On the singular Cauchy problem for a generalization of the Euler-Poisson-Darboux equation in two space variables.

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Summary. - *The present paper contains an existence and uniqueness theorem for the singular* CAUCHY *problem for the non-homogeneous* EULER-POISSON-DARBOUX *equation:*

$$u_{xx}+u_{yy}-u_{tt}-\frac{k}{t}u_t=f(x,\ y,\ t),\quad t>0,\quad k>0,$$
$$u(x,\ y,\ 0)=u_t(x,\ y,\ 0)=0.$$

The solution of this problem is used to prove an existence and uniqueness theorem for the following singular CAUCHY *problem:*

$$u_{xx}+u_{yy}-u_{tt}-\frac{k}{t}u_t-h(x,\ y,\ t)u=0,\quad t>0,\quad k>0,$$
$$u(x,\ y,\ 0)=g(x,\ y),\quad u_t(x,\ y,\ 0)=0,$$

by the method of successive approximations.

Introduction. - This paper is concerned with the solution of the following singular CAUCHY problem. Let $h(x,\ y,\ t)$ be continuous (1) for $t\geq 0$ together with its derivatives h_x, h_y, h_{xx}, h_{xy}, h_{yy}, and $g(x,\ y)$ a twice continuously differentiable function (if $k\geq 1$) or a thrice continuously differentiable function (if $0<k<1$). Then it will be shown that there exists a unique function $u(x,\ y,\ t)$, continuous for $t\geq 0$ together with its derivatives u_x, u_y, u_{xx}, u_{xy}, u_{yy}, which is twice continuously differentiable in $(x,\ y,\ t)$ for $t>0$, satisfies

$$L(u)-hu\equiv u_{xx}+u_{yy}-u_{tt}-\frac{k}{t}u_t-hu=0,\quad k>0,\quad t>0, \tag{1}$$

and takes on the regular initial data

$$u(x,\ y,\ 0)=g(x,\ y),\qquad u_t(x,\ y,\ 0)=0, \tag{2}$$

on the singular plane $t=0$.

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(1) In this paper all functions will be considered defined for all real values of their arguments, unless otherwise specified.

This constitutes a generalization (for the special case $m=2$) of the singular CAUCHY problem for the EULER-POISSON-DARBOUX equation

$$\Delta u - u_{tt} - \frac{k}{t} u_t = 0 ; \qquad \Delta \equiv \sum_{=1}^{m} \frac{\partial^2}{\partial x_i^2}, \tag{3a}$$

with data

$$\begin{aligned} u(x_1, x_2, \dots, x_m, 0) &= g(x_1, x_2, \dots, x_m), \\ u_t(x_1, x_2, \dots, x_m, 0) &= 0. \end{aligned} \tag{3b}$$

This latter problem has been solved by WEINSTEIN [11], and DIAZ and WEINBERGER [12].

The present analysis is restricted to the case of two space variables since then the definite integrals concerned (though improper) are convergent. For more than two space variables the corresponding improper definite integrals are divergent, and HADAMARD's concept [8; 9, p. 133] of the finite parts of these integrals must be used. The restriction $k > 0$ is made since for $k < 0$ the solutions of (3) itself are not unique (²), and for $k = 0$ the problem is no longer a singular one.

In section 1 the elementary solution, in the sense of HADAMARD, is found for (3) from the known fundamental solution of a corresponding elliptic equation, given by DIAZ and WEINSTEIN [15]. This elementary solution (for $m=2$) is used in section 2 to construct the solution of a regular CAUCHY problem on $t = t_0 > 0$ for $L(u) = 0$, and in section 3 bounds for this solution and certain of its derivatives are obtained. The solution of this regular problem is then used in section 4 to construct the solution of a singular CAUCHY problem on $t = 0$ for the inhomogeneous equation $L(u) = f(x, y, t)$. The method used is patterned after that of DUHAMEL (³) [2] for regular CAUCHY problems, and the result obtained represents an extension of the theory of HADAMARD [9, p. 166] to a singular initial values problem. The bounds obtained in section 3 are employed in 4 to show that certain « DUHAMEL integrals » which are improper, are nevertheless convergent. Finally in section 5, the given CAUCHY problem (1) and (2) is solved by considering an equivalent problem in integral equations, the equivalence being demonstrated by the results of section 4.

In the conclusion it is shown that the singular problem (1) and (2) is well-posed, but that the present method fails when space-derivatives of the first order (even with regular coefficients) are included in (1). The analysis indicates that difficulties may be expected in this case. Finally it is remarked that the corresponding problem for one space variable is covered, by the method of descent, in the present problem.

(²) See [11; 12].

(³) Also known as the method of variation of parameters or « Stössmethode ».

1. **The Elementary Solution.** - DIAZ and WEINSTEIN have shown [15] that the singular elliptic equation

$$\Delta u + u_{yy} + \frac{k}{y} u_y = 0; \qquad \Delta \equiv \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}, \tag{4}$$

has, when $k > 0$, a fundamental solution

$$u_b^{(k)}(\underline{x}, y) = \int_0^{\pi} \frac{\sin^{k-1}\alpha \, d\alpha}{(\rho^2 + b^2 + y^2 - 2by\cos\alpha)^{(k+m-1)/2}}, \tag{5}$$

where $\underline{x} = (x_1, x_2, \dots, x_m)$, $\rho^2 = \underline{x}^2 = \sum_{i=1}^{m} x_i^2$, and the singularity is at $(\underline{0}, b)$,

The function (5) is clearly defined for complex values of y and b also. In particular, on putting $y = it$ and $b = i\tau$, with t and τ real, it may be inferred that

$$\mathfrak{U}(\underline{x}, t; \tau) = \int_0^{\pi} \frac{\sin^{k-1}\alpha \, d\alpha}{(t^2 + \tau^2 - \rho^2 - 2\tau t\cos\alpha)^{(k+m-1)/2}}, \tag{6}$$

is a solution of the singular hyperbolic equation

$$\Delta u - u_{tt} - \frac{k}{t} u_t = 0. \tag{7}$$

It is easy to verify that for $(\tau - t) > \rho$ and $t \geq 0$, the region of $(\underline{x}, t)$ — space which is of present interest, $\mathfrak{U}(\underline{x}, t; \tau)$ is real and finite.

As was noticed by DIAZ and WEINSTEIN for $u_b^{(k)}$, the function $\mathfrak{U}$ may be expressed in terms of the hypergeometric function. Thus putting $\xi = \cos^2 \frac{\alpha}{2}$, $z = 4\tau t/[(t + \tau)^2 - \rho^2]$, (6) becomes

$$\mathfrak{U}(\underline{x}, t; \tau) = 2^{-m}(z/\tau t)^{(k+m-1)/2} \int_0^1 \xi^{\frac{k}{2}-1}(1 - \xi)^{\frac{k}{2}-1}(1 - z\xi)^{-(k+m-1)/2} d\xi,$$

$$= 2^{-m} B\left(\frac{k}{2}, \frac{k}{2}\right)(z/\tau t)^{(k+m-1)/2} F\left(\frac{k+m-1}{2}, \frac{k}{2}, k; z\right),$$

where EULER's integral for the hypergeometric function [3, p. 14] has been used.

