Relativity and quantum theory. A unified model. (*)

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Synopsis. - *By considering a distribution of complex basic spinors W , satisfying the invariant linear differential equation*

$$\text{curl}_5 W = 0,$$

in a 5-dimensional Kaluza space such that the space curvatures are determined by the concomitants of the spinors, a precisely defined mathematical model is obtained, which displays the basic properties of the physical world, including gravitational, electromagnetics, and many, if not all, of the basic quantum effects.

§ 1. - Introduction.

Since historical times men have developed precisely defined pure mathematical models for the purpose of explaining the function of the universe. Of recent times the most significant has been EINSTEIN'S General Theory of Relativity. EINSTEIN'S model, however, is deficient in two respects.

Firstly, while it explains gravity in terms of geometric curvature, it gives no explanation of electromagnetic forces. This deficiency has been met by a modification due to KALUZA ⁽¹⁾ which was later improved and modified by KLEIN. This postulates an extra fifth dimension, which is cylindrical of small period. The cylindrical connection allows an extra form of space curvature, which can be described by a 6-vector in the homomorphic 4-space, and is identified with the electromagnetic field. In the 5-space the path of an uncharged particle is a geodesic independent of this fifth dimension, while the path of a charged particle of mass m and charge e is a geodesic tracing a helical path round the cylinder at an angle $\tan^{-1}(e/m)$ with the uncharged geodesic. The formalism accounts for both gravitational and electromagnetic forces.

The general verdict on this KALUZA formalism was one of deferred judgment, since the successes seemed hardly enough to justify the extra complexity of an additional dimension. PAULI ⁽²⁾ writes:

« The question of whether KALUZA'S formalism has any future in physics is thus leading to the more general unsolved main problem of accomplishing a synthesis between the general theory of relativity and quantum mechanics ».

Since this main problem is the subject of this paper, it is thought that

(*) Entrata in Redazione il 9 aprile 1968.

⁽¹⁾ [2] see also [5] p. 227 et. seq.

⁽²⁾ [5] p. 232.

no further apology is required if the KALUZA formalism is accepted here as the first of three hypotheses upon which a model for the universe is defined. In fact the formalism receives further justification in simplifying known equations, the rather complicated and somewhat arbitrary equations of DIRAC's Relativistic Electron being obtainable from the very simple 5-dimensional equation $\text{curl}_5 W = 0$.

The second deficiency in EINSTEIN's model is that it given no explanation for the nature of matter. He shows that the RICCI space curvature is a concomitant of the matter distribution, but it is quite clear that space curvature alone is not adequate to describe the properties of matter. Some extra feature must be defined.

Physical experiments show that the electron, for example, has a wave structure. It is reasonable to assume, then, that the additional feature must have the nature of a wave, but the nature of the wave function remains open. In Dirac's Relativistic Electron equations the wave function is a basic spinor. Since spinors cannot be concomitants of true tensors, if spinors are to appear in physics at all, they must appear in the basic postulates. The simplest possible assumption is that there exists a space time distribution of basic spinors.

The second hypothesis of this paper is that there exists a distribution of basic spinors in the KALUZA 5-space. Appeal is here made to an assumption of simplicity, and the spinors will be assumed to satisfy the invariant linear differential equation $\text{curl}_5 W = 0$.

In the probabilistic interpretation of DIRAC's equations the modulus of the basic spinor, $\tilde{W}W$, is associated with the probability of finding an electron. For an individual state rather than a statistical aggregate of states, the nearest interpretation is to associate $\tilde{W}W$ with the mass density. But $\tilde{W}W$ is the time component of a concomitant 4-vector Q_i , which would have to be associated with the momentum and energy. EINSTEIN's General Theory therefore implies that the RICCI space curvature is determined by the 4-vector concomitant of the basic spinors.

This makes the third and final hypothesis of the paper not only reasonable, but almost inevitable. The fifth dimension introduces an extra form of space curvature designated by a 6-vector. There is another concomitant of the basic spinors which is a 6-vector. The final assumption is that the 6-vector concomitant of the spinors determines the 6-vector curvature in the KALUZA model which is identified with the electromagnetic field.

