On the determinant of an asymmetric hyperbolic region.

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dedicated to Prof. B. SEGRE.

Summary. - *A main problem in the geometry of numbers is the evaluation of the so-called determinant of various regions. The author derives a new estimate for the determinant of a certain two-dimensional region bounded by two hyperbolas and applies his result to a problem in the theory of automorphic star bodies.*

1. Introduction.

Let x, y be the Cartesian coordinates of a point in the plane. For given positive numbers a and b let $K_{a,b}$ denote the domain, determined by

$$K_{a,b}: -a \leqq xy \leqq b.$$

The main object of this note is to derive a new upper bound for the determinant of this domain.

We begin by recalling some of the usual definitions. Let M be an arbitrary domain in R_n and let $\wedge$ be any lattice in R_n, the determinant of which may be denoted by $d(\wedge)$. The lattice $\wedge$ is called M-admissible, if no point of $\wedge$, except for the origin, is an inner point of M. And the determinant of M, denoted by $\Delta(M)$, is defined as follows.

1°. if there exist M-admissible lattices, then $\Delta(M)$ is the lower bound of $d(\wedge)$, taken over all M-admissible lattices.

2°. if no such lattice exists, then $\Delta(M) = \infty$.

The determinant of the two-dimensional domain $K_{a,b}$ will be denoted by $\Delta(a, b)$. For reasons of symmetry and homogeneity one has the obvious relations

$$\Delta(a, b) = \Delta(b, a), \qquad (1)$$
$$\Delta(ca, cb) = c^2 \Delta(a, b) \quad if \quad c > 0.$$

Consequently, in order to evaluate or to estimate $\Delta(a, b)$, it is sufficient to consider the case

$$a = 1, \qquad b \geqq 1. \qquad (2)$$

In the following we always suppose that (2) *holds.*

In the case $b = 1$ one has the classical result

$$\Delta(1, 1) = \sqrt{5}. \qquad (3)$$

It is the merit of B. SEGRE to have first considered the more general, asymmetric case [1]. In the case (2) his result takes the following form.

THEOREM 1. - *If* $b \geqq 1$, *then*

$$\Delta(1,\ b) \geqq \sqrt{b^2 + 4b}; \tag{4}$$

furthermore in (4) *the equality sign holds if and only if* b *is a positive integer.*

We shall prove here that the inequality (4) can be sharpened as follows.

THEOREM 2. - *Suppose* $b \geqq 1$. *Let* β *be the smallest positive integer* $\geqq b$ *and let* σ *denote the fraction* $b/[b]$. *Then we have*

$$\Delta(1,\ b) \geqq \min\{\sqrt{\beta^2 + 4b},\ \sigma\sqrt{b^2 + 4b\sigma}\}. \tag{5}$$

The question whether or not the equality sign holds in (5) admits the following answer.

THEOREM 3. - *Let* β *and* σ *be defined as in theorem 2. Then in the relation* (5) *the equality sign holds if and only if either* b *satisfies the relation*

$$\sigma\sqrt{b^2 + 4b\sigma} \leqq \sqrt{\beta^2 + 4b}$$

or the fraction $(\beta + 2)/(\beta + 1 - b)$ *has an integral value.*

We put $(\beta + 2)/(\beta + 1 - b) = q$, so that $b = \beta + 1 - \frac{1}{q}(\beta + 2)$. For a fixed value of β the number q is restricted to the interval $\frac{1}{2}(\beta + 2) < q \leqq \beta + 2$. Consequently the condition that the fraction $(\beta + 2)/(\beta + 1 - b)$ is integral is equivalent with the following assertion:

there exist positive integers β and q such that

$$b = \beta + 1 - \frac{1}{q}(\beta + 2), \qquad \frac{1}{2}(\beta + 2) < q \leqq \beta + 2. \tag{6}$$

In particular b is of the form (6). if b is a positive integer. It is evident that the right hand sides of (4) and (5) are equal if and only if b is a positive integer, and that theorem 1 is included in theorems 2 and 3.

For the proof of theorems 2 and 3 it appears useful to consider a particular set of $K_{1,b}$-admissible lattices. Let $\mathfrak{B}$ be the boundary of $K_{1,b}$ and let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be the parts of $\mathfrak{B}$ which belong to the first and the second quadrant respectively. Further let P be the point $(-1,\ 1)$; clearly P lies on $\mathfrak{B}_2$. Now we denote by $\mathcal{H}_b(P)$ the set of those $K_{1,b}$-admissible lattices, which contain P as a lattice point. The principal part of the proof of theorems 2 and 3 consists in a proof of the following two lemmas (sections 2 and 3).

