On the topology of the joined point-pairs of an algebraic variety.

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Sunto. - *Data una varietà complessa non singolare V, si può definire una* coppia congiunta *di punti di V. Nel caso ove V è una varietà razionale, di un tipo molto ristretto* (§ 2), *si considerano le proprietà topologiche dell'insieme di coppie congiunte. Questo è uno spazio fibrato, il che rende possibile la determinazione dei suoi gruppi di omologia. Si costruisce esplicitamente una base per questi gruppi.*

1. An *ordered point-pair*, in a given n-dimensional complex projective space S, consists of a pair of points taken in a definite order: a *joined point-pair* consists of an ordered point-pair (P, Q), together with a line R which goes through P and through Q. To any joined pair corresponds just one ordered pair: but the converse is true only if the two points do not coincide.

An irreducible, non-singular, algebraic variety V being given in S, it is possible to define a *joined point-pair of* V. This is a joined pair (P, Q, R) in S, subject to the restrictions that:

(*a*) P and Q must be on V; and

(*b*) If P and Q coincide, R must be a tangent line to V at P.

I use, for the set of joined pairs of V, the symbol $V * V$, which resembles the symbol $V \times V$ for the set of ordered pairs. The set $S * S$ of joined pairs of S is a particular case.

VAN DER WAERDEN [6] has given some account of the joined pairs *(verbundene Punktepaare)* of S: the more general concept has hardly ever been stated explicitly. But, implicitly, it plays a large part in SCHUBERT's enumerative geometry: thus, his *Strahlenpaare* ([3], § 15) are essentially joined point-pairs of the KLEIN quadric. Again, it is implicit in B. SEGRE's recent discoveries ([5], pp. 31, 91) that the study of $V * V$ can throw light on the properties of the variety V itself. In such a study, the present topological investigation may be helpful.

In § 2, severe restrictions are imposed on the variety V. But these restrictions cover quadric varieties, Grassmannians, and other types whose topology has been studied; so that greater generality would not have much practical importance. However, some of the propositions established below could be proved more generally.

2. The variety V is supposed to have all the following properties:

(*a*) V is an irreducible, non-singular, m-dimensional algebraic variety in the n-dimensional projective space S. It follows that, topologically, V can be regarded as an oriented manifold of $2m$ « real » dimensions. It is supposed that this manifold is simply connected.

(*b*) V is rational: moreover, it can be birationally projected from an $(n-m-1)$-space T onto an m-space S', the correspondence being one-to-one as between points of V and S' not lying in a prime S_0. (S_0 will contain T.) Given any two points A, B on V, the projection can be so chosen that neither A nor B is on S_0.

(*c*) We can choose on V varieties $A_1, A_2, \ldots, A_v$; $'A_1, 'A_2, \ldots, 'A_v$, with the following properties: The homology classes determined by $A_1, \ldots, A_v$ form a basis for the homology groups of V in all dimensions. A_i and $'A_j$ have no common point if $i<j$: but (for $i=1, \ldots, v$) A_i and $'A_i$ have one common point, at which they intersect simply on V. These varieties are not uniquely determined by the properties stated: they can still be chosen, even when they are required to be in general position (on V) with respect to one another, and also with respect to a finite number of varieties given on V. The variety A_1 will be a single point.

By way of example, suppose that V is a non-singular quadric surface. Here $m=2$, $n=3$, and the conditions (*a*) and (*b*) are known to be satisfied. To verify (*c*), we have only to take A_1 and $'A_4$ to be distinct points of the surface; A_2 and $'A_3$, distinct generators of one system; A_3 and $'A_2$, distinct generators of the other system; and A_4 and $'A_1$ each consisting of the whole surface: care being taken that neither of the two points lies on any of the four generators.

3. $V * V$ is an irreducible, non-singular, $2m$-dimensional algebraic variety. This follows ([4], p. 15) if we regard $V * V$ as obtained by dilating $V \times V$, the base being the variety of elements (P, P). Another way of proving it is to set up a system of parameters, allowable in the neighbourhood of an arbitrary point of $V * V$.