Summing up, the only fundamental hypotheses to be made in this paper are that there exists a distribution of complex basic spinors in a KALUZA 5-space, that these spinors satisfy the invariant linear differential equation, and that the space curvatures are determined by the concomitants of the spinors.

These hypotheses generate a precisely defined model the development of which is pure mathematical. The properties of this model are extremely close to the properties of the physical world. There are light waves with the correct energy momentum and electromagnetic fields, which on closer examination are found to be quantised. There are wave packet solutions which behave as particles, the simplest of which has the correct field and spins to represent an electron. Many quantum effects, such as the PAULI Exclusion Principle, the frequency laws for spectrum, are displayed by the model.

MAXWELL'S equations are not exactly satisfied, but the deviation represents a small oscillation about a zero mean, which would be expected to be unobservable. The problem of constructing a statistical aggregate of possible states which would satisfy, or nearly satisfy, the usual equations of Quantum Mechanics has not been attempted, but would not seem to be impossible.

§ 2. - The Kaluza model.

A brief description is first given of the KALUZA model. It has been abstracted from PAULI'S ⁽³⁾ account to which the reader is referred for further details.

One considers a 5-space with a cylindrical metric

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu, \quad (\mu, \nu = 0, 1, 2, 3, 5)$$

the fifth dimension has a small period, in which one takes $\gamma_{55} = 1$. If the metric is put in the form

$$ds^2 = (dx^5 + \gamma_{i5} dx^i)^2 + g_{ik} dx^i dx^k \quad (i, k = 0, 1, 2, 3)$$

then transformation of the form

$$x'^5 = x^5 + f(x^0, x^1, x^2, x^3)$$

leave

$$(2.1) \quad \frac{\partial \gamma_{k5}}{\partial x^i} - \frac{\partial \gamma_{i5}}{\partial x^k} = f_{ik}$$

invariant. This represents a form of space curvature, and is identified with the electromagnetic field.

The homomorphic 4-space can be identified with the RIEMANNIAN space of EINSTEIN'S General Relativity, and thus an uncharged particle follows a geodesic which does not involve the fifth dimension.

⁽³⁾ [5] p. 227 et seq.

For a geodesic rotated through an angle θ from this direction towards the fifth dimension, the path is influenced by the term γ_{i5} in the metric, and calculation shows that the deflection of the projection of the geodesic path onto the homomorphic 4-space, is precisely the same as the deflection from a geodesic path which would be caused by the electromagnetic field

$$f_{ik} = \frac{\partial \gamma_{k5}}{\partial x^i} - \frac{\partial \gamma_{i5}}{\partial x^k}$$

on a particle with charge-mass ratio $e/m = \tan \theta$.

§ 3. - The spinor frame.

The definition of basic spinors depends on the formation of a set of anticommuting matrices. It will be developed first for four dimensions when four anticommuting matrices are required. The extension to five dimensions requires only the addition of a fifth anticommuting matrix which is uniquely determined in the algebra, apart from sign.

There is some arbitrariness in the choice of such matrices. To fit in with the metric

$$ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 - (dx_0)^2,$$

it is required that the squares of three of them shall be $+1$, and the square of the other -1 . Real matrices may be chosen with this property.

DIRAC'S frame of anticommuting matrices will not be used, as his second matrix σ_y is not real but pure imaginary. Since we shall be considering at times real spinors, a different choice will be made, making all the anticommuting matrices real.

The following set will be used

$$(3.1) \quad \begin{aligned} X_1 &= \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}, & X_2 &= \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & 1 & \\ & & & -1 \end{bmatrix}, \\ X_3 &= \begin{bmatrix} & & & -1 \\ & -1 & & \\ & & -1 & \\ -1 & & & \end{bmatrix}, & X_0 &= \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & 1 & \\ & & & -1 \end{bmatrix}. \end{aligned}$$

with $X^i = X_i$ ($i = 1, 2, 3$), $X^0 = -X_0$.