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(1) B. SEGRE, *Lattice points in infinite domains and asymmetric diophantine approximations*, « Duke Mathem. Journal », *12* (1945), 337-365.

LEMMA 1. - *Let $b \geqq 1$ be arbitrary. If Λ is a lattice of the set $\mathcal{H}_b(P)$, then*

$$d(\Lambda) \geqq \sqrt{\beta^2 + 4b}. \tag{7}$$

LEMMA 2. - *The set $\mathcal{H}_b(P)$ contains a lattice Λ_0 with*

$$d(\Lambda_0) = \sqrt{\beta^2 + 4b}, \tag{7'}$$

if and only if the fraction $(\beta + 2)/(\beta + 1 - b)$ has an integral value.

The method of proof of lemma 1 is similar to a reasoning of K. OLLERENSHAW and C. A. ROGERS given in the symmetrical case $b = 1$ (²). We note that lemma 1 does not imply that the set $\mathcal{H}_b(P)$ is not empty. But this fact, which, by the way, is wellknown, is an immediate consequence of lemma 2. For this lemma implies that the set $\mathcal{H}_b(P)$ is not empty for some values of b, for instance all positive integral values. Now the set $\mathcal{H}_{b_1}(P)$ contains the set $\mathcal{H}_b(P)$ if b and b_1 are any positive numbers with $1 \leqq b \leqq b_1$. Hence $\mathcal{H}_b(P)$ is not empty for any value of $b \geqq 1$ whatsoever.

Section 4 brings two further simple lemmas, whereafter in section 5 the proof of our theorems is given. In the last section we give an application to a pròblem in the theory of automorphic star bodies.

2. Proof of lemma 1.

Let Λ be an arbitrary lattice of the set $\mathcal{H}_b(P)$. The straight line through $O = (0, 0)$ and $P = (-1, 1)$ contains an infinity of lattice points with mutual distances $\sqrt{2}$. Consider the straight lines, which are parallel to this line and pass through a lattice point. Exactly one of these lines passes through the first quadrant and has a minimal distance to O. Call this line L. For any two points P_1 and P_2 of L we shall denote by $\delta(P_1, P_2)$ the difference between the abscissae of P_1 and P_2. For points of L, which also belong to Λ, this quantity takes all integral values.

Let $y = \lambda - x$ be the equation of L and let S and T be the points of intersection of L and the hyperbola $xy = -1$ (see fig. 1). Clearly $\delta(S, T) > 2$. Hence, by the above remark, the interior of the line-segment $\overline{ST}$ contains at least two points of Λ. Now, since Λ is $K_{1,b}$-admissible, these points do not belong to the interior of $K_{1,b}$. As a consequence, L intersects the hyperbola $xy = b$, in two different points, Q and R say (see fig. 1). Furthermore the line segment $\overline{QR}$ contains at least two lattice points. So we have

$$\delta(Q, R) \geqq 1. \tag{8}$$

(²) KATHLEEN OLLERENSHAW, *On the minima of indefinite quadratic forms*, « Journal London Math. Soc. », *23* (1948), 148-153.

Solving for x and y the pair of equations

$$xy = -1, \qquad y = \lambda - x,$$

and also the pair of equations

$$xy = b, \qquad y = \lambda - x,$$

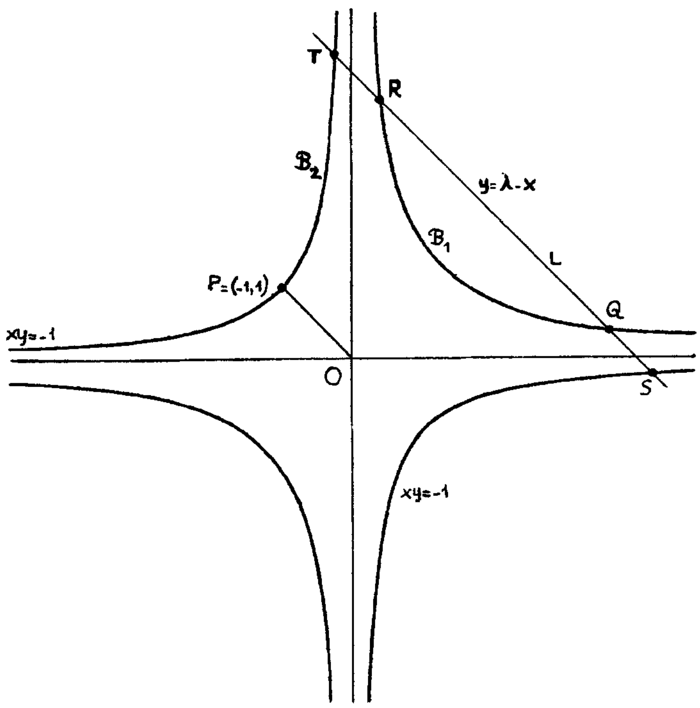

Fig. 1.