To do this, let us take the arbitrary point to be the element (A, B, G); and let us make the arrangements postulated in § 2 (*b*), neither A nor B being on S_0. Then, since T and S' are skew and S_0 contains T, we may take all these spaces to be faces of the simplex of reference: say,

$$T, \quad x_0 = \ldots = x_m = 0;$$
$$S', \quad x_{m+1} = \ldots = x_n = 0;$$
$$S_0, \quad x = 0.$$

The effect of this is that, if one point is the projection of another from T onto S', their coordinates x_i are proportional $(i=0, \ldots, m)$: and that, if one line is the projection of another, their coordinates p_{ij} are proportional $(i, j = 0, \dots, m)$.

Let projections on S' be denoted by dashes; for instance, let G' be the projection of G. Then G' goes through a point A' not on S_0, and therefore meets S_0 in one point. This point cannot have $x_1 = \dots = x_m = 0$: so we may suppose, without loss of generality, that it has $x_1 \neq 0$.

Now let M be the set of all elements (P, Q, R) of $V * V$, such that P and Q are not on S_0, and that R' meets S_0 in a point (Z', say) at which $x_1 \neq 0$. Then M is a neighbourhood of (A, B, G) on $V * V$. Suppose, for such an element, that

P and P' have	$x_0 : x_1 : \dots : x_m = 1 : u_1 : \dots : u_m$;
Q and Q' have	$x_0 : x_1 = 1 : u_{m+1}$;
and Z' has	$x_0 : x_1 : x_2 : \dots : x_m = 0 : 1 : u_{m+2} : \dots : u_{2m}$,
so that R and R' have	$p_{01} : p_{02} : \dots : p_{0m} = 1 : u_{m+2} : \dots : u_{2m}$.

Then $(u_1, \dots, u_{2m})$ is an allowable system of parameters for M. To verify this, we observe first that, for any element of M, $u_1, \dots$ have determinate finite values: we must then prove that any set of finite values for $u_1, \dots$ determines an element of M. In fact, such a set determines a point P', on S' but not on S_0, and a point Z', on S' and S_0 but having $x_1 \neq 0$. P' and Z', being distinct, have a definite join R', which does not meet the face $x_0 = x_1 = 0$ of the simplex of reference: therefore there is just one point Q' on R' having $x_0 : x_1 = 1 : u_{m+1}$. And P' and Q', not being on S_0, have definite projections P and Q on V. If P and Q are distinct, R is their join: if not, R is the unique tangent line to V at P whose projection is R'.

So it is verified that $V * V$ is an irreducible, non-singular, $2m$-dimensional variety: accordingly, $V * V$ is a $4m$-dimensional manifold.

4. The most important topological property of $V * V$ is that it is a fibre-bundle. The rigorous proof of this property occupies §§ 4–8.

Let A be a fixed point of V, and let us make the arrangements of § 3, A not being on S_0. Let N, N' be the sets of all points of V, S' which are not on S_0. By taking as coordinates the real and imaginary parts of the non-homogeneous coordinates $x_1/x_0, \dots, x_m/x_0$, let us put N' in one-one correspondence with a real Euclidean space N'' of $2m$ dimensions.

Let A'' be the point of N'' corresponding to A', and P'' another fixed point of N''. Corresponding to any point Q'' of N'', we define a point U'' by the following construction:

if $P''Q'' \leq P''A''$, draw $Q''U''$ equal and parallel (in the same sense) to $P''A''$;

if $P''A'' \leq P''Q'' < 3P''A''$, draw $Q''U''$ parallel to $P''A''$, but of length $\frac{1}{2}(3P''A'' - P''Q'')$;

if $P''Q'' \geq 3P''A''$, take U'' to coincide with Q''.

Then it is clear that the mapping $Q'' \to U''$ is a homoeomorphism of N'' with itself, and has the following properties:

(*a*) The image of P'' is A'';

(*b*) If Q'' is sufficiently near P'', we obtain the coordinates of U'' by adding constants to the coordinates of Q'';

(*c*) If Q'' is sufficiently far from P'', U'' coincides with Q'';

(*d*) If P'' were allowed to vary, U'' would depend continuously on P'' and Q'', and Q'' would depend continuously on P'' and U''.

