On a class of nonlinear fourth order differential equations. (*)

by PUI-KEI WONG (**)

Summary. - *In this paper we study the nonlinear fourth order differential equation $u^{iv} \pm F(u, x)u = 0$, where F is a positive monotone function of u . Asymptotic behaviour of certain solutions are treated in sections 2 and 4 while a two point boundary value problem is studied in section 3.*

1. - In this paper we are concerned with nonlinear differential equations of the form

$$(1.1) \quad u^{iv} - F(u, x)u = 0$$

and

$$(1.2) \quad u^{iv} + F(u, x)u = 0,$$

where $u^{iv} = d^4u/dx^4$. The function F is assumed to satisfy the following hypotheses:

H1. $F(u, x)$ is continuous in u and x for $x \geq x_0$ and $0 \leq u < \infty$,

H2. $F(u, x) > 0$ for $u > 0$ and $x \geq x_0$,

H3. $U^{-2\varepsilon}F(u, x)$ is a strictly increasing function of u for each $x \geq x_0$ and some constant $\varepsilon > 0$.

We remark that many of the results remain true if H3 is replaced by the weaker assumption that $F(u, x) < F(v, x)$ for $0 < u < v < \infty$ and $x \geq x_0$, i.e., with $\varepsilon = 0$ in H3. It will be apparent in the proofs whether the weaker hypothesis will suffice.

In sections 2 and 4 of this paper we present some necessary and sufficient conditions for solutions of equations (1.1) and (1.2) respectively to have prescribed asymptotic forms. In section 3 a two point boundary value problem for equation (1.1) is treated by a method used on second order equations [7]. For related studies on the oscillatory behavior of solutions of

$$y^{(n)} + f(x)g(y) = 0,$$

see [2, 3, 5].

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(**) Michigan State University, East Lansing, Michigan, U.S.A.

2. - In this section we shall study the asymptotic behavior of some positive proper solutions of equation (1.1). By a proper solution we understand a real-valued solution of (1.1) which is of class $C^4[a, \infty)$, where $a \geq x_0$. Analogous results have been obtained earlier [8] for the corresponding second order equation $u'' = F(u, x)u$. Actually, it will be apparent from our method of proof that these same results may be extended to the equation $u^{(n)} = F(u, x)u$.

To facilitate discussion we shall call a proper solution of (1.1) a solution of type I if $(-1)^k u^{(k)}(x) \geq 0$, $k = 0, 1, 2, 3$. A proper solution u is of type II if u is eventually positive, convex increasing, and u''' is of one sign.

THEOREM 2.1 - *A necessary and sufficient condition for equation (1.1) to have proper solutions u of type I which ultimately decrease monotonically to positive constants is that*

$$(2.1) \quad \int_{x_0}^{\infty} x^3 F(B, x) dx < +\infty$$

for some constant $B > 0$.

PROOF. - If u is such a solution, then by TAYLOR'S Theorem,

$$(2.2) \quad u(x) = \sum_{k=0}^3 (-1)^k u^{(k)}(b)(b-x)^k/k! + (1/6) \int_x^b (t-x)^3 u F(u, t) dt.$$

Since $(-1)^k u^{(k)}(x) \geq 0$ for $k = 0, 1, 2, 3$, it follows that

$$\begin{aligned} u(a) &\geq u(x) \geq (1/6) \int_x^b (t-x)^3 u F(u, t) dt \\ &\geq (C/6) \int_x^b (t-x)^3 F(C, t) dt, \end{aligned}$$

where $C = \lim_{x \rightarrow \infty} u(x) > 0$. (2.1) follows from this immediately.