we find for the coordinates of the points S, T, Q, R

$$x_S = y_T = \frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 + 4}, \quad y_S = x_T = \frac{1}{2}\lambda - \frac{1}{2}\sqrt{\lambda^2 + 4}, \tag{9}$$

$$x_Q = y_R = \frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 - 4b}, \quad y_Q = x_R = \frac{1}{2}\lambda - \frac{1}{2}\sqrt{\lambda^2 - 4b}, \tag{10}$$

with obvious notations. Consequently

$$\delta(S,\ T)] = \sqrt{\lambda^2 + 4} \tag{11}$$

$$\delta(Q,\ R) = \sqrt{\lambda^2 - 4b}. \tag{12}$$

Let us now suppose that k is a positive integer with the following properties:

1° $1 \leq k < \beta$
2° $\delta(Q,\ R) \geq k$.

Then it follows from (12) that we have

$$\lambda^2 \leqq k^2 + 4b.$$

From the definition of β and the requirement 1° we learn that:

$$k < b;$$

hence, using (11), we deduce

$$\delta(S,\ T) \geqq \sqrt{k^2 + 4b + 4} > \sqrt{k^2 + 4k + 4},$$

and so

$$\delta(S,\ T) > k + 2.$$

Consequently, the interior of the line-segment $\overline{ST}$ contains at least $k+2$ points of Λ. Since Λ is $K_{1,b}$-admissible, these points must all belong to the segment $\overline{QR}$.

This leads to

$$\delta(Q,\ R) \geqq k + 1.$$

Clearly, in virtue of (8), the requirements 1° and 2° are fulfilled with $k=1$, unless $b=1$. So, by a repeated application of the above reasoning, we conclude that:

(13) $$\delta(Q,\ R) \geqq \beta,$$

the closed line-segment $\overline{QR}$ containing at least $\beta+1$ lattice points. In view of (12), the last formula is equivalent with

(14) $$\lambda \geqq \sqrt{\beta^2 + 4b}.$$

The determinant $d(\Lambda)$ is easily found to be equal to λ. So, finally, (14) is equivalent with (7). This proves lemma 1.

3. Proof of lemma 2.

We begin with some preliminary remarks. Let λ be determined by

(15) $$\lambda = \sqrt{\beta^2 + 4b},$$

and let L be the line with equation $y = \lambda - x$. Denote by S, T and Q, R the points of intersection of L and the hyperbola $xy = -1$, $xy = b$ respectively (see fig. 2). Further let P^* be the point $(1, -1)$, which is the reflection of P in the origin and also lies on $\mathfrak{B}$. As in the proof of lemma 1, the coordinates of the points S, T, Q, R are given by (9) and (10), with λ given by (15). In particular we have

$$0 < x_R = \frac{1}{2}\sqrt{\beta^2 + 4b} - \frac{1}{2}\beta$$

$$< \frac{1}{2}\sqrt{\beta^2 + 4\beta + 4} - \frac{1}{2}\beta = 1,$$

hence

$$0 < x_R < x_{P^*}.$$

Consequently, the straight line through P and R, M say, intersects $\mathfrak{B}_1$ in one further point and $\mathfrak{B}_2$ in exactly one point; call these points R_1 and T_1 successively (see fig. 2).