5. Let us consider the images in N' of the points of N'' discussed in § 4. We have, determined by any fixed point P' of N', a homoeomorphism $Q' \to U'$ of N' with itself, having the following properties:

(*a*) The image of P' is A';

(*b*) If Q' is in a certain neighbourhood of P', we obtain the non-homogeneous coordinates of U' by adding constants to those of Q';

(*c*) If Q' is in the neighbourhood of S_0, U' coincides with Q';

(*d*) If P' were allowed to vary, U' would depend continuously on P' and Q', and Q' would depend continuously on P' and U'.

It is a consequence of property (*b*) there is a collineation of S' (determined by P') which, for points Q' in the neighbourhood of P', coincides with the above homoeomorphism. Now this collineation puts the lines R' through P' in one-one correspondence with the lines L' through A', in such a way that:

(*e*) If Q' is in the neighbourhood of P', the lines $P'Q'$ and $A'U'$ correspond;

(*f*) If P' were allowed to vary, L' would depend continuously on P' and R', and R' on P' and L'.

6. Let us consider the projections on N of the points of N' discussed in § 5. We have, determined by any fixed point P of N, a homoeomorphism $Q \to U$ of N with itself, having the following properties:

(*a*) The image of P is A;

(*c*) If Q is in the neighbourhood of S_0, U coincides with Q;

(*d*) If P were allowed to vary, U would depend continuously on P and Q, and Q on P and U.

In virtue of property (*c*), we can extend this mapping, by taking U to coincide with Q when Q is in S_0, and obtain a homoeomorphism of V with itself which has properties (*a*) and (*d*). (A and P must still be points of N.)

Further, we can set up (by projection from the correspondence $R' \to L'$) a projectivity between tangent lines R to V at P and tangent lines L to V at A, with the following properties:

(*e*) If Q tends to P in any manner, the limiting positions of the lines PQ and AU are tangent lines corresponding in the projectivity;

(*f*) If P were allowed to vary, L would depend continuously on P and R, and R on P and L.

7. If E, F are two sets of points on V, we may denote by $E*F$ the set of all elements (P, Q, R) of $V*V$ such that P is in E and Q in F. V being fixed, there is no danger of ambiguity.

We are now able to set up, for any fixed point P of N, a mapping $(P, Q, R) \rightarrow (A, U, L)$ of $P*V$ onto $A*V$, Namely, if (P, Q, R) is an element of $P*V$, we take U to be the image of Q in the homoeomorphism of § 6. If Q is distinct from P, then U is distinct from A, and L can only be the join of A and U: if Q coincides with P, then we take L to be that tangent line to V at A which corresponds to R in the projectivity of § 6.

It follows from § 6 that this mapping is a homoeomorphism between $P*V$ and $A*V$, and that, if P is allowed to vary, (A, U, L) depends continuously on P and (P, Q, R), and (P, Q, R) depends continuously on P and (A, U, L).

This is as much as to say that the mapping

$$(P, Q, R) \rightarrow (P, (A, U, L))$$

is a homoeomorphism between $N*V$ and $N \times (A*V)$, in which the image of $P*V$ always coincides with $P \times (A*V)$.

8. Now let us take a fixed point O of V, and allow A to vary. We may suppose that, for each position of the point A on V, the process of §§ 4-7 has been gone through, the apparatus being always so chosen that O, as well as A, lies in N.

It will then be clear from § 7 that there is always a homoeomorphism between $O*V$ and $A*V$, and another between $N*V$ and $N \times (A*V)$; combining these, we can construct a homoeomorphism between $N*V$ and $N \times (O*V)$. We are thus in a position to assert:

Let O be a fixed point of V, and A be any point of V. Then $A*V$ is homoeomorphic with $O*V$: and we can find a neighbourhood N of A on V, and a homoeomorphism between $N*V$ and $N \times (O*V)$, in which, for any point P of N, the image of $P*V$ must coincide with $P \times (O*V)$.