These matrices satisfy

$$(3.2) \quad (x_1 X^1 + x_2 X^2 + x_3 X^3 + x_0 X^0)^2 = x_1^2 + x_2^2 + x_3^2 - x_0^2,$$

which will be taken as the metric of space time. In tensor notation

$$(x^i X_i)^2 = (x_i X^i)^2 = g_{ij} x^i x^j = g^{ij} x_i x_j.$$

For a LORENTZ transformation

$$x'_i = \xi_i^j x_j,$$

the corresponding anticommuting matrices satisfy the same equations, e. g.

$$X'_i = \xi_i^j X_j.$$

The new matrices X'_i are transforms of the old matrices X_i by a fixed matrix U

$$(3.3) \quad \begin{aligned} X'_i &= U^{-1} X_i U \\ UX'_i &= X_i U \end{aligned}$$

a relation which defines U uniquely apart from a scalar factor, which will be determined subsequently.

The matrix $T = X_0$ clearly transforms X_i into $-X^i = -\tilde{X}_i$,

$$(3.4) \quad TX_i = -\tilde{X}_i T,$$

the symbol $\sim$ denoting transposition.

A vector W is called a *Basic spinor* if, corresponding to the above LORENTZ transformation it transforms so that

$$(3.5) \quad W' = UW.$$

Clearly

$$\tilde{W} = \tilde{W}\tilde{U}.$$

The relation between U and $\tilde{U}$ can be obtained as follows:

$$\begin{aligned} X'_i(T\tilde{U}TU) &= -T\tilde{X}'_i\tilde{U}TU = -T\tilde{U}\tilde{X}_i TU \\ &= T\tilde{U}TX_i U = (T\tilde{U}TU)X'_i. \end{aligned}$$

Thus $T\tilde{U}TU$ commutes with each X'_i and thus with every matrix. Hence it is a scalar multiple of the unit matrix.

The arbitrary scalar factor in U can be adjusted so that

$$(3.6) \quad T\tilde{U}TU = -I, \quad \tilde{U}TU = T,$$

at least for transformations which do not reverse the direction of time.

§ 4. - Concomitants of degree 2.

The types of concomitants which are linear in each of two spinors, or quadratic in a single spinor, can be determined by characteristic analysis, the formulae being ⁽⁴⁾

$$(4.1) \quad \Delta\Delta = \{0\} + \{1\} + \{1^2\} + \{1^3\} + \{1^4\}.$$

$$\Delta \otimes \{2\} = \{1\} + \{1^2\}.$$

Ignoring the difference between a relative invariant corresponding to $\{1^4\}$ and an absolute invariant corresponding to $\{0\}$, and similarly not distinguishing between $\{1\}$ and $\{1^3\}$, the concomitants of two spinors yield two invariants, two 4-vectors and one 6-vector. When the two spinors are made equal, to give concomitants of degree two in a single spinor, the two invariants and one 4-vector become zero, leaving one 4-vector and one 6-vector.

It will be shown that the concomitants of two basic spinors W and W_1 are respectively

$$(4.2) \quad \tilde{W}_1 TW, \quad \tilde{W}_1 TX_i W, \quad \tilde{W}_1 TX_i X_j W, \quad \tilde{W}_1 TX_i X_j X_k W,$$

$$\text{and } \tilde{W}_1 TX_0 X_1 X_2 X_3 W.$$

Firstly, after transformation $\tilde{W}_1 TW$ becomes

$$W_1' TW' = \tilde{W}_1 \tilde{U} T U W = \tilde{W}_1 TW$$

Secondly, $\tilde{W}_1 TX_i W$ becomes

$$\tilde{W}_1' TX_i W' = \tilde{W}_i \tilde{U} T U U^{-1} X_i U W = \tilde{W}_1 TX_i' W.$$

The invariance of the other concomitants follows similarly.

Taking

$$(4.3) \quad \tilde{W} = [\alpha, \beta, \gamma, \delta], \quad \tilde{W}_1 = [\alpha_1, \beta_1, \gamma_1, \delta_1]$$

the five concomitants are

⁽⁴⁾ See [2] p. 288 et. seq.

