Sequences of nonlinear differential equations with related solutions

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Summary. - *A second-order nonlinear differential equation which occurs (together with variants of it) in many problems of applied mathematics, physics and engineering is here reduced to a first-order equation. This equation contains a parameter which is a quadratic rational function of two parameters appearing in the original equation. By applying a certain identity for a quadratic rational function, two (finite or infinite) sequences of nonlinear differential equations are generated whose solutions are determinable whenever the solution of any equation belonging to a sequence is known. The cases amenable to exact solution by quadrature are given.*

1. - Introduction.

We consider the second-order nonlinear differential equation

$$y'' + \frac{a}{y}y'^2 + \frac{b}{x}y' + \frac{c}{x^2}y + dx^r y^s = 0, \tag{1}$$

where a, b, c, d, r, s are arbitrary constants, except that $r \neq -2$ and $s \neq 1$. This equation (or a variant of it) occurs extensively in applied mathematics, physics, astrophysics and engineering, as we will show by examples.

One variant of (1) which occurs in applications is obtained by making the substitution $y = e^\psi$, whereupon (1) becomes

$$\psi'' + (a+1)\psi'^2 + \frac{b}{x}\psi' + \frac{c}{x^2} + dx^r e^{(s-1)\psi} = 0. \tag{2}$$

A second variant of (1), an equation of the third order, is obtained by introducing an independent variable ξ and a dependent variable $f(\xi)$ by means of the substitutions

$$x = f\frac{dx}{df} \quad \text{and} \quad y = \frac{df}{d\xi}\left\{\frac{x}{f}\right\}^{(2+r)/(1-s)}.$$

Equation (1) then goes over into

$$f_3 + \frac{a-1}{f_1}f_2^2 + \frac{b}{f}f_1f_2 + \frac{c}{f^2}f_1^3 + df^r f_1^{s+2} = 0, \tag{3}$$

where $f_n = d^n f/d\xi^n$, $n = 1, 2, 3$.

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In justification for our present interest in (1), (2) and (3), we give some examples where these equations occur in applications:

(i) A special case of (2) (with $a = -1$, $b = 1$, $c = 0$, $s = 2$) occurs in problems in the flow of viscous fluids with a viscosity depending exponentially on temperature, and also in calculating the temperature distribution in a dielectric in an alternating field [1].

(ii) The Blasius equation of laminar boundary layer theory,

$$\frac{d^3\xi}{d\eta^3} + \xi\frac{d^2\xi}{d\eta^2} = 0,$$

is transformed by means of the substitution: $f = d\xi/d\eta$ into

$$\xi + \frac{f}{f_1} f_2 + f_1 = 0, \tag{4}$$

where $f_n = d^n f/d\xi^n$. Differentiating (4) with respect to ξ, we get

$$f_3 - \frac{1}{f_1} f_2^2 + \frac{2}{f} f_1 f_2 + \frac{1}{f} f_1 = 0, \tag{5}$$

which is a special case of (3).

(iii) An equation arising in the theory of precursor waves in shock tubes (equation (5.5) in [2]) is

$$\frac{d}{d\xi}(ff_1) + \xi f_1 = 0,$$

with the same subscript notation as in (ii). By differentiation with respect to ξ, we get (5).

(iv) In the study of cylindrical auto-confinement of a plasma [3], the following equation occurs:

$$\frac{1}{x}\frac{d}{dx}\left\{\frac{x}{y}\frac{d}{dx}y^{\gamma}\right\} + y = 0,$$

where γ is a parameter. Carrying out the indicated differentiations, we get for $y(x)$ the differential equation (1), with $a = \gamma - 2$, $b = 1$, $c = 0$, $d = 1/\gamma$, $r = 0$, $s = 3 - \gamma$.