To prove sufficiency, we consider the integral equation

$$(2.3) \quad u(x) = C + (1/6) \int_x^{\infty} (t-x)^3 u(t) F(u, t) dt.$$

Suppose B is a positive constant for which (2.1) holds, then there must be a point $a \geq x_0$ such that for all $x \geq a$,

$$(2.4) \quad \int_x^\infty (t-x)^3 F(B, t) dt < 3.$$

We now define an approximating sequence $\{u_k(x)\}$ by

$$(2.5) \quad \begin{cases} u_0(x) = C, \\ u_{k+1}(x) = C + (1/6) \int_x^\infty (t-x)^3 u_k F(u_k, t) dt \\ k = 0, 1, 2, \dots \end{cases}$$

If we choose C such that $0 < C < B/2$, we see immediately that, for all k , $0 < C \leq u_k(x)$. By assuming $u_k(x) < B$ and using (2.4), we find that

$$u_{k+1}(x) < (B/2) + (B/6) \int_x^\infty (t-x)^3 F(B, t) dt < B.$$

Since $u_0(x) < B$, it follows by induction that $0 < C \leq u_k(x) < B$ for all k . Moreover, for any two points x_1 and x_2 such that $a \leq x_1 < x_2 < \infty$, a simple calculation shows that

$$2|u_{n+1}(x_2) - u_{n+1}(x_1)| \leq |x_2 - x_1| \int_{x_1}^\infty (t-x_1)^2 u_n F(u_n, t) dt.$$

In view of the uniform boundedness of $\{u_n\}$ and condition (2.1), it follows that the sequence is equicontinuous also. According to H3, $F(u, x) < F(v, x)$ whenever $0 < u < v < \infty$, it follows from the assumption $u_{n+1}(t) \geq u_n(t)$ and (2.5) that

$$u_{n+2}(x) - u_{n+1}(x) = \int_x^\infty (t-x)^3 [u_{n+1} F(u_{n+1}, t) - u_n F(u_n, t)] dt > 0.$$

This, together, with the fact that $u_1(x) > u_0(x)$, shows that $\{u_k(x)\}$ is also a monotonically increasing sequence. We can therefore extract a uniformly

convergent subsequence whose limit function u is a solution of (2.3) in $[a, \infty)$.

It remains to show that the solution u so obtained is indeed a proper solution of type I which satisfies (1.1). To this end we note that for $x \geq a$ and $h > 0$,

$$\begin{aligned} & \left| h^{-1}[u(x+h) - u(x)] + (1/2) \int_x^\infty (t-x)^2 F(u, t) u(t) dt \right| \\ & \leq (h/6) \left\{ \int_{x+h}^\infty |3(t-x) - h| F(u, t) u(t) dt + \int_x^{x+h} |t-x - 3h| u F(u, t) dt \right\}. \end{aligned}$$

A corresponding inequality holds for $h < 0$. Since the solution u is continuous in $[a, \infty)$, it follows that

$$2u'(x) = - \int_x^\infty (t-x)^2 u(t) F(u, t) dt \leq 0.$$

In a similar manner one can show successively that

$$u''(x) = \int_x^\infty (t-x) u(t) F(u, t) dt \geq 0,$$

$$u'''(x) = - \int_x^\infty u(t) F(u, t) dt \leq 0,$$

and $u^{iv} = u F(u, x)$. This proves the theorem.

Before proceeding to the next theorem on solutions of type II, we first state as a lemma the following well known (see e.g., [1]). Tauberian theorem:

LEMMA 2.2. - *Let $f(x) \sim x^s$, $s > 0$. If $f'(x)$ is positive, integrable and monotonically non-decreasing, then $f'(x) \sim s x^{s-1}$.*

THEOREM 2.3. - *A necessary and sufficient condition for equation (1.1) to have positive proper solutions of type II with asymptotic form*

$$(2.6) \quad u(x) \sim C x^m, \quad C > 0, \quad m = 1, 2, 3$$

is that there exist a constant $B > 0$ such that

$$(2.7) \quad \int_x^\infty x^{m+3} F(Bx^m, x) dx = O(x^m)$$

for all sufficiently large x .