$$(4.4) \quad \left\{ \begin{array}{ll} I = \alpha_1\beta - \beta_1\alpha + \gamma_1\delta - \delta_1\gamma, & J = \alpha_1\gamma - \gamma_1\alpha - \beta_1\delta + \delta_1\beta; \\ P_1 = -\alpha_1\gamma + \gamma_1\alpha - \beta_1\delta + \delta_1\beta, & P_2 = -\alpha_1\delta + \delta_1\alpha + \beta_1\gamma - \gamma_1\beta; \\ P_3 = \alpha_1\beta - \beta_1\alpha - \gamma_1\delta + \delta_1\gamma, & P^0 = \alpha_1\delta - \delta_1\alpha + \beta_1\gamma - \gamma_1\beta; \\ Q_1 = \alpha_1\beta + \beta_1\alpha + \gamma_1\delta + \delta_1\gamma, & Q_2 = \alpha_1\alpha - \beta_1\beta - \gamma_1\gamma + \delta_1\delta; \\ Q_3 = \alpha_1\gamma + \gamma_1\alpha - \beta_1\delta - \delta_1\beta, & Q^0 = (\alpha_1\alpha + \beta_1\beta + \gamma_1\gamma + \delta_1\delta); \\ R_{01} = -\alpha_1\alpha + \beta_1\beta - \gamma_1\gamma + \delta_1\delta, & R_{02} = \alpha_1\beta + \beta_1\alpha - \gamma_1\delta - \delta_1\gamma; \\ R_{03} = \alpha_1\delta + \delta_1\alpha + \beta_1\gamma + \gamma_1\beta, & R_{23} = \alpha_1\delta + \delta_1\alpha - \beta_1\gamma - \gamma_1\beta; \\ R_{31} = -\alpha_1\gamma - \gamma_1\alpha - \beta_1\delta - \delta_1\beta, & R_{12} = \alpha_1\alpha + \beta_1\beta - \gamma_1\gamma - \delta_1\delta. \end{array} \right.$$

The 6-vector concomitant R_{ij} will sometimes be identified with an electromagnetic field. It will be convenient sometimes to put

$$R_{0i} = E_i, \quad R_{23} = H_1, \quad R_{31} = H_2, \quad R_{12} = H_3.$$

The alternating tensor Γ^{ijkp} is defined to be equal to $+1$, if (i, j, k, p) is a positive permutation of $(1, 2, 3, 0)$, equal to -1 if a negative permutation, and equal to zero if any suffix is repeated. It is sometimes convenient to express the 6-vector R_{ij} in its dual form

$$(4.5) \quad \tilde{R}^{ij} = \Gamma^{ijkp} R_{kp}, \quad \tilde{R}_{ij} = g_{iv} g_{jv} \Gamma^{v'j'kp} R_{kp}.$$

This has the effect of interchanging E and H , with a change of sign in one case, i.e.

$$(4.5 \text{ a}) \quad \tilde{E}_i = H_i, \quad \tilde{H}_i = -E_i$$

For the concomitants of degree 2 of a single spinor, W_1 is made equal to W . Only the 4-vector Q_i , and the 6-vector R_{ij} survive.

$$(4.6) \quad \begin{aligned} Q_1 &= 2(\alpha\beta + \gamma\delta), & Q_2 &= \alpha^2 - \beta^2 - \gamma^2 + \delta^2, \\ Q_3 &= 2(\alpha\gamma - \beta\delta), & Q_0 &= -(\alpha^2 + \beta^2 + \gamma^2 + \delta^2), \\ E_1 &= -\alpha^2 + \beta^2 - \gamma^2 + \delta^2, & E_2 &= 2(\alpha\beta - \gamma\delta), & E_3 &= 2(\alpha\delta + \beta\gamma), \\ H_1 &= 2(\alpha\delta - \beta\gamma), & H_2 &= -2(\alpha\gamma + \beta\delta), & H_3 &= \alpha^2 + \beta^2 - \gamma^2 - \delta^2. \end{aligned}$$

Of special interest concerning the concomitants of two spinors is the case when W is a complex spinor while $W_1 = \bar{W}$ is the complex conjugate. Thus,

e. g.

$$\begin{aligned} I &= \bar{\alpha}\beta - \bar{\beta}\alpha + \bar{\gamma}\delta - \bar{\delta}\gamma, & P_1 &= -\bar{\alpha}\gamma + \bar{\gamma}\alpha - \bar{\beta}\delta + \bar{\delta}\beta, \\ Q_1 &= \bar{\alpha}\beta + \bar{\beta}\alpha + \bar{\gamma}\delta + \bar{\delta}\gamma, & E_1 &= -\bar{\alpha}\alpha + \bar{\beta}\beta - \bar{\gamma}\gamma + \bar{\delta}\delta. \end{aligned}$$

In this case the concomitants I, J, P_i are pure imaginary, while Q_i and R_{ij} are real. In fact Q_i and R_{ij} are the sum of corresponding concomitants of the real part and the imaginary part taken separately. Also Q_i, R_{ij} are independent of a change of phase,

$$W' = e^{i\theta} W.$$

The operators $e^{t\varphi}, e^{it\varphi}$.