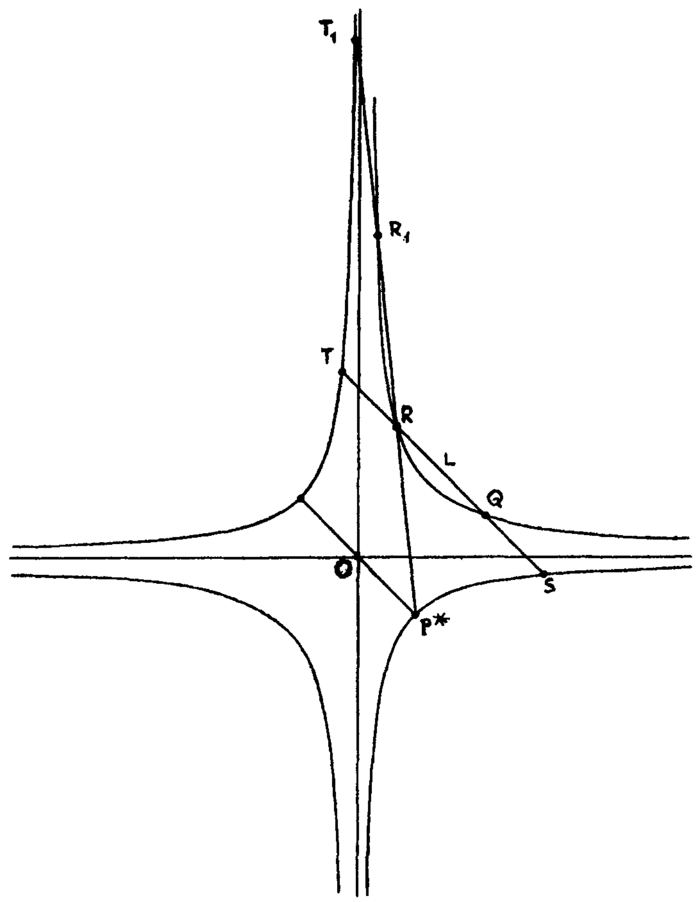

Fig. 2.

We wish to determine these points R_1 and T_1. An arbitrary point Z of the line M can be written as

$$Z = P^* + t(R - P^*), \tag{16}$$

where t runs through all real numbers. This point lies on the hyperbola $xy = b$, if and only if t satisfies the equation

$$\{1 + t(x_R - 1)\} \cdot \{-1 + t(y_R + 1)\} = b.$$

On account of (10) and (15) this relation reduces to

$$\left\{1+t\left(\frac{1}{2}\sqrt{\beta^2+4b}-\frac{1}{2}\beta-1\right)\right\}\cdot\left\{-1+t\left(\frac{1}{2}\sqrt{\beta^2+4b}+\frac{1}{2}\beta+1\right)\right\}=b,$$

i. e.

$$(\beta+1-b)t^2-(\beta+2)t+(b+1)=0.$$

One solution of this equation is given by $t=1$, corresponding with the point $P^*+(R-P^*)=R$; the other solution is

$$t=(b+1)/(\beta+1-b).$$

Clearly this number is greater than 1. We find that R_1 is given by

$$R_1=P^*+\frac{b+1}{\beta+1-b}\cdot(R-P^*) \tag{17}$$

and that R_1 and P^* lie on opposite sides of R.

As a fact of elementary geometry, the line-segments $\overline{PR}$ and $\overline{R_1T_1}$, cut off from the line M by the hyperbolas $xy=-1$ and $xy=b$, have equal lengths. Hence we find

$$T_1=R_1+(R-P^*)=P^*+\frac{\beta+2}{\beta+1-b}\cdot(R-P^*). \tag{18}$$

Now suppose that there exists a lattice Λ_0 in the set $\mathcal{H}_b(P)$, for which (7′) holds. Then, on account of (15) and (7′), this lattice is generated by P and some point of L. The lattice points on L have mutual distances $\sqrt{2}$. For the points Q, R, S, T we have, on account of (9), (10) and (15),

$$\delta(Q,\ R)=\beta,\qquad \delta(S,\ T)=\sqrt{\beta^2+4b+4}>\beta+1.$$

By the same argument as used in the proof of lemma 1, it follows that the closed line-segment $\overline{QR}$ contains at least $\beta+1$ lattice points. Consequently Q and R are lattice points. Now, turning our attention to the line M, we first remark that the open line-segments $\overline{PR}$ and $\overline{R_1T_1}$ belong entirely to the interior of $K_{1,b}$ and so are free from lattice points. But P and R belong to Λ_0. It follows that a point Z of the form (16) belongs to Λ_0 if (and only if) t is integral. From this we infer, taking into account the first half of the relation (18), that R_1 and T_1 are points of Λ_0. Henceforth, by (18), the fraction $(\beta+2)/(\beta+1-b)$ has an integral value.

Conversely suppose that $q=(\beta+2)/(\beta+1-b)$ is integral. Consider the lattice Λ_0, generated by P and R. We shall prove that Λ_0 is $K_{1,b}$-admissible.

Since β is integral, from $\delta(Q,\ R)=\beta$ it follows that Q is a point of Λ_0. On account of (17) and (18) also R_1 and T_1 are points of Λ_0. Further P^* is a point of Λ_0. Now we have

$$\delta(S,\ T)=\sqrt{\beta^2+4b+4}\leq\sqrt{\beta^2+4\beta+4}=\beta+2,$$

hence

$$\delta(R,\ T) = \frac{1}{2}\{\delta(S,\ T) - \delta(Q,\ R)\} \leqq 1\,;$$

so T does not belong to the interior of $K_{1,b}$.