Now $v = t^k u$ defines a correspondence between solutions u of the equation (7) and solutions v of its adjoint equation ([4])

$$\Delta v - v_{tt} + k(v/t)_t = 0. \tag{8}$$

([4]) A similar fact has been noted by OLEVSKIĬ [5] for the elliptic equation (4), and by TRICOMI [1] for a particular case of the same equation.

Hence for any $\underline{\xi} = (\xi_1, \xi_2, \ldots, \xi_m)$ a singular solution of this latter equation is

$$\mathfrak{V}(\underline{x}, t; \underline{\xi}, \tau) = t^k \mathfrak{U}(\underline{x} - \underline{\xi}, t; \tau),$$

(9)
$$= 2^{k-1} B\left(\frac{k}{2}, \frac{k}{2}\right) \frac{t^k}{r'^{(k+m-1)}} F\left(\frac{k+m-1}{2}, \frac{k}{2}, k; 1 - \frac{r^2}{r'^2}\right),$$

where $r = \sqrt{(t-\tau)^2 - (\underline{x} - \underline{\xi})^2}$ is the hyperbolic distance of $(\underline{x}, t)$ from $(\underline{\xi}, \tau)$ and $r' = \sqrt{(t+\tau)^2 - (\underline{x} - \underline{\xi})^2}$ is the hyperbolic distance from the image point $(\underline{\xi}, -\tau)$, see figure 1. A short calculation shows that $\mathfrak{V}$ becomes infinite [5]

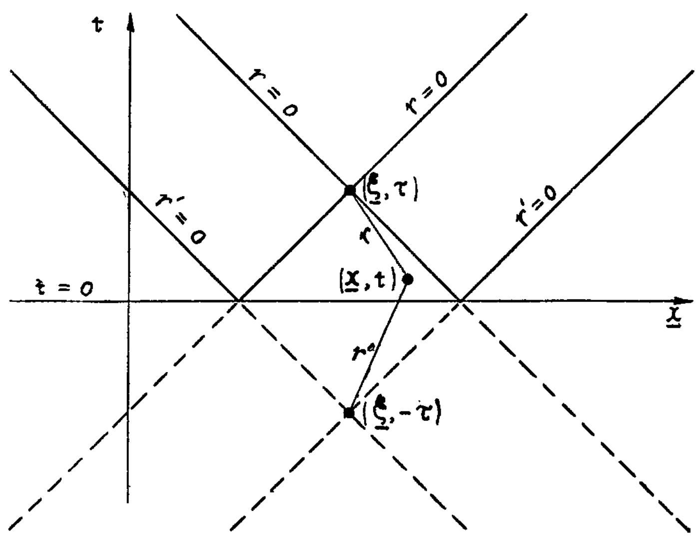

Fig. 1.

like $1/r^{(m-1)}$ and $1/r'^{(m-1)}$ as $r \to 0$ and $r' \to 0$ respectively, and these singularities are indicated in the figure.

Now $\mathfrak{V}$ has the same singularities as the elementary solution of Hadamard [9, p. 104]. For m odd this elementary solution is not unique and in fact $\mathfrak{V}$ itself is such a solution. For m even, however, the elementary solution is unique and may be extracted from $\mathfrak{V}$ as follows. A well known transformation of the hypergeometric function yields [3, p. 15]

(10)
$$F\left(\frac{k+m-1}{2}, \frac{k}{2}, k; 1 - \frac{r^2}{r'^2}\right) = \left(\frac{r'}{r}\right)^{m-1} F\left(\frac{k-m+1}{2}, \frac{k}{2}, k; 1 - \frac{r^2}{r'^2}\right),$$

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(5) For $m > 1$. For $m = 1$ the singularities are logarithmic.

and, provided m is even, another [3, p. 21] gives

$$F\left(\frac{k-m+1}{2},\frac{k}{2},k;1-\frac{r^2}{r'^2}\right)=\frac{\Gamma(k)\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{k+m-1}{2}\right)\Gamma\left(\frac{k}{2}\right)}F\left(\frac{k-m+1}{2},\frac{k}{2},\frac{3-m}{2};\frac{r^2}{r'^2}\right)+ \tag{11}$$
$$+\frac{\Gamma(k)\Gamma\left(\frac{1-m}{2}\right)}{\Gamma\left(\frac{k-m+1}{2}\right)\Gamma\left(\frac{k}{2}\right)}\left(\frac{r}{r'}\right)^{m-1}F\left(\frac{k+m-1}{2},\frac{k}{2},\frac{m+1}{2};\frac{r^2}{r'^2}\right).$$

Both of the hypergeometric functions appearing on the right hand side of (11) are regular at $r=0$, when $\tau\neq0$. Hence, it follows from (10) that, when m is even, the first forms a multiple of the elementary solution, whilst the second yields an additive regular solution, of the differential equation (8). Thus, the required elementary solution (6) is, see (9),

$$V(\underline{x},\ t;\ \underline{\xi},\ \tau)=\frac{2^k t^k}{r'^k r^{(m-1)}}F\left(\frac{k-m+1}{2},\ \frac{k}{2},\ \frac{3-m}{2};\ \frac{r^2}{r'^2}\right), \tag{12}$$

for m even and $\tau\neq0$. The present paper is concerned with the case $m=2$, for which

$$V(\underline{x},\ t;\ \underline{\xi},\ \tau)=\frac{2^k t^k}{r'^k r}F\left(\frac{k-1}{2},\ \frac{k}{2},\ \frac{1}{2};\ \frac{r^2}{r'^2}\right).$$

It is well known [9, p. 179] that the elementary solution (12) satisfies the original differential equation (7) in the pole variables $(\underline{\xi},\ \tau)$.

2. **A Regular Cauchy Problem for $\tau=t_0>0$.** - When there are only two space variables it is more convenient to write $(x_1, x_2)=(x, y)$, $(\xi_1, \xi_2)=(\xi, \eta)$ and thus avoid the use of subscripts. With this change, consider the problem of determining that solution $u=w(\underline{\xi},\ \tau;\ t_0)$ of the hyperbolic differential equation

$$u_{\xi\xi}+u_{\eta\eta}-u_{\tau\tau}-\frac{k}{\tau}u_\tau=0, \tag{13}$$

which assumes on the plane $\tau=t_0>0$, see figure 2, the boundary values

$$w(\underline{x},\ t_0;\ t_0)=0,\qquad w_\tau(\underline{x},\ t_0;\ t_0)=f(\underline{x}), \tag{14}$$

where it is assumed that $f(\underline{x})$ possesses continuous derivatives of the second order. According to HADAMARD [9, p. 166] the function w is unique and is

(6) The normalization is that $r^{m-1}V\rightarrow1$ as $r\rightarrow0$ through positive values.

given for $\tau > t_0$ by

$$w(\underline{\xi}, \tau; t_0) = \frac{1}{2\pi} \iint\limits_{(\underline{x}-\underline{\xi})^2 \leq (\tau - t_0)^2} V(\underline{x}, t_0; \underline{\xi}, \tau) f(\underline{x}) dx dy, \tag{15}$$

where the « finite part » sign may be omitted in this case since, as will be shown immediately, the integral is convergent. The absence of this sign indicates that the direct verification of (15) is simpler than for the case of general m.