PROOF. - If u is such a solution, then (2.2) shows that

$$x^{-m}u(x) \geq x^{-m}[u'(b)(x-b) + u'''(b)(x-b)^3/3!] \\ + (1/6x^m) \int_x^\infty (t-x)^3 u(t) F(u, t) dt.$$

In view of (2.2) there exists a point $a \geq x_0$ such that

$$1/2 \leq u(x)/Cx^m \leq (3/2)$$

for all $x \geq a$. Moreover, for $m = 1, 2, 3$, Lemma 2.2 shows successively that the quantity

$$|x^{-m}[u'(b)(x-b) + u'''(b)(x-b)^3/6]|$$

is bounded uniformly so that

$$3C/2 + M \geq (C/12x^m) \int_x^b (t-x)^3 t^m F(Ct^m/2, t) dt$$

for all $b > x \geq a$. The necessity of (2.7) follows immediately.

As in the proof of Theorem 2.1, the sufficiency of condition (2.7) depends on our solving the integral equation

$$(2.8) \quad v(x) = C + (1/6x^m) \int_x^\infty (t-x)^3 t^m v(t) F(t^m v, t) dt$$

by iteration. Except for the obvious computational differences, the arguments involved are entirely analogous and will therefore be omitted. If $v(x)$ is the solution of (2.8) so obtained, the function $u(x) = x^m v(x)$ will be the desired solution of (1.1) having asymptotic form (2.6).

3. - In this section we shall consider a two-point boundary value problem associated with equation (1.1), namely,

$$(3.1) \quad \begin{aligned} u^{iv} - uF(u, x) &= 0 \\ u(a) = u'(a) = u(b) = u'(b) &= 0, \end{aligned}$$

where $x_0 \leq a < b < \infty$. With suitable modifications, the methods used here can be applied to the problem with a triple zero at a and a simple zero at b .

In earlier papers MOORE and NEHARI [6] and NEHARI [7] have studied analogous problems for the equations $u'' + p(x)u^{2n+1} = 0$ and $u'' + uF(u^2, x) = 0$ respectively.

Problem (3.1) is equivalent to a variational problem for which the direct methods of the calculus of variations are applicable. The results given here are direct extensions of those obtained by NEHARI in [7].

We begin with the functional J defined by

$$(3.2) \quad \int_a^\infty [u'']^2 - 2h(u, x) dx,$$

where

$$(3.3) \quad h(u, x) = \int_0^u tF(t, x) dt.$$

It is easy to see that the EULER-LAGRANGE equation associated with the functional J is precisely equation (1.1). Let Ω denote the family of twice differentiable function on $[a, b]$ for which $u(a) = u'(a) = 0$, $u(x) \geq 0$ on (a, b) and $u(x) \equiv 0$. Moreover, we further suppose that u satisfies the normalization condition

$$(3.4) \quad \int_a^b (u'')^2 dx = \int_a^b u^2 F(u, x) dx.$$

If v is any function of Ω then H3 implies the existence of a positive constant k such that $u = kv$ will satisfy (3.4). We can now state the principal result of this section as follows:

THEOREM 3.1. - *The variational problem*

$$(3.5) \quad \min_{\Omega} J(u) = \lambda(a, b)$$

has a solution $u \in C^4[a, b]$ such that u is also a solution of problem (3.1).

PROOF. - Since we are concerned with a minimum problem we may further restrict ourselves to those functions of Ω for which

$$(3.6) \quad \int_a^b (u'')^2 dx \leq M,$$

where M is some fixed positive constant. Let Ω^* denote this subset of Ω .

By H3, $u > t > 0$ implies $u^{-2\varepsilon}F(u, x) > t^{-2\varepsilon}F(t, x)$ so that

$$\begin{aligned} h(u, x) &= \int_0^u tF(t, x)dt \leq u^{-2\varepsilon}F(u, x) \int_0^u t^{1+2\varepsilon}dt \\ &= u^2F(u, x)/(2 + 2\varepsilon). \end{aligned}$$

It follows that

$$u^2F(u, x) - 2h(u, x) \geq u^2F(u, x)/(1 + \varepsilon).$$

Combining this with condition (3.4) we see that

$$(3.7) \quad J(u) \geq (1 + \varepsilon)^{-1} \int_a^b (u'')^2 dx > 0.$$

Hence, $\lambda(a, b) = \inf_{\Omega^*} J(u)$ exists and the variational problem (3.5) is therefore meaningful.