Let Σ_0, Σ_1 be the closed line-segments $\overline{PQ}$, $\overline{R_1T_1}$ respectively, and denote by $\mathcal{K}_0$ the closed part of $K_{1,b}$ contained between Σ_0 and Σ_1. We split $\mathcal{K}_0$ into two parts, viz. the intersections of $\mathcal{K}_0$ and the triangles PQT and TRT_1 respectively. Since the pair of points R, P and likewise the pair R, P^* constitute a basis for Λ_0, it is clear that the interior of each of these strips is free from lattice points. The points of Λ_0 which lie on the boundary of one of these strips are easily identified. One arrives at the conclusion that the only lattice points which belong to Λ_0 are lying on the boundary $\mathcal{B}$ of $K_{1,b}$.

Next, let t be the ordinate of T_1 and let Ω be the affine transformation determined by

$$\Omega : x' = t^{-1}x, \qquad y' = ty. \tag{19}$$

The domain $K_{1,b}$ is left invariant under this transformation, as well as the boundary $\mathcal{B}$. We further have

$$\Omega P = T_1.$$

We denote for a moment by R_1^*, T_1^*, M^* the reflections into the line $y = x$ of R_1, T_1, M respectively. Clearly $\Omega^{-1}P^* = T_1^*$, i. e. $\Omega T_1^* = P^*$. Hence te line M^* is transformed into M by the transformation Ω. Then the same is true for the points of intersection of these lines and the hyperbola $xy = b$. In particular, we find

$$\Omega Q = R_1.$$

As a consequence the line-segment Σ_0 is transformed by Ω into Σ_1. We now put

$$\mathcal{K}_m = \Omega^m \mathcal{K}_0,$$

where m runs through all integers and the right hand side is defined in an obvious way. Clearly the union of all domains $\mathcal{K}_m$ coincides with the part of $K_{1,b}$ above the x-axis.

The pair of points P, R, as well as the pair of points P, Q, constitutes a basis for Λ_0. The same is true for the pair of points $\Omega P = T_1$, $\Omega Q = R_1$ of Λ_0. For, on account of (18) and the definition of the positive integer q, we have

$$- T_1 + q(T_1 - R_1) = - T_1 + q(R - P^*) = - P^* = P,$$
$$T_1 - (q-1)(T_1 - R_1) = T_1 - (q-1)(R - P^*) = P^* + (R - P^*) = R.$$

Consequently, not only the domain $K_{1,b}$ and its boundary $\mathcal{B}$, but also the lattice Λ_0, is left invariant under the transformation Ω. Hence, since the lattice points, which belong to $\mathcal{K}_0$, are lying on $\mathcal{B}$, the same is true for each domain $\mathcal{K}_m$. Hence the region bounded by $\mathcal{B}_1$, $\mathcal{B}_2$ and the x-axis is free from lattice points.

Finally we remark that $K_{1,b}$ as well as Λ_0 are symmetric with respect to the origin and also with respect to the line $y = x$. It follows that the origin is the only point of Λ_0 interior to $K_{1,b}$, i. e. that Λ_0 is $K_{1,b}$-admissible.

This completes the proof of lemma 2.

4. Some further lemmas.

The set of $K_{1,b}$-admissible lattices is not empty, as we have proved that even the set $\mathcal{H}_b(P)$ is not empty. It follows from the general theory of star bodies that there exists a so-called critical lattice of $K_{1,b}$, i. e. a $K_{1,b}$-admissible lattice Λ' with determinant $d(\Lambda') = \Delta(1, b)$ [3]. An analogous property holds for certain subsets of the set of $K_{1,b}$-admissible lattices. In fact we shall prove the following

LEMMA 3. - *Let $\mathcal{H}_b(\mathcal{B}_2)$ be the set of $K_{1,b}$-admissible lattices Λ with*

$$\inf_{xy<0,\ (x,y)\in\Lambda} |xy| = 1.$$

Put

$$\Delta^*(1, b) = \inf_{\Lambda\in\mathcal{H}_b(\mathcal{B}_2)} d(\Lambda). \tag{20}$$

Then there exists a $K_{1,b}$-admissible lattice Λ_0 with

$$P = (1, 1) \in \Lambda_0, \qquad d(\Lambda_0) = \Delta^*(1, b).$$

REMARK. - Clearly a lattice Λ_0 with the above property belongs to the set $\mathcal{H}_b(\mathcal{B}_2)$. In general, however, a lattice of this set may not contain any point of the curve $\mathcal{B}_2$.