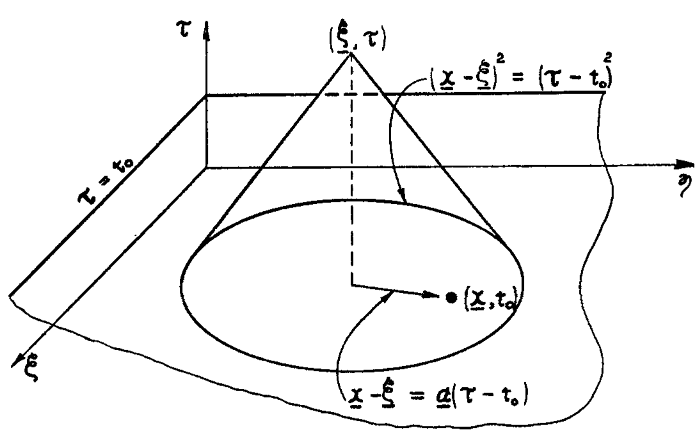

Fig. 2.

This verification can be best carried out by writing $\underline{x} - \underline{\xi} = \underline{\alpha}(\tau - t_0)$, with $\underline{\alpha} = (\alpha, \beta)$, and

$$\tilde{V}(\gamma, \tau; t_0) = (\tau - t_0)^2 V(\underline{x}, t_0; \underline{\xi}, \tau)$$

$$= \frac{2^k t_0^{\,k}(\tau - t_0)}{(1-\gamma^2)^{1/2}[(\tau+t_0)^2 - \gamma^2(\tau - t_0)^2]^{k/2}} F\left(\frac{k-1}{2}, \frac{k}{2}, \frac{1}{2}; \frac{1-\gamma^2}{\left(\frac{\tau+t_0}{\tau-t_0}\right)^2 - \gamma^2}\right), \tag{16}$$

where $\gamma^2 = \alpha^2 + \beta^2$. Then (15) becomes

$$w(\xi, \eta, \tau; t_0) = \frac{1}{2\pi} \iint\limits_{\alpha^2+\beta^2 \leq 1} \tilde{V}(\gamma, \tau; t_0) f(\xi + \alpha\,\overline{\tau - t_0}, \eta + \beta\,\overline{\tau - t_0}) d\alpha d\beta, \tag{17}$$

the integration now taking place over a fixed region in the (α, β) — plane. Differentation under the integral sign now yields

$$(18)\quad \begin{aligned} w_\tau &= \frac{1}{2\pi}\iint [\tilde V_\tau f + \tilde V(\alpha f_x + \beta f_y)]d\alpha d\beta, \\ w_{\tau\tau} &= \frac{1}{2\pi}\iint [\tilde V_{\tau\tau} f + 2\tilde V_\tau(\alpha f_x + \beta f_y) + \tilde V(\alpha^2 f_{xx} + 2\alpha\beta f_{xy} + \beta^2 f_{yy})]d\alpha d\beta, \end{aligned}$$

$$(19)\quad \begin{cases} w_{\xi\xi} = \dfrac{1}{2\pi}\iint \tilde V f_{xx} d\alpha d\beta, \\ w_{\eta\eta} = \dfrac{1}{2\pi}\iint \tilde V f_{yy} d\alpha d\beta, \end{cases}$$

where the convergence of these integrals, as well as that in (17), is assured for $\tau > t_0$ by the form [7] (16) of $\tilde V$. The left hand side of (13) may therefore be replaced by

$$(20)\quad \begin{aligned} \frac{1}{2\pi}\iint \Big[& f\Big(\tilde V_{\tau\tau} + \frac{k}{\tau}\tilde V_\tau\Big) + (\alpha f_x + \beta f_y)\Big(2\tilde V_\tau + \frac{k}{\tau}\tilde V\Big) \\ & + (\overline{\alpha^2 - 1} f_{xx} + 2\alpha\beta f_{xy} + \overline{\beta^2 - 1} f_{yy})\tilde V\Big] d\alpha d\beta. \end{aligned}$$

Now from the fact, noted at the end of the last section, that V satisfies the equation (7) in the pole variables $(\underline{\xi}, \tau)$, it may be deduced, see (16), that $\tilde V$ satisfies the equation

$$(21)\quad \begin{aligned} &(1-\gamma^2)\tilde V_{\gamma\gamma} + 2\gamma(\tau - t_0)\tilde V_{\gamma\tau} - (\tau - t_0)^2\tilde V_{\tau\tau} + \left[\frac{1}{\gamma} - 6\gamma + \frac{k\gamma(\tau - t_0)}{\tau}\right]\tilde V_\gamma + \\ &+ (\tau - t_0)\left[4 - \frac{k(\tau - t_0)}{\tau}\right]\tilde V_\tau - \left[6 - \frac{2k(\tau - t_0)}{\tau}\right]\tilde V = 0, \end{aligned}$$

and hence that the integral (20) may be written in the form

$$(22)\quad \frac{1}{2\pi}\iint [(Af + Bf_x + Cf_y)_\alpha + (Df + Ef_x + Ff_y)_\beta] d\alpha d\beta,$$

where

$$A = \frac{\alpha}{(\tau - t_0)^2}\left\{\frac{(1-\gamma^2)}{\gamma}\tilde V_\gamma + 2(\tau - t_0)\tilde V_\tau - \left[3 - \frac{k(\tau - t_0)}{\tau}\right]\tilde V\right\},$$

$$B = \frac{(\alpha^2 - 1)}{(\tau - t_0)}\tilde V, \qquad C = \frac{\alpha\beta}{(\tau - t_0)}\tilde V,$$

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(7) Thus the argument of the hypergeometric function is less than one for $0 \leq \gamma \leq 1$, and $(\tau + t_0)^2 - \gamma^2(\tau - t_0)^2$ cannot vanish in the same range.

and D, F, E are obtained from A, B, C respectively by interchanging α and β. In detail, identification of the coefficients of f_{xx}, f_{xy} and f_{yy} in the integrals (20) and (22) yields B, $(C+E)$ and F, and from symmetry it is clear that $C=E$. Knowing B and E, identification of the coefficients of f_x yields A, and similarly knowing C and F, identification of the coefficients of f_y yield D. Finally, the equality of (20) and (22) is established by computing from these values of A and D the coefficient $\frac{1}{2\pi}(A+D_\beta)$ of f in the latter, and proving that it is equal to the coefficient of f in the former by virtue of (21).