According to TAYLOR'S theorem, we have

$$u(x) = \int_a^x (x - t)u''(t)dt.$$

Applying SCHWARZ'S inequality and (3.6) we arrive at the estimate

$$(3.8) \quad |u(x)|^2 \leq M(b - a)^3/3, \quad x \in [a, b].$$

Moreover, for any two points x_1 and x_2 such that $a \leq x_1 < x_2 \leq b$,

$$\begin{aligned} |u(x_2) - u(x_1)| &\leq |x_2 - x_1| \int_a^{x_2} |u''(t)| dt + \int_{x_1}^{x_2} |x_1 - t| |u''(t)| dt \\ &\leq |x_2 - x_1| [2M(b - a)]^{1/2} \end{aligned}$$

so that the family Ω^* of admissible function is in fact an uniformly bounded and equi-continuous family. Consequently, there exists a sequence $\{u_n\}$ in Ω^* which converges uniformly on $[a, b]$ to a continuous function u such that

$$(3.9) \quad \lim J(u_n) = J(u) = \lambda(a, b).$$

Let u be any member of the minimizing sequence given above. We now construct a new function y of class $C^4[a, b]$ by solving the linear inhomogeneous system

$$(3.10) \quad \begin{aligned} y^{iv} &= \alpha u F(u, x) \\ y(a) &= y'(a) = y(b) = y'(b) = 0, \end{aligned}$$

where the constant α is determined by requiring y to satisfy condition (3.4). Integration by parts shows that

$$(3.11) \quad \int_a^b u y^{iv} dx = \int_a^b u'' y'' dx.$$

Combining this with (3.4) and (3.10) we have

$$\begin{aligned} \left[\alpha \int_a^b u^2 F(u, x) dx \right]^2 &= \left[\int_a^b u'' y'' dx \right]^2 \leq \int_a^b (u'')^2 dx \int_a^b (y'')^2 dx \\ &= \int_a^b u^2 F(u, x) dx \int_a^b y^2 F(y, x) dx. \end{aligned}$$

Consequently,

$$(3.12) \quad \alpha^2 \int_a^b u^2 F(u, x) dx \leq \int_a^b y^2 F(y, x) dx.$$

On the other hand,

$$\begin{aligned} \left[\int_a^b y^2 F(y, x) dx \right]^2 &= \left[\int_a^b (y'')^2 dx \right]^2 = \left[\int_a^b y^{iv} y dx \right]^2 \\ &= \alpha^2 \left[\int_a^b y u F(u, x) dx \right]^2 \\ &\leq \alpha^2 \int_a^b u^2 F(u, x) dx \int_a^b y^2 F(y, x) dx \end{aligned}$$

so that, by (3.12), we have

$$(3.13) \quad \int_a^b y^2 F(y, x) dx \leq \int_a^b y^2 F(u, x) dx.$$

Now $F(y, x)$ is an increasing function of y so that (3.3) implies

$$2[h(u, x) - h(y, x)] \leq F(u, x)(u^2 - y^2).$$

Combining this with (3.4) and (3.13) we finally arrive at

$$(3.14) \quad J(u) \geq J(y).$$

This argument shows that the minimizing sequence $\{u_n\}$ may be replaced by another minimizing sequence $\{y_n\}$ which is of class $C^4[a, b]$. In view of (3.6) and (3.12), we have

$$\alpha_n^2 \lambda(a, b) \leq M.$$

Moreover, we know that

$$y_n(x) = \alpha_n \int_a^b g(x, t) u_n(t) F(u_n, t) dt,$$

where $g(x, t)$ is the GREEN'S function for the linear operator

$$\begin{aligned} Lu &= -u^{iv} \\ u(a) &= u'(a) = u(b) = u'(b) = 0. \end{aligned}$$

