence this case cannot arise.

In conclusion, we observe that the fact that $X_0(W_3)$ is of order zero is a *consequence* of the hypotheses of our theorem; this is in strict analogy with *(ii)*.

11. We shall now examine briefly the problems which arise in trying to extend the above results to the case of a threefold for which *(a)* X_2 is effective and X_1 virtual, of order zero, *(b)* X_2 and X_1 are both virtual of order zero.

In case *(a)* we evidently have $q = 1$, since, when $q = 2$, X_1 is always effective of order zero. Also, to secure the conditions stated, there must be no surfaces $B_2^{(s)}$, but a certain number of curves $B_1^{(s)}$ must be present. The congruence $\{V_1\}$ cuts on each surface V_2 an involution i_d whose coincidences arise from these curves; precisely, each curve $B_1^{(s)}$ cuts V_2 in d/s points which are $(s-1)$-ple coincidences of i_d.

As in § 10, V_2 is either Picardian or regular of genus 1. In the former case, i_d is of course Abelian and W_3 is Abelian of the first species, being mapped by an involution without coincidences on a PICARD threefold; thus, by § 2, both X_2 and X_1 certainly have order zero. But while X_2 is effective, since $X_2(W_3^*)$ is so, X_1 must be virtual, on account of the presence of the curves $B_1^{(s)}$. The values of the numbers d and s which are *a priori* possible, and the nature of the corresponding involutions i_d, may be inferred from the work of BAGNERA and DE FRANCHIS (2).

In the case where V_2 is regular, we should first have to obtain all the involutions with a finite (non-zero) number of coincidences which V_2 can carry; such involutions must be regular and be endowed with at most a canonical curve of order zero. The only cases that have so far been discussed are those for which the involutions have genus 1 (9, 10); and even these results need further analysis before they can be applied here.

Consider next problem (*b*). First, let $q=2$: then the equivalence for X_2 given in (21) shows that the pencil $\{V_2\}$ of trajectories must be either elliptic or rational. If it is elliptic, then the congruence $\{V_1\}$ must be Abelian of the first species, since otherwise X_2 would be effective of order zero; also there can be no surfaces $B_2^{(s)}$, since X_2 would otherwise have order greater than zero. And since $\{V_2\}$ then cuts on V_1 an elliptic involution without coincidences, it follows that V_1 is elliptic.

If instead $\{V_2\}$ is rational, the surfaces $B_2^{(s)}$ are certainly present, and the various numbers s must be such as to give a virtual surface X_2 of order zero; also $\{V_1\}$ is either Picardian or Abelian of the first species. In either case we find that $\{V_2\}$ cuts on V_1 a rational involution whose coincidences are such as to make V_1 elliptic. Thus in all cases W_3 is an Abelian threefold.

Next, let $q=1$: then, since X_2 cuts any surface of the pencil $\{V_2\}$ in a virtual or effective canonical curve (in this case of order zero), it follows that, if V_2 is regular, either $p_g=1$, or $p_g=0$ (in which case V_2 is an Enriques surface); and that, if V_2 is irregular, the surface is either Picardian or Abelian of the first species. In all cases we have to consider the possibility of both surfaces $B_2^{(s)}$ and curves $B_1^{(s)}$ being present, giving rise to united curves and united points in the corresponding involutions i_d. The only case for which the necessary data are at present available is that in which V_2 is Picardian; in the remaining cases only general statements concerning the nature of the relative congruences $\{V_1\}$ can be made.

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