Hence in order to show that w is a solution of (13) there only remains to be shown that for $\underline{\xi}$, τ and t_0 fixed, with $\tau > t_0$, $\alpha(Af+Bf_x+Cf_y)+$ $+\beta(Df+Ef_x+Ff_y)$ tends uniformly (8) to zero as $\gamma \to 1$. For the integral (22) is, by the divergence theorem, the limit of the line integral of $1/2\pi\gamma$ times this quantity, taken around the circle $\alpha^2+\beta^2=\gamma^2<1$. Now for fixed τ, t_0, see (9) (16),

$$\begin{aligned}\tilde{V}(\gamma,\ \tau;\ t_0) &= K(\tau,\ t_0)\ (1-\gamma^2)^{-1/2}[1+0(1-\gamma^2)],\\ \tilde{V}_\gamma(\gamma,\ \tau;\ t_0) &= K(\tau,\ t_0)\gamma(1-\gamma^2)^{-3/2}[1+0(1-\gamma^2)],\\ \tilde{V}_\tau(\gamma,\ \tau;\ t_0) &= L(\tau,\ t_0)\ (1-\gamma^2)^{-1/2}[1+0(1-\gamma^2)],\end{aligned}$$

where

$$K(\tau,\ t_0)=(\tau-t_0)(t_0/\tau)^{k/2}; \qquad L(\tau,\ t_0)=(t_0/\tau)^{k/2}\left[1-\frac{k(\tau-t_0)}{2\tau}\right].$$

Hence

$$\frac{1-\gamma^2}{\gamma}\tilde{V}_\gamma+2(\tau-t_0)\tilde{V}_\tau-\left[3-\frac{k(\tau-t_0)}{\tau}\right]V=0(1-\gamma^2)^{1/2},$$

and $\alpha A+\beta D$, which is $\gamma^2/(\tau-t_0)^2$ times this quantity, tends to zero uniformly as $\gamma\to 1$. Similarly it can be shown that $\alpha B+\beta E$ and $\alpha C+\beta F$ are each $0(1-\gamma^2)^{\frac{1}{2}}$ and hence tend to zero uniformly as $\gamma \to 1$.

Finally it may be seen from (16) that $(1-\gamma^2)^{\frac{1}{2}}\ \tilde{V}$ tends to zero, and that $(1-\gamma^2)^{\frac{1}{2}}\ \tilde{V}_\tau$ tends to unity, both uniformly with respect to γ, as $\tau\to t_0>0$. Hence letting $\underline{\xi}\to\underline{\xi}_0$ and $\tau\to t_0>0$ in (17) and (18) one obtains

$$\lim_{\underline{\xi}\to\underline{\xi}_0,\ \tau\to t_0} w(\underline{\xi},\ \tau;\ t_0)=0,$$

$$\lim_{\underline{\xi}\to\underline{\xi}_0,\ \tau\to t_0} w_\tau(\underline{\xi},\ \tau;\ t_0)=\frac{1}{2\pi}\iint\frac{f(\xi_0,\ \eta_0)}{(1-\gamma^2)^{1/2}}d\alpha d\beta=f(\underline{\xi}_0),$$

so that w takes on the correct CAUCHY data.

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(8) With respect to $\theta=\tan^{-1}\beta/\alpha$.

(9) See also (24) for the explicit expression of $\tilde{V}_\tau$.

This completes the proof that (15) is a solution of (13), taking on the boundary data (14) ([10]).

3. **Bounds on w and its Derivatives.** - In order to obtain certain bounds on w and its derivatives which will be required in the next section, it is first necessary to find bounds for

$$\tilde{V}(\gamma,\ \tau;\ t_0)=\frac{2^k t_0^{\,k}(\tau-t_0)}{(1-\gamma^2)^{1/2}[(\tau+t_0)^2-\gamma^2(\tau-t_0)^2]^{k/2}}F\left(\frac{k-1}{2},\ \frac{k}{2},\ \frac{1}{2};\ \zeta\right), \tag{23}$$

and

$$\begin{aligned}\tilde{V}_\tau(\gamma,\ \tau;\ t_0)=&\left[\frac{1}{\tau-t_0}-\frac{k\{(\tau+t_0)-\gamma^2(\tau-t_0)\}}{\{(\tau+t_0)^2-\gamma^2(\tau-t_0)^2\}}\right]\tilde{V}+\\&+\frac{2^{k+1}k(k-1)t_0^{\,k+1}(\tau-t_0)^2(\tau+t_0)(1-\gamma^2)^{1/2}}{\{(\tau+t_0)^2-\gamma^2(\tau-t_0)^2\}^{2+k/2}}F\left(\frac{k+1}{2},\ \frac{k}{2}+1,\ \frac{3}{2};\ \zeta\right),\end{aligned} \tag{24}$$

where

$$\zeta=\frac{1-\gamma^2}{\left(\dfrac{\tau+t_0}{\tau-t_0}\right)^2-\gamma^2}. \tag{25}$$

It will be shown immediately that for $0<\gamma<1,\ 0<t_0<\tau$,

$$|\ \tilde{V}(\gamma,\ \tau;\ t_0)\ |\leq A\tau/(1-\gamma^2)^{1/2}, \tag{26}$$

$$|\ \tilde{V}_\tau(\gamma,\ \tau;\ t_0)\ |\leq B\tau^{\bar{\alpha}}/t_0^{\,\bar{\alpha}}(1-\gamma^2)^{1/2}\quad\begin{cases}\bar{\alpha}=1-\dfrac{k}{2},\ 0<k<1,\\[2mm]\bar{\alpha}=\dfrac{1}{2},\ k\geq 1,\end{cases} \tag{27}$$

where A and B are constants (depending on k). These results are based on the following bounds on $F(a,\ b,\ c;\ \zeta)$ for $0\leq\zeta<1$, see [3, pp. 10, 15]:

$$|\ F(a,\ b,\ c;\ \zeta)\ |\leq M,\qquad c-a-b>0, \tag{28}$$

$$|\ F(a,\ b,\ c;\ \zeta)\ |\leq \overline{M}(1-\zeta)^{c-a-b},\qquad c-a-b<0. \tag{29}$$

The constants M and $\overline{M}$ depend on a, b and c; in what follows a subscript on M will denote a particular application of (28), and a subscript on $\overline{M}$ a particular application of (29). Notice that for $0<\gamma<1,\ 0<t_0<\tau$, the variable ζ given by (25) lies between 0 and 1, so that these bounds apply.

In each of the three cases given below, the estimates are obtained by using either (28) or (29) and replacing γ by 0 or 1, and $\pm t_0$ by τ in appropriate places.

([10]) As a matter of fact if (17) is taken as the definition of w, rather than (15), then w is defined for $\tau>0$, and it is twice continuously differentiable and satisfies the differential equation for $\tau>0$. Limit signs are then superfluous in the last two formulas.