This is as much as to say that $V*V$ *is a fibre-bundle, with base-space* V *and fibre* $O*V$*: the projection of the bundle onto the base-space is given by* $(P, Q, R) \rightarrow P$.

9. Now $O*V$ is an irreducible, non-singular, m-dimensional algebraic variety (being obtained by dilating V with O as base). It is, accordingly, an orientable manifold of $2m$ dimensions.

The correspondence $(O, Q, R) \rightarrow Q$ is a continuous mapping of $O*V$ onto V; the whole set $O*O$ maps on the point O, but elsewhere the mapping takes the form of a homoeomorphism.

Since V is a $2m$-manifold, there is a neighbourhood E of O on V which is a $2m$-cell, and whose frontier D is a $(2m-1)$-sphere. Let F be the closure and G the complement of E in V: let E', D', F', G' be the respective inverse images in $O * V$ of E, D, F, G.

$O * O$ is homoeomorphic with the set of tangent lines to V at O, and therefore with complex projective $(m-1)$-space; so that its p'th homology group, $H_p(O * O)$, is free cyclic for $p = 0, 2, 4, \dots, 2m-2$, and null for other values of p. The relative homology group $H_p(F', O * O)$, being isomorphic with $H_p(F, O)$, is null for every p: therefore $H_p(F')$, for every p, is isomorphic with $H_p(O * O)$.

Again, $H_p(G', D')$ is isomorphic (for every p) with $H_p(V, F)$, and therefore $(p \neq 0)$ with $H_p(V)$.

Now $H_p(D')$ is null for $p = 1, 2, \dots, 2m-2$. Therefore, for $p = 2, 3, \dots, 2m-2$, $H_p(F', D')$ is isomorphic with $H_p(F')$, and $H_p(O * V)$ with $H_p(O * V, D')$. But the set $O * V - D'$ is the union of the disjoint sets $F' - D'$ and $G' - D'$: therefore $H_p(O * V, D')$ is (for every p) the direct sum of $H_p(F', D')$ and $H_p(G', D')$. And so, for $p = 2, 3, \dots, 2m-2$, $H_p(O * V)$ is the direct sum of $H_p(V)$ and $H_p(O * O)$.

Again, $H_{2m-1}(O * O)$ and $H_{2m-1}(O * V, O * O)$ are null, and $H_{2m}(O * V, O * O)$ free cyclic: therefore $H_{2m-1}(O * V)$ is null, and $H_{2m}(O * V)$ free cyclic.

It follows by duality that $H_0(O * V)$ is free cyclic, and $H_1(O * V)$ null.

Combining these results, we conclude that $O * V$ has no torsion, and that its Betti numbers are given by

$$R_p(O * V) = R_p(V) + 1 \quad (p = 2, 4, 6, \dots, 2m-2);$$
$$R_p(O * V) = R_p(V) \quad \text{(all other } p\text{)}.$$

The sum of the Betti numbers of V is v (§ 2): therefore the sum of those of $O * V$ is $v + m - 1$.

10. Thus $V * V$ is a fibre-bundle, and the homology groups of the base-space V and the fibre $O * V$ are known. From a theorem of Kudo [2] we know that the homology groups of $V * V$ will be isomorphic with those of $V \times (O * V)$, if the following conditions are satisfied:

(*a*) The base-space is a finite complex;

(*b*) The fibre is an orientable manifold;

(*c*) The group of transformations of the fibre is connected;

(*d*) The injection homomorphisms of the homology groups of the fibre into those of the bundle are isomorphisms into.

Now conditions (*a*) and (*b*) are, in fact, satisfied, and (*c*) must be because the base-space is simply connected. To verify (*d*) it will suffice (the fibre having no torsion) to construct a collection of linearly independent homology classes of $V * V$, all containing cycles on $O * V$, the number of classes being the sum of the Betti numbers of $O * V$.

We shall, in fact, construct such a collection of homology classes: but it will be only a part of another collection, which will be large enough to serve as a basis for the homology groups of $V * V$.