(v) The generalized Duffing equation with a damping term is

$$\ddot{z}_1 + \alpha\dot{z}_1 + \beta z_1 + \gamma z_1^m = 0, \qquad (\dot{z}_1 = dz_1/dt)$$

the original Duffing equation having $m = 3$. By means of the substitutions:

$$z_1 = ye^t \quad \text{and} \quad x = e^t,$$

we get for $y(x)$ equation (1) with $a = 0$, $b = \alpha + 3$, $c = \alpha + \beta + 1$, $d = \gamma$, $r = m - 3$, $s = m$.

(vi) A special case of (1) is the EMDEN-LANE-FOWLER equation:

$$y'' + \frac{2}{x}y' + y^{\mu} = 0,$$

which occurs in astrophysics.

(vii) The equation

$$y'' = kx^r y^s,$$

a simplified form of (1), has been considered by BELLMAN [4], and a special case of it is the THOMAS-FERMI equation of nuclear physics:

$$\sqrt{x}y'' = y\sqrt{y}.$$

Another equation which may be brought to the same form is:

$$\xi\frac{d}{d\xi}\left[\frac{1}{\xi}\frac{d}{d\xi}\left\{\xi^2\gamma\right\}\right] = \frac{(\lambda^2 - \xi^2)^2}{8\xi^2\gamma^2},$$

which occurs in the analysis of large deflection of an annular membrane [5].

For the substitutions

$$y = \xi^2\gamma \quad \text{and} \quad x = \xi^2 - \lambda^2$$

bring the equation to the form

$$32y^2y'' = x^2.$$

(viii) The equation

$$\sqrt{y}y'' = e^x,$$

a space-charge equation in one dimension [6], goes over by means of the substitution $t = e^x$ into the LANGMUIR-BLODGETT space-charge equation for cylinders:

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} = \frac{1}{t\sqrt{y}},$$

which is a special case of (1).

(ix) The LANGMUIR-BOGUSLAVSKI equation [7]

$$\frac{d}{dx}\left\{x^n \frac{d\varnothing}{dx}\right\} = \frac{1}{\sqrt{\varnothing}},$$

where $n = 0$ for the plane, $n = 1$ for the cylinder, and $n = 2$ for the sphere, is also a special case of (1).

(x) The nonlinear Poisson equation for the potential with linear, cylindrical or spherical symmetry:

$$\frac{1}{x^n}\frac{d}{dx}\left\{x^n \frac{dy}{dx}\right\} = g(x, y),$$

where $g(x, y)$ is the space-charge density, is a special case of (1) when $g(x, y)$ has the proper form, such as in Dow's equation [8], p. 66:

$$\frac{1}{x}\frac{d}{dx}\left\{x \frac{dy}{dx}\right\} = \frac{1}{\sqrt{y}}.$$

(xi) Another equation occurring in space-charge theory is IVEY'S equation [8], p. 242:

$$y'' - \frac{1}{y}y'^2 + \frac{2}{x}y' + ky^2 = 0,$$

which is a special case of (1).

(xii) In the theory of internal ballistics of guns [9], the following equation occurs:

$$y'' + \frac{a}{y}y'^2 + \beta = \gamma\frac{x}{y}, \tag{6}$$

and for the special practical cases where $\beta = 0$, it has the form of equation (1). When $a = \beta = 0$ and $\gamma = -1$, (6) becomes an equation to which are reducible both the BLASIUS equation and the LANGMUIR-BLODGETT space-charge equation for cylinders [10].

(xiii) In the equation

$$\frac{d}{dz}\left\{z\frac{d}{dz}F^2\right\}+2zF=0,$$

which occurs in the analysis of plasma diffusion in a magnetic field (equation (13) in [11]), we make the substitutions

$$x=z^2 \quad \text{and} \quad y=F^2,$$

and get

$$y''+\frac{1}{x}y'+\frac{\sqrt{y}}{2x}=0,$$

which is a special case of (1).

2. - Reduction to a first-order differential equation.