Case I: $0 < k < 1$. - For this range of k, $c - a - b = 1 - k > 0$ for the hypergeometric function in (23). Hence

$$
\begin{aligned}
|\tilde{V}| &\leq \frac{2^k t_0^k(\tau - t_0)}{(1-\gamma^2)^{1/2}(4\tau t_0)^{k/2}} \cdot M_1 \\
&\leq \frac{(\tau - t_0)t_0^{k/2}M_1}{(1-\gamma^2)^{1/2}\tau^{k/2}} \leq \frac{2\tau M_1}{(1-\gamma^2)^{1/2}},
\end{aligned} \tag{30}
$$

so that (26) holds with $A = 2M_1$. For the hypergeometric function in (24), $c - a - b = -k < 0$ so that (29) must be used. This together with the middle estimate for $\tilde{V}$ in (30) yields

$$
\begin{aligned}
|\tilde{V}_\tau| &\leq \left[\frac{1}{\tau - t_0} + \frac{k(\tau + t_0)}{4\tau t_0}\right] \frac{(\tau - t_0)t_0^{k/2}M_1}{(1-\gamma^2)^{1/2}\tau^{k/2}} + \\
&\quad + \frac{2^{k+1}k(1-k)t_0^{k+1}(\tau - t_0)^2(\tau + t_0)(1-\gamma^2)^{1/2}}{[(\tau + t_0)^2 - \gamma^2(\tau - t_0)^2]^{2-k/2} \cdot (4\tau t_0)^k} \overline{M}_1, \\
&\leq (k+1)\left[\frac{\tau + t_0}{2t_0(\tau - t_0)}\right] \cdot \frac{(\tau - t_0)t_0^{k/2}M_1}{(1-\gamma^2)^{1/2}\tau^{k/2}} + \\
&\quad + \frac{2^{k+1}k(1-k)t_0^{k+1}(\tau - t_0)^2(\tau + t_0)(1-\gamma^2)^{1/2}}{[4\tau t_0]^{2+k/2}} \overline{M}_1, \\
&\leq \frac{(k+1)\tau^{1-k/2}M_1}{(1-\gamma^2)^{1/2}t_0^{1-k/2}} + \frac{k(1-k)(1-\gamma^2)^{1/2}\tau^{1-k/2}\overline{M}_1}{t_0^{1-k/2}},
\end{aligned} \tag{31}
$$

so that (27) holds with $B = (k+1)M_1 + k(1-k)\overline{M}_1$.

Case II: $k = 1$. - For this value of k the hypergeometric function in (23) is identically unity. Thus the estimates (30) hold with $M_1 = 1$, so that (26) holds with $A = 2$. Again, the second term on the right hand side of (24) is identically zero, so that the bounds (31) hold with $M_1 = 1$ and $\overline{M}_1 = 0$. Thus finally, (27) holds with $B = 2$.

Case III: $k > 1$. - For these values of k, $c - a - b = 1 - k < 0$ in the hypergeometric function in (23). Hence from (29)

$$
\begin{aligned}
|\tilde{V}| &\leq \frac{2^{2-k}t_0\tau^{1-k}(\tau - t_0)\overline{M}_2}{(1-\gamma^2)^{1/2}(\tau + t_0)^{2-k}\left[1 - \gamma^2\left(\frac{\tau - t_0}{\tau + t_0}\right)^2\right]^{1-k/2}}, \\
&\leq \frac{2^{2-k}t_0\tau^{1-k}(\tau - t_0)\overline{M}_2}{(1-\gamma^2)^{1/2}(\tau + t_0)^{2-k}\left[1 - \gamma^2\left(\frac{\tau - t_0}{\tau + t_0}\right)^2\right]^{1/2}}, \\
&\leq \frac{2^{2-k}t_0\tau^{1-k}(\tau - t_0)(\tau + t_0)^{k-1}\overline{M}_2}{(1-\gamma^2)^{1/2}(4\tau t_0)^{1/2}}, \\
&\leq \frac{(\tau - t_0)t_0^{1/2}\overline{M}_2}{(1-\gamma^2)^{1/2}\tau^{1/2}} \leq \frac{2\tau\overline{M}_2}{(1-\gamma^2)^{1/2}},
\end{aligned}
$$

so that (26) holds with $A = 2\overline{M}_2$. For the hypergeometric function in (24), $c - a - b = -k < 0$ as before, so that (29) together with the last but one estimate above for $\tilde{V}$ gives

$$
\begin{aligned}
|\tilde{V}_\tau| \leq & \left[\frac{1}{\tau - t_0} + \frac{k(\tau + t_0)}{4\tau t_0}\right]\frac{(\tau - t_0)t_0^{1/2}\overline{M}_2}{(1-\gamma^2)^{1/2}\tau^{1/2}} + \\
& + \frac{2^{k+1}k(k-1)t_0^{k+1}(\tau - t_0)^2(1-\gamma^2)^{1/2}\overline{M}_1}{(\tau + t_0)^{3-k}\left[1 - \gamma^2\left(\frac{\tau - t_0}{\tau + t_0}\right)^2\right]^{2-k/2}(4\tau t_0)^k}, \\
\leq & (k+1)\left[\frac{(\tau + t_0)}{2t_0(\tau - t_0)}\right]\cdot\frac{(\tau - t_0)t_0^{1/2}\overline{M}_2}{(1-\gamma^2)^{1/2}\tau^{1/2}} + \\
& + \frac{2^{1-k}k(k-1)t_0(\tau - t_0)^2(1-\gamma^2)^{1/2}\overline{M}_1}{(\tau + t_0)^{3-k}\tau^k\left[1 - \gamma^2\left(\frac{\tau - t_0}{\tau + t_0}\right)^2\right]^{3/2}}, \\
\leq & \frac{(k+1)\tau^{1/2}\overline{M}_2}{(1-\gamma^2)^{1/2}t_0^{1/2}} + \frac{k(k-1)(\tau - t_0)^2(\tau + t_0)^k(1-\gamma^2)^{1/2}\overline{M}_1}{2^{2+k}t_0^{1/2}\tau^{k+3/2}}, \\
\leq & \frac{(k+1)\tau^{1/2}\overline{M}_2}{(1-\gamma^2)^{1/2}t_0^{1/2}} + \frac{k(k-1)\tau^{1/2}(1-\gamma^2)^{1/2}\overline{M}_1}{t_0^{1/2}},
\end{aligned}
$$

so that (27) holds with $B = (k+1)\overline{M}_2 + k(k-1)\overline{M}_1$.