It follows from the uniform convergence of $\{u_n\}$ that the sequence $\{y_n\}$ will necessarily converge to a function $y \in C^4[a, b]$. Moreover, in view of (3.14) and (3.5), we have

$$\lambda = \lim J(u_n) \geq \lim J(y_n) = J(y) = \lambda$$

so that

$$J(y) = \lambda(a, b) = \inf_{\Omega^*} J(u).$$

We now show that $y(x) \neq 0$ on $[a, b]$. To do this we set

$$(3.15) \quad A^2 = \int_a^b (y'')^2 dx$$

so that

$$[y'(x)]^2 = \left(\int_a^x y''(t) dt \right)^2 \leq A^2(x - a) \leq A^2(b - a)$$

$$\begin{aligned} y(x)^2 &= \left(\int_a^x y'(t) dt \right)^2 \leq A^2(b-a)(x-a)^2 \\ &= B^2(x-a)^2, \end{aligned}$$

where $B^2 = A^2(b-a)$. Thus, $y(x) \leq B(x-a)$. Combining this with (3.4) and (3.15) we arrive at

$$1 \leq (b-a) \int_a^b (x-a) F[B(x-a), x] dx.$$

In view of the monotonicity hypothesis H3, this implies A is bounded from below by A_0 . Consequently, we have

$$\int_a^b (y'')^2 dx \geq A_0^2 > 0.$$

From this we conclude that $y(x) \equiv 0$.

Finally we note that equality can hold in (3.14) only if equality holds in (3.13) which in turn requires equality in SCHWARZ'S inequality, i.e., $y = u$ on $[a, b]$. In this case y must in fact be a solution of

$$\begin{aligned} y^{iv} &= \alpha y F(y, x) \\ y(a) &= y'(a) = y(b) = y'(b) = 0. \end{aligned}$$

Integration by parts show that

$$\int_a^b (y'')^2 dx = \alpha \int_a^b y^2 F(y, x) dx.$$

Comparing this with normalization (3.4) we conclude that $\alpha = 1$. Thus, equality is attained in (3.14) only if y is a solution of problem (3.5). This completes the proof of the theorem.

Our next result shows the effect on the minimum value $\lambda(a, b)$ when the function F is altered. Before we state this theorem we first note that if $a' \leq a < b \leq b'$, then it follows trivially that

$$(3.16) \quad \lambda(a', b') \leq \lambda(a, b).$$

THEOREM 3.2. - *Let F_1 and F_2 be two functions satisfying hypotheses H1-3. Moreover, we suppose that for each $t > 0$ and $x \in [a, b]$, $F_1(t, x) \leq F_2(t, x)$. If $\lambda_1(a, b)$ and $\lambda_2(a, b)$ are the respective minima associated with the equations*

$$(3.17) \quad \begin{aligned} u_k'' &= u_k F_k(u_k, x), \\ u_k(a) &= u_k'(a) = u_k(b) = u_k'(b) = 0, \quad k = 1, 2, \end{aligned}$$

then $\lambda_1(a, b) \geq \lambda_2(a, b)$.

PROOF. - Let u_1 be a solution of the system (3.17) associated with the function F_1 . Then by the monotonicity property of F_2 we can find a positive constant A such that

$$\int_a^b (u_1'')^2 dx = \int_a^b u_1^2 F_2(Au_1, x) dx.$$

The function $w(x) = Au_1(x)$ will clearly satisfy the normalization

$$\int_a^b (w'')^2 dx = \int_a^b w^2 F_2(w, x) dx$$

and is therefore a member of the admissible family Ω_2 . It follows that

$$\lambda_2(a, b) = \min_{\Omega_2} J_2(u) \leq \int_a^b [(w'')^2 - 2h_2(w, x)] dx.$$

Since $F_2(t, x) \geq F_1(t, x)$, (3.3) shows that $h_2(w, x) \geq h_1(w, x)$. Moreover,

$$h_1(Au_1, x) - h_1(u_1, x) = \int_{u_1}^{Au_1} t F_1(t, x) dt \geq \frac{1}{2} (A^2 - 1) u_1^2 F_1(u_1, x)$$

so that

$$2 \int_a^b h_1(Au_1, x) dx \geq 2 \int_a^b h_1(u_1, x) dx + (A^2 - 1) \int_a^b u_1^2 F_1(u_1, x) dx.$$