11. Let us begin by making the arrangements of § 2(c) — and, in fact, by making them twice over. That is, let there be varieties $A_1, \ldots, A_v$; $'A_1, \ldots, 'A_v$ on V, with the properties of § 2: and let there be other varieties, named respectively $B_1, \ldots, B_v$; $'B_1, \ldots, 'B_v$, with corresponding properties. Let the whole collection of $4v$ varieties be in general position among themselves: let no three of them have a common point if the sum of their dimensions is less than $2m$.

Let us likewise choose, for $k = 1, 2, \ldots, m-1$, linear spaces S_k and $'S_k$ in S: the dimension of S_k is to be $n-m+k$, and that of $'S_k$ to be $n-1-k$. These spaces are all required to be in general position in S.

The point O of § 8 was arbitrary; let us suppose, for convenience of statement, that O is the single point of which the variety A_1 consists.

12. Let us define, by means of the varieties of § 11, various sets of elements (P, Q, R) of $V * V$.

Let P_i, $'P_i$ $(i = 1, 2, \ldots, v)$ be defined by the property that P lies on A_i, A_i respectively. Let Q_j, $'Q_j$ $(j = 1, \ldots, v)$ be defined by the property that Q lies on B_j or $'B_j$ respectively. Let R_k, $'R_k$ $(k = 1, \ldots, m-1)$ be defined by the property that R meets S_k or $'S_k$ respectively. And let C be defined by the property that P and Q coincide.

All these are subvarieties of $V * V$. If the dimensions of A_i, A_j are g, h, those of P_i, $'P_i$, Q_j, $'Q_j$ are $m+g$, $2m-g$, $m+h$, $2m-h$: and the dimensions of R_k, $'R_k$, C are $m+k+1$, $2m-k$, $2m-1$. (These statements are fairly obvious, and will be verified in §§ 15, 16, where the varieties will be represented by equations involving local parameters.)

According to the supposition made at the end of § 11, P_1 is the same set that we have been calling $O * V$.

13. Each of the varieties of § 12 defines a homology class (of twice as many dimensions) of the $4m$-dimensional oriented manifold $V * V$: this class may be represented by the same symbol as the variety, only with a lower-case letter instead of the capital.

By taking the intersections of classes so determined, we define other homology classes of $V * V$ as follows:

$$d_{ij} = p_i \cdot q_j \qquad (i, j = 1, 2, \ldots, v);$$
$$'d_{ij} = 'p_i \cdot 'q_j \qquad (i, j = 1, 2, \ldots, v);$$
$$e_{ik} = p_i \cdot r_k \cdot c \qquad (i = 1, \ldots, v;\ k = 1, \ldots, m-1);$$
$$'e_{ik} = 'p_i \cdot 'r_k \qquad (i = 1, \ldots, v;\ k = 1, \ldots, m-1).$$

The dimensions of these classes are respectively $2(g+h)$, $2(2m-g-h)$, $2(g+k)$, $2(2m-g-k)$, where g and h are the dimensions of A_i and A_j.

Also, for $j=1, \dots, v$; $k=1, \dots, m-1$, the class d_{1j} or e_{1k} must contain cycles of $O * V$.

14. It has to be proved that the homology classes d_{ij}, e_{ik} of § 13 are all linearly independent. In order to do this, we determine some of their intersection numbers with the classes $'d_{ij}$, $'e_{ik}$.

If d_{ij} and d_{st} are classes of the same dimension, and $i<s$, then the intersection number $d_{ij} \cdot 'd_{st}$ is zero.

For A_i and $'A_s$ have no common point; therefore P_i and $'P_s$ have no common element; therefore $p_i \cdot 'p_s$ is a zero homology class; therefore the intersection number $p_i \cdot q_j \cdot 'p_s \cdot 'q_t$ is zero, which was to be proved.

If e_{ik} and e_{st} are of the same dimension, and $i<s$, then $e_{ik} \cdot 'e_{st}$ is zero, for similar reasons.

If d_{ij} and d_{st} are of the same dimension, and $j<t$, then $d_{ij} \cdot 'd_{st}$ is zero, for similar reasons (relating to Q).

If e_{ik} and d_{st} are of the same dimension, then $e_{ik} \cdot 'd_{st}$ is zero.