Put

$$(4.7) \quad \varphi = X_1 X_2 X_3 X_0 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$$

so that $\varphi^2 = -1$. Then

$$e^{t\varphi} = \cos t + \varphi \sin t.$$

The transformation

$$W' = e^{t\varphi} W$$

changes I into $I' = I \cos 2t - J \sin 2t$, J into $J' = J \cos 2t + I \sin 2t$,

$$(4.8) \quad \begin{cases} E_i \text{ into } E'_i = E_i \cos 2t - H_i \sin 2t, \\ H_i \text{ into } H'_i = H_i \cos 2t + E_i \sin 2t, \end{cases}$$

but leaves P_i and Q_i unaltered.

The operator

$$e^{it\varphi} = \cosh t + i\varphi \sinh t$$

has a different effect because of the extra change of sign in taking the transposed conjugate. The concomitants which are changed by $e^{t\varphi}$ are left unchanged by $e^{it\varphi}$, and conversely. The concomitants I, J, E_i, H_i are unaltered, but

$$W' = e^{it\varphi} W$$

$$(4.9) \quad \begin{cases} \text{changes } P \text{ into } P' = P \cosh t + iQ \sinh t, \\ Q \text{ into } Q' = Q \cosh t + iP \sinh t. \end{cases}$$

§ 5. - Syzygies.

The set of concomitants of a set of ground forms is not algebraically independent, but the forms are connected with each other by a set of relations which are called syzygies. In the case of the concomitants of degree 2 of a single spinor the set of a syzygies is easily seen to be

$$(5.1) \quad \left\{ \begin{array}{l} Q_1^2 + Q_2^2 + Q_3^2 = Q_0^2 \\ E_1^2 + E_2^2 + E_3^2 = Q_0^2 \\ H_1^2 + H_2^2 + H_3^2 = Q_0^2 \\ Q_1 E_1 + Q_2 E_2 + Q_3 E_3 = 0 \\ E_1 H_1 + E_2 H_2 + E_3 H_3 = 0 \end{array} \right.$$

Thus, relative to any assigned time axis, the concomitants of a single spinor define a set of three mutually perpendicular axes given by the special directions of Q_i , E_i , H_i .

The canonical form is easily obtained by transforming into the form $(k, 0, 0, 0)$, when the nonzero components are easily seen to be

$$Q_2 = k^2, \quad E_1 = -k^2, \quad H_3 = k^2, \quad Q_0 = -k^2.$$

The physical significance of a real spinor may therefore be interpreted as an orientation of 3-space relative to any assigned time axis, together with a scalar factor, Q_0 , which transforms like the time component of a 4-vector.

Especially it may be noticed that if R_{ij} is interpreted as the electromagnetic field, the concomitants of a real spinor exactly describe a uniform light wave, Q_i representing the POYNTING vector. This will be referred to again later in connection with differential equations.

The syzygies concerning the concomitant of two spinors are naturally more complicated. To make sure that the complete set is obtained use will be made of characteristic analysis.

From (3.1) the concomitants linear in each of two spinors correspond to the product $\Delta\Delta$ and the concomitants of degree 2 of these concomitants to

$$(\Delta \cdot \Delta) \otimes (2) = (\Delta \otimes \{2\})(\Delta \otimes \{2\}) + (\Delta \otimes \{1^2\})(\Delta \otimes \{1^2\}).$$

But these concomitants of concomitants are themselves of degree 2 in each of the basic spinors. The non-zero terms therefore correspond to

$$(\Delta \otimes \{2\})(\Delta \otimes \{2\}),$$

the terms of

$$(5.2) \quad (\Delta \otimes \{1^2\})(\Delta \otimes \{1^2\}) = ([0] + [0] + [1^2])([0] + [0] + [1^2]) \\ = 5[0] + 4[1] + [2] + [1^2],$$

are therefore zero terms and represent the syzygies.