These three inequalities together with normalization (3.4) for u_1^- show that

$$\begin{aligned}
 \lambda_2(a, b) &\leq \int_a^b [(w'')^2 - 2h_1(w, x)] dx \\
 &= A^2 \int_a^b (u_1')^2 dx - 2 \int_a^b h_1(Au_1, x) dx \\
 &\leq A^2 \int_a^b (u_1')^2 dx - 2 \int_a^b h_1(u_1, x) dx + (1 - A^2) \int_a^b u_1^2 F_1(u_1, x) dx \\
 &= \int_a^b [(u_1')^2 - 2h_1^-(u_1, x)] dx \\
 &= \lambda_1(a, b).
 \end{aligned}$$

This proves our assertion.

4. - In this section we shall treat briefly some asymptotic properties of positive solutions of equation (1.2). Oscillation criteria for equations of this type have been given recently [2] and [3]. Theorem 4.4 given here is an extension of a result of LEIGHTON and NEHARI [4] for linear fourth-order equations.

LEMMA 4.1. - *Let $p(x)$ be positive and continuous for $x \geq x_0$. Then no nontrivial solution of the differential inequality*

$$(4.1) \quad w^{iv} + p(x)w \leq 0$$

can have more than one double zero on $[a, \infty)$, where $a \geq x_0$.

PROOF. - Let w be a nontrivial solution of (4.1), and we set

$$(4.2) \quad G(x) = w'w'' - ww''''.$$

Then,

$$\begin{aligned}
 G'(x) &= (w'')^2 - ww^{iv} \\
 &\leq (w'')^2 + p(x)w^2 > 0
 \end{aligned}$$

so that, for $b > a$,

$$(4.3) \quad G(b) - G(a) = \int_a^b G'(x) dx > 0.$$

If w has a double zero at a point $c \in [a, \infty)$, then (4.2) shows that $G(c) = 0$. By (4.3), G is a strictly increasing function of x for $x > c$ so that G cannot vanish for $x > c$. This proves the assertion.

LEMMA 4.2. - *The boundary value problem*

$$(4.4) \quad \begin{cases} u^{iv} + F(u, x)u = 0 \\ u(a) = A, \quad u(b) = B, \\ u'(a) = A', \quad u'(b) = B', \end{cases}$$

with $b > a$ and $A, B > 0$ has at most one positive solution in $[a, b]$.

PROOF. - Suppose the contrary. Let u and y be two positive solutions of (4.4). We further assume that (a, A) and (b, B) are consecutive points of intersection of u and y . Set $u(x) > y(x)$ for $a < x < b$, then

$$(u - y)^{iv} + uF(u, x) - yF(y, x) = 0$$

or

$$(u - y)^{iv} + (u - y)F(y, x) + u[F(u, x) - F(y, x)] = 0.$$

Now, $u(x) > y(x)$ implies $F(u, x) > F(y, x)$ so that, by setting $w(x) = u(x) - y(x)$, we obtain

$$(4.5) \quad w^{iv} + wF(y, x) \leq 0.$$

Since both u and y are solutions of (4.4), $w(x)$ clearly has double zeros at a and b which is impossible as Lemma (4.1) is applicable to inequality (4.5). This being true for each consecutive pairs of points of intersection, the result follows.

LEMMA 4.3. - *Let u be a solution of (1.2) such that $u(x) > 0$ for $x > a$. Then*

$$(4.6) \quad \lim_{x \rightarrow \infty} 6x^{-3}u(x) = \lim_{x \rightarrow \infty} u'''(x).$$

PROOF. - Let $P_3(x)$ be a polynomial of degree 3, then every solution u of equation (1.2) can be written as

$$6P_3(x) = 6u(x) + \int_a^x (x - t)^3 u(t) F(u, t) dt.$$

Hence,

$$\begin{aligned} 6P_3(x) &\leq 6u(x) + (x-a)^3 \int_a^x u(t)F(u, t)dt \\ &= 6u(x) + (x-a)^3[u'''(a) - u'''(x)]. \end{aligned}$$