Proof. - Each transformation Ω of the type (19) leaves $K_{1,b}$ invariant and so transforms any $K_{1,b}$-admissible lattice into another $K_{1,b}$-admissible lattice. By suitable choice of t we can obtain that an arbitrarily chosen point $S \models X$ of any lattice Λ is transformed by Ω into some point of the line $y = -x$. So, on account of the definitions of the quantity $\Delta^*(1, b)$ and the set $\mathcal{H}_b(\mathcal{B}_2)$, it is clear that there exist a sequence of lattices Λ_n and a sequence of positive numbers a_n ($n = 1, 2, \ldots$) such that:

a) $\Lambda_n \in \mathcal{H}_b(\mathcal{B}_2)$ *for* $n = 1, 2, \ldots$ *and* $d(\Lambda_n) \to \Delta^*(1, b)$ *as* $n \to \infty$,

b) $a_n \geq 1$ *for* $n = 1, 2, \ldots$ *and* $a_n \to 1$ *as* $n \to \infty$,

c) $A_n = (-a_n, a_n) \in \Lambda_n$.

For each $n = 1, 2, \ldots$ there certainly exists a point B_n in the first quadrant, such that B_n is a point of Λ_n and furthermore A_n and B_n constitute a basis of Λ_n. The sequence of points B_n is restricted to some bounded part of the plane. Hence we can select a subsequence $\{B_{n_k}\}$ which converges to some point B. On the other hand, the points A_n tend to the point $P = (-1, 1)$.

Let Λ_0 be the lattice generated by P and B. It follows from a wellknown theorem of MAHLER ([3]) that Λ_0 is $K_{1,b}$-admissible, whereas we have $d(\Lambda_0) = \lim_{n \to \infty} d(\Lambda_n)$. The lattice Λ_0 possesses therefore the desired properties.

It may be remarked that lemma 3 can easily be generalized in the following way.

LEMMA 3'. - *Let S be a n-dimensional automorphic star body and let Γ be its group of automorphisms. Let $\mathfrak{B}^*$ be a closed part of its boundary, which is left invariant under the transformations of Γ. Further let $\mathcal{H}(\mathfrak{B}^*)$ be the set of S-admissible lattices which have points arbitrarily near to $\mathfrak{B}^*$ and suppose that $\mathcal{H}(\mathfrak{B}^*)$ is not empty. Then there exists a lattice Λ_0 with*

$$d(\Lambda_0) = \inf_{\Lambda \in \mathcal{H}(\mathfrak{B}^*)} d(\Lambda),$$

which is S-admissible and contains a point of $\mathfrak{B}^*$.

We conclude this section with the following

LEMMA 4. - *Let b be a real number $\geqq 1$. For $z > 0$ denote by $\psi(z)$ the least positive integer $\geqq z$. Then in the interval $1 \leqq x \leqq b$ the function $f(x)$, defined by*

$$f(x) = x^2 \sqrt{\{\psi(b/x)\}^2 + 4b/x},$$

takes its minimal value in one or both points $x = 1$, $x = b/[b]$ and not elsewhere.

Proof. - Let $x_1, x_2, \ldots, x_k$ be the points of the interval $1 \leqq x \leqq b$, where b/x is integral. The function $f(x)$ is positive in the whole interval $1 \leqq x \leqq b$ and continuous and monotoneously increasing in each point $x \neq x_\nu$ ($\nu = 1, 2, \ldots, k$). In each point $x = x_\nu$ it undergoes a negative jump and there takes as value the right hand limit in that point. Furthermore, if $k \geqq 2$ and $x_\nu > x_\mu$, we have

$$\begin{aligned} f(x_\nu)^2 &= x_\nu^2 \cdot (b^2 + 4bx_\nu) \\ &> x_\mu^2 \cdot (b^2 + 4bx_\mu) = f(x_\mu)^2, \end{aligned}$$

hence $f(x_\nu) > f(x_\mu)$.

It follows that $f(x)$ attains its lower bound in one or both points $x = 1$, $x = b/[b]$, and not elsewhere.

We remark that, if we put $b/[b] = \sigma$ and denote by β the least positive integer $\geqq b$, the values $f(1)$ and $f(b/[b])$ are given by

$$\left\{ \begin{aligned} & f(1) = \sqrt{\beta^2 + 4b} \\ & f(b/[b]) = (b/[b])^2 \sqrt{[b]^2 + 4[b]} = \sigma \sqrt{b^2 + 4b\sigma}. \end{aligned} \right. \tag{21}$$

([3]) K. MAHLER, *On lattice points in n-dimensional star bodies I. Existence theorems*, « Proc. Royal Soc. », A, *187* (1946), 151-187, in particular theorem 8, p. 159 and theorem 19, p. 173.

5. Proof of theorems 2 and 3.

We begin with a preliminary consideration. Let Λ be a $K_{1,b}$-admissible attice. Put

$$a = \inf_{xy<0,\ (x,y)\in\Lambda} |xy|.$$

Clearly $a \geqq 1$.