The actual forms of the syzygies can be determined by considering the frame of reference which makes $W_1 = [1, 0, 0, 0]$, $W = [\alpha, \beta, \gamma, \delta]$. This gives

$$I = \beta, \quad J = \gamma \\ Q = (\beta, \alpha, \gamma, -\alpha), \quad P = (-\gamma, -\delta, \beta, \delta) \\ E = (-\alpha, \beta, \delta), \quad H = (\delta, -\gamma, \alpha).$$

By evaluation the 5 invariant syzygies are seen to be

$$(5.3) \quad \begin{cases} P_i Q^i = 0 \\ -R_{ij} R^{ij} = E^2 - H^2 = I^2 - J^2 \\ \frac{1}{2} \tilde{R}_{ij} R^{ij} = E \cdot H = -IJ \\ P_i P^i = I^2 + J^2 \\ Q_i Q^i = I^2 + J^2. \end{cases}$$

The 4 syzygies of type [1] are

$$(5.4) \quad \begin{cases} P^i R_{ij} = -J Q_j, & P^i \tilde{R}_{ij} = -I Q_j \\ Q^i R_{ij} = J P_j, & Q^i \tilde{R}_{ij} = I P_j. \end{cases}$$

The 6-vector syzygy is

$$(5.5) \quad P_i Q_j - P_j Q_i = I \tilde{R}_{ij} + J R_{ij}.$$

The syzygy of type [2] ⁽⁵⁾ is perhaps the most interesting, as it is associated with energy-momentum-tension tensor. There are 4 concomitants of type [2],

$$R_{ij} R_k^i + \tilde{R}_{ij} \tilde{R}_k^i \\ P_j P_k - \frac{1}{4} g_{jk} P_i P^i, \quad P_j Q_k + P_k Q_j, \quad Q_j Q_k - \frac{1}{4} g_{jk} Q_i Q^i.$$

⁽⁵⁾ See [2], appendix.

The first is invariant for the rotation group of transformations of the form

$$W' = e^{i\varphi} W.$$

The other three are linearly transformed according to the representation of this rotation group corresponding to

$$\{2\} = \{2\} + [0].$$

The invariant part is

$$P P_k + Q_j Q_k - \frac{1}{4} g_{jk} (P_i P^i + Q_i Q^i).$$

The syzygy must therefore identify this with $R_{ij} R_k^i + \tilde{R}_{ij} \tilde{R}_k^i$. Comparison of coefficients show that

$$(5.6) \quad \frac{1}{2} (R_{ij} R_k^i + \tilde{R}_{ij} \tilde{R}_k^i) = P_j P_k + Q_j Q_k - \frac{1}{4} g_{jk} (P_i P^i + Q_i Q^i).$$

§ 6. - Differential equations.

Differentiation in space time involves the operator $\frac{\partial_i}{\partial x_i}$ which is of transformation type $\{1\}$. The derivatives of a spinor are therefore of type

$$\Delta[1] = \left[\frac{1}{2}, \frac{1}{2} \right] [1] = \left[\frac{3}{2}, \frac{1}{2} \right] + \left[\frac{1}{2}, \frac{1}{2} \right].$$

To obtain a linear differential equation there are three possibilities. Firstly the concomitant of type $\left[\frac{3}{2}, \frac{1}{2} \right]$ may be equated to zero, secondly the concomitant of type $\left[\frac{1}{2}, \frac{1}{2} \right]$ may be equated to zero, or thirdly the concomitant of type $\left[\frac{1}{2}, \frac{1}{2} \right]$ may be equated to a multiple of the original spinor, possibly by multiplication by an invariant operator.

The first case is rather too restrictive to be of much interest.