Having now shown that the differential equations (1), (2) and (3) are of sufficiently general applicability to warrant independent study, we shall reduce these equations to a single first-order nonlinear equation. We shall concentrate on equation (1), since equations (2) and (3) are reducible to it. Introducing an independent variable z and a dependent variable $w(z)$ by means of the substitutions

$$y=zx^{(2+r)/(1-s)} \quad \text{and} \quad w(z)=xz^a\frac{dz}{dx}, \tag{7}$$

(remembering that $r\neq-2$ and $s\neq1$), equation (1) is transformed into the first-order nonlinear equation

$$\frac{dw}{dz}=F(z)+\frac{G(z)}{w}, \tag{8}$$

where

$$F(z)=\left\{1-b-2(a+1)\frac{2+r}{1-s}\right\}z^a,$$

and

$$G(z)=\left\{(1-b)\frac{2+r}{1-s}-(a+1)\left[\frac{2+r}{1-s}\right]^2-c\right\}z^{2a+1}$$
$$-dz^{2a+s}.$$

Equation (8) is integrable only in special cases, such as when the coefficient of z^a in $F(z)$ vanishes, or when $d = 0$ (two other cases of integrability will be given below). So, assuming that $d \neq 0$ and setting

$$A = 1 - b - 2(a+1)\frac{2+r}{1-s} \neq 0,$$

we make the following changes of scale of w and z:

$$z = Z\{A^2/d\}^{1/(s-1)}$$

and

$$w = Wd^{(a+1)/(1-s)}A^{1+2(a+1)/(s-1)},$$

so that (8) becomes

$$\frac{dW}{dZ} = Z^a + \frac{1}{W}\left\{KZ^{2a+1} - Z^{2a+s}\right\}, \tag{9}$$

where K is a function of a, b, c, r, s. We shall treat a, c and s as fixed constants, and K will then appear as a quadratic rational (Q.R.) function of both b and r. Considering K as a Q.R. function of b, we have

$$K(b, r) = \frac{\{(1-s)(2+r) - (a+1)(2+r)^2 - c(1-s)^2\} - (1-s)(2+r)b}{[\{1-s-2(a+1)(2+r)\} - (1-s)b]^2}, \tag{10}$$

and considering it as a Q.R. function of r, we have

$$K(b, r) = \frac{\{2(1-s)(1-b) - 4(a+1) - c(1-s)^2\} + \{(1-s)(1-b) - 4(a+1)\}r - (a+1)r^2}{[\{(1-s)(1-b) - 4(a+1)\} - 2(a+1)r]^2}. \tag{11}$$

3. - Sequences of equations.

We now come to the main objective of this paper, which consists in the application of an identity for a Q.R. function first given in the solution of a proposed problem [10], and applied in [12] to classical polynomials and BESSEL functions, and in [13] to electric circuits. (For brevity we shall not reproduce the identity here, and refer the reader to any of the above-mentioned references).

Applying the identity first to $K(b, r)$ as given by (10) we get: $K(b, r) \equiv$ $\equiv K(b', r)$, where

$$D_1 b' = b\{(1-s)(2+r) - (a+1)(2+r)^2 - c(1-s)^2\} + \{2c(1-s) - (2+r)\}\{1-s-2(a+1)(2+r)\}, \tag{12}$$

and

$$D_1 = b(1-s)(2+r) - \{(1-s)(2+r) - (a+1)(2+r)^2 - c(1-s)^2\}.$$

Next, applying the same identity to $K(b, r)$ as given this time by (11), we get: $K(b, r) \equiv K(b, r')$, where

$$r' = \frac{(1-s)(1-b)}{a+1} - r - 4. \tag{13}$$

Thus, when b is replaced throughout by b', or else r by r', the numerical value of K in (9) remains the same, and since a and s are fixed, we have the *same* solution $W(Z)$ for the two values b and b' (or r and r'). However, $y(x)$ as a solution of (1) will *not* be the same as for the equation which is similar to (1) except that it has b' instead of b (or r' instead of r), although knowledge of the solution of one enables one to derive that of the other.