First suppose $a \leqq b$. By the transformation

$$x' = a^{-1}x, \qquad y' = ay$$

Λ is transformed into a lattice Λ' with the following properties:

a) Λ' is admissible for the domain $K_{1,b/a}$,

b) $\inf_{xy<0,\ (x,y)\in\Lambda'} |xy| = 1$,

c) $d(\Lambda') = a^{-2}d(\Lambda)$.

Hence Λ' belongs to the set $\mathcal{H}_{b/a}(\mathfrak{B}_2)$, defined with respect to the domain $K_{1,b/a}$. In virtue of lemma 3, there exists a $K_{1,b/a}$-admissible lattice Λ_0 which contains $P=(-1,\ 1)$ and has determinant $d(\Lambda_0) = \Delta^*(1,\ b/a)$, hence $d(\Lambda_0) \leqq d(\Lambda')$. To the lattice Λ_0 we can apply lemma 1. So we get, recalling the definitions of $\psi(z)$ and $f(z)$,

$$d(\Lambda_0) \geqq \sqrt{\{\psi(b/a)\}^2 + 4b/a},$$

hence

$$d(\Lambda) = a^2 d(\Lambda') \geqq a^2\sqrt{\{\psi(b/a)\}^2 + 4b/a} = f(a). \tag{22}$$

Next suppose $a > b$. Then Λ is also $K_{b,b}$-admissible. Now by (3) (which result could be obtained with the help of lemmas 1 and 2) we have $\Delta(1,\ 1) = \frac{1}{2}\sqrt{5}$. So by (1) we have

$$\Delta(b,\ b) = b^2\Delta(1,\ 1) = b^2\sqrt{5},$$

hence

$$d(\Lambda) \geqq \Delta(b,\ b) \geqq b\sigma\sqrt{5} \geqq \sigma\sqrt{b^2 + 4b\sigma}. \tag{23}$$

Theorem 2 now follows at once. For, consider any $K_{1,b}$-admissible lattice Λ, and define a as above. In the case $a \leqq b$, on account of (22) and lemma 4, the number $d(\Lambda)$ can be minorized as follows:

$$d(\Lambda) \geqq \min\{f(1),\ f(b/[b])\};$$

hence, from (21),

$$d(\Lambda) \geqq \min\{\sqrt{\beta^2 + 4b},\ \sigma\sqrt{b^2 + 4b\sigma}\}.$$

Clearly, on account of (23), the last relation also holds in the case $a > b$. The same estimate holds for $\Delta(1,\ b)$, Λ being arbitrary. This proves theorem 2.

In order to prove theorem 3, we first treat the case

$$\sigma\sqrt{b^2+4b\sigma} \leqq \sqrt{\beta^2+4b}. \tag{24}$$

Since b/σ is integral, by lemma 2 there certainly exists a $K_{1,b/\sigma}$-admissible lattice Λ_0 with determinant $d(\Lambda_0)=\sqrt{(b/\sigma)^2+4b/\sigma}$. Hence there exists a $K_{\sigma,b}$-admissible lattice Λ' with determinant $d(\Lambda')=\sigma^2\sqrt{(b/\sigma)^2+4b/\sigma}=\sigma\sqrt{b^2+4b\sigma}$. This lattice is also $K_{1,b}$-admissible. It follows from theorem 2 that, in the actual case, theorem 3 is valid.

Next we consider the case

$$\sigma\sqrt{b^2+4b\sigma} > \sqrt{\beta^2+4b}. \tag{25}$$

This implies, on account of lemma 4 and the relations (21),

$$f(x) > f(1) \quad \text{if} \quad x > 1.$$

Assume that in (5) the equality sign holds, and consider a critical lattice Λ of $K_{1,b}$. Let a be defined as in the beginning of this section. If we had $a > b$, then we might apply (23); so by (25) we should have $d(\Lambda) > \sqrt{\beta^2+4b}$, contrary to our assumption. Hence we have $a \leqq b$ and we may apply (22). Hence $d(\Lambda) \geqq f(a)$. But, by assumption, we have $d(\Lambda)=\sqrt{\beta^2+4b}=f(1)$. Hence, in virtue of the above remark, a is equal to 1. Then lemma 3 learns us that there even exists a $K_{1,b}$-admissible lattice Λ_0 with

$$P=(-1,\ 1)\in\Lambda_0, \qquad d(\Lambda_0)=\Delta(1,\ b)=\sqrt{\beta^2+4b}.$$

Applying lemma 2 we conclude that the fraction $(\beta+2)/(\beta+1-b)$ is integral.

Conversely, assume that this fraction is integral. Then by lemma 2 there exists a certain $K_{1,b}$-admissible lattice Λ with determinant $d(\Lambda)=\sqrt{\beta^2+4b}$. Hence in (5) the equality sign holds.