For, if A_i, A_s, A_t have dimensions f, g, h (and accordingly $'A_s$, $'B_t$ have dimensions $m-g$, $m-h$), we have

$$f+k=g+h, \qquad k>0,$$

and so

$$f+(m-g)+(m-h)<2m;$$

therefore (§ 11) A_i, $'A_s$, $'B_t$ have no common point: but a common element of P_i, C, $'P_s$, $'P_t$ would have P and Q coinciding at such a common point.

15. *For any i, j, the intersection number $d_{ij} \cdot 'd_{ij}$ is $+1$.*

In fact, A_i and $'A_i$ have one common point, X say: and B_j and $'B_j$ have one common point, Y say, distinct from X. Thus P_i, $'P_i$, Q_j, $'Q_j$ have one common element (X, Y, XY).

Let $(t_1, t_2, \dots, t_m)$ be an allowable system of parameters, representing V in the neighbourhood of X: and let $(t_{m+1}, \dots, t_{2m})$ be such a system for the neighbourhood of Y. Then there is a neighbourhood of (X, Y, XY) on $V * V$ (not containing any elements of C) in which $(t_1, \dots, t_{2m})$ is an allowable system of parameters representing $V * V$. (Namely, $t_1, \dots, t_m$ determine a point P; $t_{m+1}, \dots, t_{2m}$ determine a distinct point Q; and R is the join PQ).

In the neighbourhood of X, the varieties A_i and $'A_i$ are determined by equations in the parameters $t_1, \dots, t_m$: there are altogether m of these equations, and their Jacobian in the m parameters is not zero at X. (This is implicit in the postulate of § 2 that A_i and $'A_i$ intersect simply on V.) The same equations (regarded as equations in $t_1, \dots, t_{2m}$, in which $t_{m+1}, \dots, t_{2m}$ happen not to appear) determine P_i and $'P_i$ in the neighbourhood of (X, Y, XY).

After applying similar reasoning to B_j and $'B_j$, we conclude that, in the neighbourhood of (X, Y, XY), the varieties $P_i, 'P_i, Q_j, 'Q_j$ are determined by equations in the parameters $t_1, \dots, t_{2m}$: there are altogether $2m$ of these equations, and their Jacobian in the $2m$ parameters is not zero at (X, Y, XY).

Therefore [1] the intersection number $p_i \cdot 'p_i \cdot q_j \cdot 'q_j$ is $+1$, which was to be proved.

16. *For any i, k, the intersection number $e_{ik} \cdot 'e_{ik}$ is $+1$.*

In fact, A_i and $'A_i$ have one common point, X say: let U be the tangent m-space to V at X. Then U meets S_k and $'S_k$ in spaces, of dimensions k and $m-k-1$ respectively, which are in general position in U; so that there is one transversal line, W say, from X in U to these spaces. We see that (X, X, W) is the only common element of the varieties P_i, R_k, C, $'P_i$, $'R_k$.

It is required to prove that the five varieties intersect simply at this point. This amounts to proving that some determinant does not vanish, S_k and $'S_k$ being supposed general: and it will suffice to prove that the determinant does not vanish when S_k and $'S_k$ have some special position.

Let us, accordingly, make the arrangements of § 3, X having $x_0 \neq 0$ and W having $p_{01} \neq 0$; thus the parameters $u_1, \dots, u_{2m}$ (§ 3) will be allowable for $V * V$ in the neighbourhood of (X, X, W). Let us then specialise S_k to be

$$x_0 = x_2 = x_3 = \dots = x_{m-k} = 0$$

and $'S_k$ to be

$$x_0 = x_{m-k+1} = x_{m-k+2} = \dots = x_m = 0.$$

This means that a line (p_{ij}) having $p_{01} \neq 0$ will meet S_k if and only if

$$p_{02} = \dots = p_{0,\,m-k} = 0,$$

and $'S_k$ if and only if

$$p_{0,\,m-k+1} = \dots = p_{0m} = 0.$$

Now x_r/x_0 $(r = 1, \dots, m)$ will be an allowable system of parameters for V in the neighbourhood of X. And it will follow, as in § 15, that P_i and $'P_i$ are represented, in the neighbourhood of (X, X, W), by equations in the parameters $u_1, \dots, u_m$: these equations will be m in number altogether, with a non-vanishing Jacobian at (X, X, W).