Suppose (ξ, τ) is restricted to a finite closed region R of interest. Then the region of the plane $\tau = t_0$ for which f influences the function w, given by (15), in R will be closed and finite also, and hence f, f_x, f_y, f_{xx}, f_{xy}, f_{yy} will be bounded there by N, say. Using (26) to obtain estimates from (17), (19) and the similar expressions for w_ξ, w_η, and $w_{\xi\eta}$, it is clear that in R

$$
(32) \quad |w|,\ |w_\xi|,\ |w_\eta|,\ |w_{\xi\xi}|,\ |w_{\xi\eta}|,\ |w_{\eta\eta}| \leq \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^1 \frac{AN\tau}{(1-\gamma^2)^{1/2}}\gamma d\gamma = AN\tau.
$$

Similarly, using (27) in (18), one has the estimate

$$
(33) \qquad |w_\tau| \leq \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^1 \frac{BN\tau^\alpha}{t_0^\alpha(1-\gamma^2)^{1/2}}\gamma d\gamma + 2AN\tau = \frac{BN\tau^\alpha}{t_0^\alpha} + 2AN\tau.
$$

Since w satisfies (13), it is clear from these estimates that one has the additional estimate

$$
(34) \qquad |w_{\tau\tau}| \leq \frac{kBN\tau^{\alpha-1}}{t_0^\alpha} + 2AN(\tau + k).
$$

4. A Singular Cauchy Problem for the Non-homogeneous Euler-Poisson-Darboux Equation. - Consider the problem of determining a function $u(\underline{\xi}, \tau)$ which is twice continuously differentiable for $\tau > 0$, once continuously differentiable for $\tau \geq 0$, and has the properties that

$$u_{\xi\xi} + u_{\eta\eta} - u_{\tau\tau} - \frac{k}{\tau} u_\tau = f(\underline{\xi}, \tau), \qquad \tau > 0, \tag{35}$$

$$u(\underline{x}, 0) = 0, \qquad u_\tau(\underline{x}, 0) = 0, \tag{36}$$

where f, f_ξ, f_η, $f_{\xi\xi}$, $f_{\xi\eta}$, $f_{\eta\eta}$ are assumed continuous in the half space $\tau \geq 0$. It will be shown that the unique solution of this problem is given by the formulas

$$u(\underline{\xi}, \tau) = \int_0^\tau w(\underline{\xi}, \tau; t_0)dt_0, \qquad \tau > 0, \tag{37}$$

$$u(\underline{\xi}, 0) = 0, \tag{38}$$

where $w(\underline{\xi}, \tau; t_0)$ is given by the equation (17) with $f(\underline{x})$ replaced by $f(\underline{x}, t_0)$. Moreover it will be shown that the solution (37), (38) has continuous « space derivatives » u_ξ, u_η, $u_{\xi\xi}$, $u_{\xi\eta}$, $u_{\eta\eta}$ for $\tau \geq 0$.

One has formally, from (37) for $\tau > 0$

$$u_\tau = \int_0^\tau w_\tau(\underline{\xi}, \tau; t_0)dt_0 + w(\underline{\xi}, \tau; \tau),$$

$$= \int_0^\tau w_\tau(\underline{\xi}, \tau; t_0)dt_0,$$

from (14). Again,

$$u_{\tau\tau} = \int_0^\tau w_{\tau\tau}(\underline{\xi}, \tau; t_0)dt_0 + w_\tau(\underline{\xi}, \tau; \tau),$$

$$= \int_0^\tau w_{\tau\tau}(\underline{\xi}, \tau; t_0)dt_0 + f(\underline{\xi}, \tau),$$

from (14). Finally

$$u_{\xi\xi} = \int_0^\tau w_{\xi\xi}(\underline{\xi}, \tau; t_0)dt_0, \qquad u_{\eta\eta} = \int_0^\tau w_{\eta\eta}(\underline{\xi}, \tau; t_0)dt_0,$$

so that since w satisfies (13) for each t_0, $0 < t_0 < \tau$, u satisfies (35) for $\tau > 0$. There remains to be shown that each of the integrals appearing in this formal presentation is convergent. Now if $(\underline{\xi}, \tau)$ is restricted to some finite closed region $\mathcal{R}$ of interest (bounded partly by $\tau = 0$), then the region of $(\underline{\xi}, \tau)$ — space for which $f(\underline{x}, t_0)$ influences the function u, given by (37), in $\mathcal{R}$ will be closed and finite; hence f, f_x, f_y, f_{xx}, f_{xy}, f_{yy} will be bounded there by $\mathfrak{N}$, say. The convergence of each of the integrals above now follows from the estimates (32), (33) and (34), with N replaced by $\mathfrak{N}$, since $\alpha < 1$ for $k > 0$, see (27).

It is also clear from these estimates that

$$|u| \leq A\mathfrak{N}\tau^2, \qquad |u_\tau| \leq \frac{B\mathfrak{N}\tau}{1-\alpha} + 2A\mathfrak{N}\tau^2,$$

so that u, $u_\tau \to 0$ as $\tau \to 0$. Hence, see the second formula in (38), u is continuous for $\tau \geq 0$ and since $|u/\tau| \leq A\mathfrak{N}\tau$ (which tends to zero as $\tau \to 0$) u_τ is also. In addition, the conditions (36) are satisfied.

The space derivatives $(u_\xi, u_\eta, u_{\xi\xi}, u_{\xi\eta}, u_{\eta\eta})$ are given, for $\tau > 0$, by $\int_0^\tau (w_\xi, w_\eta, w_{\xi\xi}, w_{\xi\eta}, w_{\eta\eta})\, dt_0$ respectively, these integrals being convergent and each tending to 0 as $\tau \to 0$, from (32). Hence since these space derivatives are all zero for $\tau = 0$, see (38), they are continuous for $\tau \geq 0$.

The uniqueness follows from the fact that (35), with $f \equiv 0$, and (36) imply $u \equiv 0$, see [11, 12].

5. **Equivalence of the Given Cauchy Problem and a Problem in Integral Equations.** - Let $h(\underline{\xi}, \tau)$ be continuous for $\tau \geq 0$ together with its « space derivates » h_ξ, h_η, $h_{\xi\xi}$, $h_{\xi\eta}$, $h_{\eta\eta}$, and $g(\underline{x})$ a twice continuously differentiable function (if $k \geq 1$) or a thrice-continuously differentiable function (if $0 < k < 1$). Then it will be shown that *there exists a unique function* $u(\underline{\xi}, \tau)$, *continuous for* $\tau \geq 0$ *together with its space derivatives* u_ξ, u_η, $u_{\xi\xi}$, $u_{\xi\eta}$, $u_{\eta\eta}$, *which satisfies*

$$\text{(39)} \qquad L(u) - hu \equiv u_{\xi\xi} + u_{\eta\eta} - u_{\tau\tau} - \frac{k}{\tau} u_\tau - hu = 0, \qquad \tau > 0,$$

and takes on the initial data

$$\text{(40)} \qquad u(\underline{x}, 0) = g(\underline{x}), \qquad u_\tau(\underline{x}, 0) = 0.$$

This will be proved by demonstrating the equivalent (second) proposition in integral equations, namely: *There exists a unique solution of* (¹¹)

$$\text{(41)} \qquad u(\underline{\xi}, \tau) = u_0(\underline{\xi}, \tau) + \frac{1}{2\pi}\int_0^\tau dt_0 \iint_{\alpha^2+\beta^2 \leq 1} \tilde{V}(\gamma, \tau, t_0) h(\underline{x}, t_0) u(\underline{x}, t_0)\, d\alpha\, d\beta,$$

(¹¹) The convention is made that for $\tau = 0$ the integral is zero.

where $x = \xi + \alpha(\tau - t_0)$, *which is continuous for* $\tau \geq 0$. *Here* $u_0(\xi, \tau)$ *is the solution of the initial values problem, posed first, for* $h \equiv 0$. In order to demonstrate this equivalence it will be sufficient to show that any solution of the first problem is a solution of the second and conversely.