This completes the proof of theorem 3.

6. **Application to a problem of Mahler.**

Let Γ be the group of the affine transformations Ω of the type (17). Let us denote by $|X|$ the distance of any point X from the origin. The group Γ has the following properties:

a) $K_{1,b}$ is left invariant under each transformation Ω in Γ;

b) if X is any point of the domain $K_{1,b}$, then there exists a transformation Ω in Γ such that $|\Omega X|$ is smaller than a fixed number c (for instance $c=b$);

c) if d is any positive number and X is any point of $K_{1,b}$, then there exists a transformation Ω in Γ with $|\Omega X| > d$.

With the usual terminology, we can therefore say that $K_{1,b}$ is a *full automorphic star body* [4].

We consider now the set of numbers b with

$$1 < b < 2, \qquad \sigma\sqrt{b^2 + 4b\sigma} < \sqrt{\beta^2 + 4b}. \tag{26}$$

Since in this case we have $\sigma = b$, $\beta = 2$, the last inequality reduces to $b^2\sqrt{5} < 2\sqrt{1+b}$. Consequently, the above set is the interval $1 < b < b_0$, where b_0 is determined by

$$b_0^4 = \frac{4}{5}(1 + b_0), \qquad b_0 > 1. \tag{27}$$

Further let s be any positive number and let $K_{1,b}^s$ be the set of points X with

$$X \in K_{1,b}, \qquad |X| \leqq s.$$

Let us consider a critical lattice $\wedge$ and let a be defined as in section 5. From theorem 3 and (26) it follows that we have $d(\wedge) = \sigma\sqrt{b^2 + 4b\sigma} = b^2\sqrt{5}$. Suppose $a > b$. Then we have $d(\wedge) \geqq \Delta(a, b)$, hence, on account of the relations (1), $d(\wedge) \geqq b^2\Delta(1, a/b)$. Here a/b is > 1. Now the right hand side of the inequality (5) is a strictly increasing function of b. Consequently we have $d(\wedge) > b^2\Delta(1, 1) = b^2\sqrt{5}$. This is a contradiction, and so we must have $a \leqq b$. Then we can apply the relation (22). This gives $d(\wedge) \geqq f(a)$. On account of lemma 4 and the relations (21) and (26) the function $f(x)$, defined in lemma 4, takes its minimal value in the interval $1 \leqq x \leqq b$ in the point $x = b/[b]$ and not elsewhere. Since in our case $[b] = 1$, it follows $a = b$. This result may be stated in the following equivalent form:

If $1 < b < b_0$, *then each critical lattice of* $K_{1,b}$ *is also a critical lattice of* $K_{b,b}$.

In particular, the part $\mathfrak{B}_2$ of the boundary of $K_{1,b}$ does not contain a point of any critical lattice of $K_{1,b}$. Consequently, for each critical lattice of $K_{1,b}$ and each value of $s > 0$ there exist lattices with a smaller determinant which are infinitely near to it and are admissible for the domain $K_{1,b}^s$. In particular, we see that:

$$\text{If} \quad 1 < b < b_0, \quad \text{then}$$
$$\Delta(K_{1,b}^s) < \Delta(K_{1,b}) = \Delta(1, b) \quad \text{for each} \quad s > 0.$$

Using the appropriate terminology, we can say that $K_{1,b}$ is *boundedly irreducible* if $1 < b < b_0$. So we have found a star body, which at the same time is full automorphic and boundedly irreducible. Thus is answered one of the problems raised by MAHLER [5].

[4] H. DAVENPORT-C. A. ROGERS, *Diophantine inequalities with an infinity of solutions*, « Phil. Transactions Royal Soc. London », *242* (1950), 311-344.

[5] K. MAHLER, *Lattice points in n-dimensional star bodies II. Reducibility theorems*, « Proc. Kon. Ned. Akad. v. Wet. », *49* (1946), in particular p. 629.

We remark that another example of a star body of the above type is given by J. W. S. CASSELS [6]. CASSELS also states in a footnote of the paper cited that ROGERS-DAVENPORT have answered the problem, considering the domain $K_{1,b}$ for positive irrational b (for $b = b_0$, however, $K_{1,b}$ is boundedly reducible, as is immediately deduced from our considerations).

One could easily establish an analogous result for other b-intervals. But, since out knowledge of the quantity $\Delta(1, b)$ is yet imperfect, we can not determine them all and so we content ourselves with the result given above.

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[6] J. W. S. CASSELS, *On two problems of Mahler*, « Proc. Kon. Ned. Akad. v. Wet. », *51* (1948), 282-285.