The other three varieties are represented (in the same neighbourhood) by the following equations:

$$\begin{array}{ll} C, & u_{m+1} - u_1 = 0; \\ R_k, & u_{m+2} = \dots = u_{2m-k} = 0; \\ 'R_k, & u_{2m-k+1} = \dots = u_{2m} = 0. \end{array}$$

These m equations, with the equations of P_i and $'P_i$, make up a set of $2m$ equations in the $2m$ parameters, with a Jacobian which is obviously not zero. Therefore, in the case where S_k, $'S_k$ are general, we must likewise have $2m$ equations with a non-zero Jacobian.

Therefore the intersection number

$$p_i \cdot 'p_i \cdot c \cdot r_k \cdot 'r_k = +1,$$

which was to be proved.

17. In order to state more briefly the results of §§ 14-16, let us take, from the set of all the homology classes d_{ij}, e_{ik}, all those of dimension s (s being an arbitrary integer). Let us arrange these in a definite order, as follows: First the classes e_{ik}, in ascending order of the suffix i (there cannot be two of dimension s with equal i); then the classes d_{ij}, in ascending order of the suffix i, and (for equal i) in ascending order of the suffix j. For convenience of statement, let us give to these classes new names f_1, $f_2, \ldots$, the suffixes running in the order described: and let the corresponding classes $'d_{ij}$, $'e_{ik}$ (which are all of dimension $4m - s$) have new names $'f_1$, $'f_2, \ldots$ respectively.

Then we conclude from §§ 14-16 that *the intersection number* $f_i \cdot 'f_j$ *is zero if* $i < j$, *and unity if* $i = j$.

It follows immediately that *all the classes* d_{ij}, e_{ik} *are linearly independent.*

In particular, the classes d_{1j}, e_{1k} ($j = 1, \ldots, v$; $k = 1, \ldots, m - 1$) are linearly independent: but the number of these classes, $v + m - 1$, is the sum of the BETTI numbers of $O * V$ (§ 9), and each of them contains cycles of $O * V$ (§ 13).

Therefore the conditions of § 10 are satisfied; and we conclude that $V * V$ and $V \times (O * V)$ have isomorphic homology groups. That is, $V * V$ *has no torsion, and the sum of its Betti numbers is* $v(v + m - 1)$.

18. But, in fact, we have $v(v + m - 1)$ linearly independent homology classes d_{ij}, e_{ik}. Therefore, for any dimension s, the number of the classes f_1, $f_2, \ldots$ must be the s'th BETTI number of $V * V$. It follows, these classes being linearly independent, that any s-dimensional integral homology class h can be expressed in the form

$$h = t_1 f_1 + t_2 f_2 + \ldots,$$

where t_1, $t_2, \ldots$ are rational numbers.

We then see, moreover, that t_1, $t_2, \ldots$ are all integers. Suppose, in fact, that some coefficients, of which the last one is t_i, are not integers. Then, according to § 17, the integral classes h and $'f_i$ will have a non-integral intersection number: which is impossible.

Therefore *the classes* d_{ij}, e_{ik} $(i, j = 1, \ldots, v;\ k = 1, \ldots, m-1)$ ***form a basis for the integral homology groups of*** $V * V$ ***in all dimensions.***

(An alternative basis is formed by the classes $'d_{ij}$, $'e_{ik}$: this may be similarly proved.)

Consider, by way of example, the case of the quadric surface (§ 2), in which $m = 2$, $v = 4$, and A_1, A_2, A_3, A_4 have dimensions 0, 1, 1, 2. There are 20 basic homology classes:

d_{11}	of dimension 0
$e_{11}, d_{12}, d_{13}, d_{21}, d_{31}$	2
$e_{21}, e_{31}, d_{14}, d_{22}, d_{23}, d_{32}, d_{33}, d_{41}$	4
$e_{41}, d_{24}, d_{34}, d_{42}, d_{43}$	6
d_{44}	8.

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