Let $u(\xi, \tau)$ be a solution of the first problem. Then $\bar{u} = u - u_0$ satisfies

$$L(\bar{u}) = L(u) = hu,$$

and takes on the initial data

$$\bar{u}(x, 0) = 0, \qquad \bar{u}_\tau(x, 0) = 0.$$

Hence putting $f = hu$, the last section shows that

$$\bar{u}(\xi, \tau) = u(\xi, \tau) - u_0(\xi, \tau) = \frac{1}{2\pi}\int_0^\tau dt_0 \iint_{\alpha^2+\beta^2\leq 1} \tilde{V} hu\, d\alpha\, d\beta,$$

where the arguments of $\tilde{V}$, h, u in this integrand are the same as those in (41) respectively. Thus u is a solution of (41).

Conversely, let $u(\xi, \tau)$ be a solution of the second problem. Then it will be shown immediately that it has continuous space derivatives u_ξ, u_η, $u_{\xi\xi}$, $u_{\xi\eta}$, $u_{\eta\eta}$ for $\tau \geq 0$. Hence, as in section 4, the operator L acting on the integral yields $f(\xi, \tau) = h(\xi, \tau)u(\xi, \tau)$ so that

$$L(u) = L(u_0) + hu = hu.$$

Thus u is a solution of (39). Moreover it has been shown that the integral in (41) and its τ-derivative vanish for $\tau = 0$, so that u satisfies the same Cauchy data as u_0 on $\tau = 0$, namely (40).

In order to show that integral equation (41) possesses a unique continuous solution for $\tau \geq 0$, and that this solution has the property that its space derivatives up to and including the second order are continuous for $\tau \geq 0$ also the method of successive approximation due to Picard will be used. A sequence of functions $u_n(\xi, \tau)$, $n = 1, 2, 3, \ldots$ is defined by the recurrence relation ([12])

$$(42) \qquad u_n(\xi, \tau) = \frac{1}{2\pi}\int_0^\tau dt_0 \iint_{\alpha^2+\beta^2\leq 1} \tilde{V}(\gamma, \tau;\, t_0)\, hu_{n-1} d\alpha\, d\beta, \qquad n = 1, 2, 3, \ldots,$$

where the arguments of h and u_{n-1} in integrand are the same as for h and u in (41), and $u_0(\xi, \tau)$ is the function occuring in that equation. Then since u_0 has continuous space derivatives up to and including the second order for

([12]) Thus, as will be clear immediately, $L(u_n) = hu_{n-1}$ and u_n has zero Cauchy data on $\tau = 0$.

$\tau \geq 0$, see [12] ([13]), it follows by induction that the same is true for u_n, for $n = 1, 2, 3, \dots$. For it was shown in the preceding section that these derivatives can be formed from an integral such as that in (42) provided $h(x, t_0)$ and $u_{n-1}(x, t_0)$ have such continuous derivatives for $t_0 \geq 0$, and the integral is defined to be zero for $\tau = 0$. These derivatives $\partial u_n/\partial\xi$, $\partial u_n/\partial\eta$, $\partial^2 u_n/\partial\xi^2$, $\partial^2 u_n/\partial\xi\partial\eta$, $\partial^2 u_n/\partial\eta^2$ will be given by a formula similar to (42) with hu_{n-1} replaced by

$$(43)\qquad \begin{cases} \dfrac{\partial h}{\partial x} \quad u_{n-1} + h\dfrac{\partial u_{n-1}}{\partial x}, \\[2mm] \dfrac{\partial h}{\partial y} \quad u_{n-1} + h\dfrac{\partial u_{n-1}}{\partial y}, \\[2mm] \dfrac{\partial^2 h}{\partial x^2} \quad u_{n-1} + 2\dfrac{\partial h}{\partial x}\,\dfrac{\partial u_{n-1}}{\partial x} + h\dfrac{\partial^2 u_{n-1}}{\partial x^2}, \\[2mm] \dfrac{\partial^2 h}{\partial x\partial y}\, u_{n-1} + \dfrac{\partial h}{\partial x}\,\dfrac{\partial u_{n-1}}{\partial y} + \dfrac{\partial h}{\partial y}\,\dfrac{\partial u_{n-1}}{\partial x} + h\dfrac{\partial^2 u_{n-1}}{\partial x\partial y}, \\[2mm] \dfrac{\partial^2 h}{\partial y^2} \quad u_{n-1} + 2\dfrac{\partial h}{\partial y}\,\dfrac{\partial u_{n-1}}{\partial y} + h\dfrac{\partial^2 u_{n-1}}{\partial y^2}, \end{cases}$$

respectively. There remains to be shown that the series $\sum\limits_{n=0}^{\infty} u_n(\xi, \tau)$ converges uniformly in any arbitrarily large retrograde half-cone with base on $\tau = 0$, together with the corresponding series of space derivatives. The solution of the integral equation will then be the sum function of this series, and it will have the stated continuity properties.

In such a cone, $u_n(\xi, \tau)$ and its space derivatives are determined by the values of $u_{n-1}(\xi, \tau)$ and its space derivatives in the same cone. Let H be an upper bound for the absolute values of $h(\xi, \tau)$ and its space derivatives, and M an upper bound for the absolute values of $u_0(\xi, \tau)$ and its space derivatives. Then

$$(44)\qquad |u_n|, \left|\frac{\partial u_n}{\partial\xi}\right|, \left|\frac{\partial u_n}{\partial\eta}\right|, \left|\frac{\partial^2 u_n}{\partial\xi^2}\right|, \left|\frac{\partial^2 u_n}{\partial\xi\partial\eta}\right|, \left|\frac{\partial^2 u_n}{\partial\eta^2}\right| \leq \frac{2^{2n}A^n H^n M \tau^{2n}}{(2n-1)(2n-3)\dots 1}, \quad n \geq 1.$$

For clearly from (26), (42) and the corresponding integrals involving (43),

$$|u_1| \leq \frac{1}{2\pi}\int_0^\tau dt_0 \int_0^{2\pi} d\theta \int_0^1 \gamma d\gamma \left[\frac{A\tau}{(1-\gamma^2)^{1/2}} \cdot H \cdot M\right] = AHM\tau^2,$$

$$\left|\frac{\partial u_1}{\partial\xi}\right|, \left|\frac{\partial u_1}{\partial\eta}\right| \leq 2AHM\tau^2,$$

$$\left|\frac{\partial^2 u_1}{\partial\xi^2}\right|, \left|\frac{\partial^2 u_1}{\partial\xi\partial\eta}\right|, \left|\frac{\partial^2 u_1}{\partial\eta^2}\right| \leq 4AHM\tau^2,$$

([13]) Although it is not stated explicitly in that paper, it follows from the formulas given there that u_0 has continuous space derivatives up to and including the second order for $\tau \geq 0$, provided that for (*i*) $0 < k < 1$, i. e. $m - 2 < k < m - 1$, g has continuous third order derivates, and for (*ii*) $k \geq 1$, i. e. $k \geq m - 1$, g has continuous second order derivatives.