This, however, is not the end of the matter. For we can now do two alternative operations: (i) Replace b by $b_1 = b'$ using (12), and *then* replace r by $r_1 = r'$ using (13) with b_1 in place of b; next, replace b_1 by b_2 using (12) with r_1 in place of r; and so on. (ii) Replace r by $r_1 = r'$ from (13), and *then* replace b by $b_1 = b'$ using (12) with r_1 in place of r; next, replace r_1 by r_2 using (13) with b_1 in place of b; and so on.

The first chain of replacements gives:

$$K(b, r) \equiv K(b_1, r) \equiv K(b_1, r_1) \equiv K(b_2, r_1) \equiv K(b_2, r_2) \equiv K(b_3, r_2) \equiv K(b_3, r_3) \equiv K(b_4, r_3) \equiv \ldots, \tag{14}$$

where $b_1, b_2, b_3, \ldots$, and $r_1, r_2, r_3, \ldots$, are determined from the recurrence relations:

$$D_2 b_{n+1} = b_n\{(1-s)(2+r_n) - (a+1)(2+r_n)^2 - c(1-s)^2\} + \{2c(1-s) - (2+r_n)\}\{1-s-2(a+1)(2+r_n)\},$$

where

$$D_2 = b_n(1-s)(2+r_n) - \{(1-s)(2+r_n) - (a+1)(2+r_n)^2 - c(1-s)^2\},$$

and

$$r_{n+1} = \frac{1-s}{a+1}(1 - b_{n+1}) - r_n - 4, \qquad n = 0,\ 1,\ 2,\ \ldots$$

with $b_0 = b$ and $r_0 = r$.

The second chain of replacements gives:

$$\begin{aligned}(15) \qquad & K(b,\ r) \equiv K(b,\ r_1) \equiv K(b_1,\ r_1) \equiv K(b_1,\ r_2) \equiv K(b_2,\ r_2) \\ & \equiv K(b_2,\ r_3) \equiv K(b_3,\ r_3) \equiv K(b_3,\ r_4) \equiv \ldots,\end{aligned}$$

where

$$r_{m+1} = \frac{1-s}{a+1}(1 - b_m) - r_m - 4, \qquad m = 0,\ 1,\ 2,\ \ldots$$

and

$$\begin{aligned} D_3 b_{m+1} = {} & b_m \{(1-s)(2 + r_{m+1}) - (a+1)(2 + r_{m+1})^2 - c(1-s)^2\} \\ & + \{2c(1-s) - (2 + r_{m+1})\}\{1 - s - 2(a+1)(2 + r_{m+1})\},\end{aligned}$$

where

$$\begin{aligned} D_3 = {} & b_m(1-s)(2 + r_{m+1}) \\ & - \{(1-s)(2 + r_{m+1}) - (a+1)(2 + r_{m+1})^2 - c(1-s)^2\},\end{aligned}$$

with $r_0 = r$ and $b_0 = b$.

The n-sequence (14) and the m-sequence (15) are infinite unless: (*i*) any value of b_ν or r_ν becomes infinite, (ii) a fixed point is reached, i.e. $b_{\nu+1} = b_\nu$ or $r_{\nu+1} = r_\nu$, $(\nu = m,\ n)$; this follows from a property of the linear fractional transformation associated with the identity [13], or (iii) a previous pair of values $(b,\ r)$ is again reached, so that the sequence becomes cyclic, repeating itself indefinitely.

Thus, from any given differential equation (1) with fixed values of a, c and s (d, which is assumed non-zero, may be varied by a change of scale of x or y), we have generated two (finite or infinite) sequences of differential equations with related solutions and having the same values of a, c and s, but different values of b and r. Knowledge of the solution of any equation belonging to these sequences (including the generating equation) enables one to derive the solutions of all the others.