Now, it is clear that

$$\lim_{x \rightarrow \infty} (x-a)^3 P_3(x) = u'''(a)$$

and that $u'''(x)$ is a monotonically decreasing function as x tends to ∞ so that

$$(4.7) \quad \lim u'''(x) \leq \liminf 6u(x)(x-a)^{-3}.$$

On the other hand, by the mean value theorem, we can find a point $s \in (a, x)$ such that

$$\begin{aligned} 6P_3(x) &\geq 6u(x) + \int_a^s (x-t)^3 u(t)F(u, t)dt \\ &\geq 6u(x) + (x-s)^3 \int_a^s u(t)F(u, t)dt \\ &\geq 6u(x) + (x-s)^3[u'''(a) - u'''(s)]. \end{aligned}$$

Fixing s , we find that

$$6(x-a)^{-3}P_3(x) \geq (x-a)^{-3}6u(x) + [u'''(a) - u'''(s)][(x-s)/(x-a)]^{-3}.$$

By taking x through a sequence of points $\{x_n\}$ such that $(x-a)^{-3}u(x)$ tends to its upper limit, we see that for $s > a$.

$$u'''(s) \geq \limsup (x-a)^{-3}6u(x)$$

so that

$$(4.8) \quad \limsup 6(x-a)^{-3}u(x) \leq \lim u'''(s).$$

Combining (4.7) with (4.8) we obtain the desired result.

THEOREM 4.4. - *A necessary and sufficient condition for equation (1.2) to have a positive solution u such that*

$$(4.9) \quad u(x) \sim Ax^3, \quad A > 0,$$

is that there exists a constant $B > 0$ for which

$$(4.10) \quad \int_a^\infty x^3 F(Bx^3, x) dx < +\infty.$$

PROOF. - To prove sufficiency we suppose that (4.10) is satisfied for some $B > 0$. We can then find a point a large enough so that

$$(4.11) \quad \int_a^\infty x^3 F(Bx^3, x) dx < 5.$$

Consider the solution u given by

$$6B(x-a)^3 = 6u(x) + \int_a^x (x-t)^3 u(t) F(u, t) dt.$$

If we can find a point $b > a$ such that, for $a < x < b$, $u(x) > 0$ and $u(b) = 0$, then

$$(4.12) \quad Bx^3 \geq B(x-a)^3 \geq u(x) \quad x \in (a, b).$$

Now,

$$\begin{aligned} 6B(b-a)^3 &= 6u(b) + \int_a^b (b-t)^3 u(t) F(u, t) dt \\ &= \int_a^b (b-t)^3 u(t) F(u, t) dt \\ &\leq (b-a)^3 \int_a^b u(t) F(u, t) dt \end{aligned}$$

so that by (4.11),

$$6B \leq \int_a^b u(t) F(u, t) dt \leq B \int_a^b x^3 F(Bx^3, x) dx < 5B$$

which is clearly absurd. We therefore conclude that $u(x) \neq 0$ for $x \in (a, \infty)$, i.e., $u(x)$ is nonoscillatory.

With the help of the VOLTERRA equation equivalent to (1.2), we see from (4.12) that for $a \leq x < +\infty$,

$$6B(x-a)^3 \leq 6u(x) + B(x-a)^3 \int_a^x (t-a)^3 F(Bt^3, t) dt.$$

Combining this with (4.11) and (4.12), we have

$$B(x-a)^3 \leq 6u(x) \leq 6B(x-a)^3.$$

In view of Lemma 4.3, this establishes the sufficiency.

To prove necessity, we let $u(x)$ be a positive solution of (1.2) such that (4.9) holds. Then

$$\int_a^\infty u(x) F(u, x) dx = u'''(a) - 6A.$$

Now, $u(x) \geq 0$ implies $u''(x) \leq 0$ so that $u(x) \geq (A - \varepsilon)x^3$ for some $\varepsilon > 0$. Hence,

$$u'''(a) \geq (A - \varepsilon) \int_a^\infty x^3 F[(A - \varepsilon)x^3, x] dx.$$

This completes the proof of the theorem.

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