so that (44) is true for $n=1$. Moreover assuming (44) is true for n, one has in a similar manner

$$| u_{n+1} | \leq \frac{1}{2\pi}\int_0^\tau dt_0 \int_0^{2\pi} d\theta \int_0^1 \gamma d\gamma \left[\frac{A\tau}{(1-\gamma^2)^{1/2}} \cdot H \cdot 2^{2n} A^n H^n M \frac{t_0^{2n}}{(2n-1)(2n-3)\dots 1}\right],$$

$$= \frac{2^{2n} A^{n+1} H^{n+1} M \tau^{2n+2}}{(2n+1)(2n-1)\dots 1},$$

$$\left|\frac{\partial u_{n+1}}{\partial \xi}\right|, \left|\frac{\partial u_{n+1}}{\partial \eta}\right| \leq \frac{2 \cdot 2^{2n} A^{n+1} H^{n+1} M \tau^{2n+2}}{(2n+1)(2n-1)\dots 1},$$

$$\left|\frac{\partial^2 u_{n+1}}{\partial \xi^2}\right|, \left|\frac{\partial^2 u_{n+1}}{\partial \xi \partial \eta}\right|, \left|\frac{\partial^2 u_{n+1}}{\partial \eta^2}\right| \leq \frac{4 \cdot 2^{2n} A^{n+1} H^{n+1} M \tau^{2n+2}}{(2n+1)(2n-1)\dots 1},$$

so that (44) is true for $n+1$. The uniform convergence in the half-cone of the six series concerned now follows from that of the series whose general term is the right hand side of the inequality (44).

The uniqueness follows by induction in the usual way. Suppose $u^{(1)}, u^{(2)}$ are continuous solutions and let $\tilde{u} = u^{(1)} - u^{(2)}$. Then if $\tilde{M}$ is an upper bound on $| \tilde{u} |$ in the half-cone, the integral equation

$$\tilde{u}(\underline{\xi}, \tau) = \frac{1}{2\pi}\int_0^\tau dt_0 \iint_{\alpha^2+\beta^2\leq 1} \tilde{V}(\gamma, \tau, t_0) h(\underline{x}, t_0) \tilde{u}(\underline{x}, t_0) d\alpha d\beta, \tag{45}$$

satisfied by $\tilde{u}$, implies

$$| \tilde{u}(\underline{\xi}, \tau) | \leq \frac{A^n H^n \tilde{M} \tau^{2n}}{(2n-1)(2n-3)\dots 1}, \tag{46}$$

for any integer $n \geq 1$, in this half-cone. For if (46) is true for n, the equation (45) gives

$$| \tilde{u}(\underline{\xi}, \tau) | \leq \frac{1}{2\pi}\int_0^\tau dt_0 \int_0^{2\pi} d\theta \int_0^1 \gamma d\gamma \left[\frac{A\tau}{1-\gamma^2)^{1/2}} \cdot H \cdot \frac{A^n H^n \tilde{M} t_0^{2n}}{(2n-1)(2n-3)\dots 1}\right],$$

$$= \frac{A^{n+1} H^{n+1} \tilde{M} \tau^{2n+2}}{(2n+1)(2n-1)\dots 1},$$

so that (46) is true for $n+1$. That it is true for $n=1$ follows on using the original bound $\tilde{M}$ in the integral of (45). Since for each fixed τ the right hand side of (46) tends to zero as $n \to \infty$, $\tilde{u} \equiv 0$ in an arbitrarily large half-cone, and hence everywhere for $\tau \geq 0$.

Concluding Remarks. - In section 2, a certain regular CAUCHY problem, with data (14), for the equation (13) was solved. As was remarked in the introduction, HADAMARD's method, upon which this solution was based, leads to « finite parts » when the corresponding CAUCHY problem in $m > 2$ space variables, for the EULER-POISSON-DARBOUX equation (3a), is considered. As is well-known, the relation between the concept of « finite part » and analytic continuation has been developed by RIESZ and applied to the solution of regular CAUCHY problems [4] for second order equations, and these ideas have been extended to linear partial differential equations of arbitrary order with constant coefficients by GÅRDING [6]. DAVIS [13], in a doctoral dissertation now in preparation, has applied a modified version of RIESZ's method to solve the regular CAUCHY problem

$$\Delta u - u_{\tau\tau} - \frac{k}{\tau} u_\tau = 0, \qquad \tau > t_0 > 0,$$
$$u(\underline{x}, t_0) = f(\underline{x}), \qquad u_\tau(\underline{x}, t_0) = 0,$$

with $\Delta \equiv \sum_{i=1}^{m} \frac{\partial^2}{\partial \xi_i^2}$ and m arbitrary.

The following three remarks concern the CAUCHY problem treated in the present paper. First, it is clear that the restriction made that the given function f, appearing in (14) of section 2, be defined for all $\underline{x}$ is adopted for convenience. Only unessential modifications arise if the function f is not defined for all $\underline{x}$. A similar remark applies to the functions $f(\underline{x}, t)$, $g(\underline{x})$ and $h(\underline{x}, t)$. Secondly, it should be noticed that the CAUCHY problem (39), (40) of section 5 is wellposed in the sense of HADAMARD. By this is meant that if (in the notation of (39))

$$L(u^{(i)}) - hu^{(i)} = 0, \qquad \tau > 0.$$
$$u^{(i)}(\underline{x}, 0) = g^{(i)}(\underline{x}), \qquad u^{(i)}_\tau(\underline{x}, 0) = 0,$$

where $i = 1, 2$, and if $| g^{(1)} - g^{(2)} | < \varepsilon$ on the base (in the plane $\tau = 0$) of a given retrograde half-cone, then $| u^{(1)} - u^{(2)} | < K\varepsilon$ throughout this half-cone, where the constant K depends only on k, the upper bound H for $| h |$ in this half-cone, and the height τ_0 of the cone. The proof follows in the usual way by successively bounding $| u^{(1)}_n - u^{(2)}_n |$ and noting that a similar theorem holds for $| u^{(1)}_0 - u^{(2)}_0 |$. Thirdly it should be pointed out that the present method breaks down when linear terms in u_ξ and u_η are introduced into (39). For then the integrand in (41) will include terms in u_x and u_y, and upon differentation of the integral equation there will always be derivatives on the right hand side which are of one higher order that on the left. This means, see (42), that from a knowledge of bounds on u_0 and its derivatives up to the second order, only bounds on u_1 and its derivatives up to the first order can be obtained.

In particular, the theorem of section 5 holds when the functions g and h of (39) and (40) depend on only one space-variable, say $g(\underline{x}) = g(x)$ and

$\underline{h}(\xi, \tau) = h(\xi, \tau)$. Then clearly the unique solution $\underline{u}(\xi, \tau) = u(\xi, \tau)$ depends only on one space variable ξ. Thus as a special case a corresponding theorem for the differential equation

$$u_{\xi\xi} - u_{\tau\tau} - \frac{k}{\tau} u_\tau - h(\xi, \tau) u = 0,$$

in one space variable ξ, is obtained (cf. GERMAIN and BADER [7] for a related theorem for $k = 1/3$).

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