An intriguing problem is to track the successive values of b_ν and r_ν for the two sequences on the $(b,\ r)$ plane (the paths are parallel to the axes), and find out whether a limit point $(B,\ R)$ is attained as $\nu \to \infty$, or whether B and R do not exist. Of particular interest are the changes which are produced in the character of the paths when the initial point is varied. This problem is similar in outline to that of the nonlinear iterative transformations of a point, which have been studied by ULAM and STEIN using electronic computers [14].

As an example, the equation

$$x^4 y'' + y^2 = 0$$

generates the class of equations of the form

$$y'' + \frac{b}{x} y' + x^r y^2 = 0$$

which, incidentally, go over into

$$\gamma'' + \frac{b(2-b)}{4x^2}\gamma + x^{r-(b/2)}\gamma^2 = 0$$

by means of the substitution $\gamma = yx^{b/2}$.

For the n-sequence the (b, r) values are: $(0, -4)$, $(3, -4)$, $(3, 2)$, $(-3, 2)$, $(-3, -10)$, $(9, -10)$, $(9, 14)$, $(-15, 14)$, $(-15, -34)$, $(33, -34)$, $(33, 62)$, $(-63, 62)$, $(-63, -130)$, ..., and there is no limit point.

For the m-sequence the (b, r) values are: $(0, -4)$, $(0, -1)$, $\left(\frac{3}{2}, -1\right)$, $\left(\frac{3}{2}, -\frac{5}{2}\right)$, $\left(\frac{3}{4}, -\frac{5}{2}\right)$, $\left(\frac{3}{4}, -\frac{7}{4}\right)$, $\left(\frac{9}{8}, -\frac{7}{4}\right)$, $\left(\frac{9}{8}, -\frac{17}{8}\right)$, $\left(\frac{15}{16}, -\frac{17}{8}\right)$, $\left(\frac{15}{16}, -\frac{31}{16}\right)$, $\left(\frac{33}{32}, -\frac{31}{16}\right)$, $\left(\frac{33}{32}, -\frac{65}{32}\right)$, $\left(\frac{63}{64}, -\frac{65}{32}\right)$, ..., and the limit point is $B = 1$, $R = -2$.

4. - Exact solutions.

We finally give the four cases for which we were able to obtain the exact solutions of our nonlinear equations.

CASE 1. - If a, b, r, s satisfy the relation

$$1 - b - 2(a+1)\frac{2+r}{1-s} = 0,$$

equation (8) becomes separable and is thus integrable by a quadrature.

CASE 2. - If $d = 0$, the general solution of (1) is

$$y^{a+1} = Cx^{\tau - b + 1} + Dx^{-\tau}, \qquad (a + 1 \neq 0)$$

where

$$\tau^2 + (1 - b)\tau + c(a + 1) = 0,$$

and C, D are arbitrary constants.

CASE 3. - If $b = 0$ and $s = -a \neq 1$, the general solution of (1) is given by

$$x^{-\sigma}y^{a+1} = C + D\int x^{-2\sigma}dx$$

$$- d(a + 1)\int x^{-2\sigma}\left\{\int x^{\sigma+r}\,dx\right\}dx\,,$$

where

$$\sigma^2 - \sigma + c(a + 1) = 0.$$

CASE 4. - If a, b, c, r, s satisfy the relation

$$K = -2\frac{(2a + s + 1)}{(4a + s + 3)^2} \neq (0,\ \infty), \tag{16}$$

with K given by (10) or (11), we make in (9) the substitutions:

$$Z^{s-1} = KU^2$$

and

$$v = \frac{1}{U} + kWU^{(2a+s+1)/(1-s)}\,,$$

where

$$k = -\frac{1}{2}(4a + s + 3)K^{(1+a)/(1-s)}\,.$$

Equation (9) then goes over into the *linear* equation for $U(v)$:

$$\frac{dU}{dv} = \frac{1 - s}{2a + s + 1}\,\frac{vU - 1}{v^2 - 1}\,,$$

which has the general solution

$$(v^2 - 1)^{\delta}U = \frac{s - 1}{2a + s + 1}\int (v^2 - 1)^{-\theta}dv + C\,,$$