The basic spinor derivative which will be written as $\text{curl } W$ is clearly given by

$$(6.1) \quad \text{curl } W = \frac{\partial}{\partial x} X_i W.$$

It $\tilde{W} = [\alpha, \beta, \gamma, \delta]$, $\text{curl } W = [\alpha', \beta', \gamma', \delta']$, this gives

$$(6.2) \quad \begin{cases} \alpha' = \alpha_1 - \beta_2 - \delta_3 + \beta_0, \\ \beta' = -\beta_1 - \alpha_2 - \gamma_3 - \alpha_0, \\ \gamma' = \gamma_1 + \delta_2 - \beta_3 + \delta_0, \\ \delta' = -\delta_1 + \gamma_2 - \alpha_3 - \gamma_0. \end{cases}$$

Clearly

$$(6.3) \quad \text{curl curl } W = W_{11} + W_{22} + W_{33} - W_{00}$$

and the operator curl curl operating on a basic spinor may be identified with the second order invariant differential operator,

$$(6.4) \quad \square^2 = g_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}$$

When dealing with 3-space it will be found convenient to use the symbol

$$(6.5) \quad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

Especially important is the identity

$$(6.6) \quad \text{curl curl } W = \square^2 W.$$

$$\text{Curl } W = 0.$$

The simplest equation, $\text{curl } W = 0$, is clearly a wave equation. For a plane wave, suppose that the solution is independent of x_1 and x_3 . The equations become

$$\beta_2 = \beta_0, \quad \alpha_2 = -\alpha_0, \quad \gamma_2 = \gamma_0, \quad \delta_2 = -\delta_0.$$

Fields corresponding to β and/or γ are propagated with unit velocity, i.e., light velocity, in one direction, while fields corresponding to α and/or δ are propagated with the same velocity in the opposite direction. The propagation in either case is with light velocity in the direction of the POYNTING vector Q_i . Circularly polarized light corresponds to a rotation in the $\alpha\delta$ -plane in spin space, when the electric and magnetic fields rotate about the POYNTING vector in space time.

For a simple plane wave, the solutions of $\text{curl } W = 0$ give quite adequate representations of electromagnetic radiation or light waves. For a superposition

of independent waves, however, there is a distinction. On the classical theory the electromagnetic terms are linearly additive, but for solutions of $\text{curl } W = 0$ it is the spinors which are linearly additive.

Of special interest are solutions of $\text{curl } W = 0$ which are independent of time. Assuming time derivatives to be zero, then

$$\text{curl curl } W = \nabla^2 W.$$

Hence if W_1 is any solution of $\nabla^2 W_1 = 0$, and

$$W = \text{curl } W_1,$$

then

$$\text{curl } W = \text{curl curl } W_1 = \nabla^2 W_1 = 0.$$

Putting

$$\tilde{W}_1 = \left[\frac{1}{r}, 0, 0, 0 \right]$$

then

$$\tilde{W} = [-x_1/r^3, x_2/r^3, 0, x_3/r^3]$$

which is a basic solution of $\text{curl } W = 0$ with a single isolated singularity.

More generally, if Ω is a fixed spinor, put

$$W_1 = -\frac{1}{r}\Omega.$$

Then

$$(6.7) \quad W = x_i X_i \Omega / r^3$$

is a solution of

$$\text{curl } W = 0.$$

Another point of interest is the effect of the equation $\text{curl } W = 0$ on the concomitants. Taking two basic spinors W_1 and W , consider the concomitant

$$Q_i = \tilde{W}_1 T X_i W.$$

Then

$$(6.8) \quad \frac{\partial}{\partial x_i} Q_i = \tilde{W}_1 T (X_i \partial W / \partial x_i) - (\partial \tilde{W}_1 / \partial x_i \tilde{X}_i) T W$$

$$\partial Q_i / \partial x_i = 0 \quad \text{if} \quad \text{curl } W = 0 = \text{curl } W_1.$$

§ 7. - Relation between spinors and 6-vectors.

Since the 6-vector concomitant of a real spinor is restricted so that E and H are of equal magnitude and orthogonal, clearly in the general case the 6-vector must be expressed as the concomitant

$$R_{ij} = \tilde{W} T X_i X_j W$$

of a complex spinor.

THEOREM I. - Given any electromagnetic field at a given point there exist a complex spinor W for which this is equal to the 6-vector concomitant

$$R_{ij} = \tilde{W} T X_i X_j W.$$

The matrix $[R_{st}]$ has a determinant

$$|R_{st}| = \Delta^2$$

where

$$\Delta = R_{12}R_{30} + R_{23}R_{10} + R_{31}R_{20}.$$

Replacement of R_{st} by the dual $\tilde{R}_{st}$ changes Δ into $-\Delta$. Thus the function Δ of $[R_{st} + \lambda \tilde{R}_{st}]$ changes sign between $-\infty$ and 0 and 0 and $+\infty$. Thus

$$|R_{st} + \lambda \tilde{R}_{st}| = 0$$

has one negative and one positive root. Choosing a LORENTZ frame so that the factors of zero of the matrix for these two roots corresponds to the x_1x_2 -plane and the x_3x_0 -plane, this implies that the only non-zero terms of R_{ij} are

$$E_3 = k_1, \quad H_3 = k_2.$$

Consider a spinor for which α and δ are real, and

$$\beta = i\alpha, \quad \gamma = i\delta,$$

the 6-vector concomitant has the following non-zero terms

$$E_3 = -4\alpha\delta, \quad H_3 = 2(\alpha^2 - \delta^2).$$

A suitable choice of α and δ satisfies the theorem.

Exception occurs only if $|R_{st} + \lambda \tilde{R}_{st}|$ is identically zero. In this case E and H are of equal magnitude and orthogonal. The field is then the concomitant of a real spinor.

The complex spinor has 8 independent terms, the 6-vector only 6. There is an ambiguity involving two terms to be accounted for. The first ambiguity is very simple.

THEOREM II. – All concomitants of a complex spinor are independent of a change of phase.

If W is replaced by $e^{i\theta} W$, then $\tilde{W}$ is replaced by $e^{-i\theta} \tilde{W}$. The extra factors cancel.

The other ambiguity is more interesting. If W is replaced by

$$W' = e^{i\phi} W,$$

as explained in § 4, then the concomitants I , J , and the 6-vector R_{ij} will be unchanged. Hence if R_{ij} is specified, then an arbitrary factor of the form $e^{i\phi}$ may be introduced. There is, however, a change in the concomitants P_i and Q_i , namely

$$P' = P \cosh t + iQ \sinh t,$$

$$Q' = Q \cosh t + iP \sinh t.$$

Notice that if R_{ij} is given there is not one, but a continuous infinity of time axes which will make E and H collinear, and hence make the momentum terms of T_{jk} equal to zero. Taking the LORENTZ frame as for Theorem I so that

$$E_3 = k_1, \quad H_3 = k_2$$

a LORENTZ transformation corresponding to a velocity in the x_3 -direction will leave R_{ij} and hence T_{jk} unaltered. The momentum terms introduced from the energy are exactly counterbalanced by the corresponding terms introduced from the stress. These frames will be called *apparent rest frames*.

The choice of the factor $e^{i\phi}$ in W picks out exactly one of these apparent rest frames for which Q_i is in the direction of the time axis and P_i is purely spatial.

From (4.6) the energy-momentum-tension tensor may be expressed as

$$\begin{aligned} T_{jk} &= \frac{1}{2} (R_{ij} R_k^i + \tilde{R}_{ij} \tilde{R}_k^i) \\ &= \left(P_j P_k - \frac{1}{4} g_{jk} P_i P^i \right) + \left(Q_j Q_k - \frac{1}{4} g_{jk} Q_i Q^i \right). \end{aligned}$$

This effects a separation of T_{jk} into two parts, one corresponding to the stress, the other to the energy, with a corresponding separation of the momentum terms.

This separation is not uniquely determined by R_{ij} , but the choice of one of the apparent rest frames does render it unique and fixes the arbitrary factor $e^{i\phi}$.

THEOREM III. - If the electromagnetic field R_{ij} is given, and also one of the apparent rest frames is chosen as an absolute rest frame, then the corresponding spinor W is uniquely determined apart from an arbitrary phase factor $e^{i\theta}$.

§ 8. - The topological Barrier.

This section is still concerned with the background rather than with the development of the main theory. But the result has a significant application in the main theory. During the original investigation an attempt was made to represent the electromagnetic field of a classical electron as the concomitant of a distribution of basic spinors. The first impression is that this problem should present little difficulty, since the field and the rest frame, which may be taken as the rest frame of the electron, determine the spinor at each point unambiguously apart from phase. The following theorem, therefore, is somewhat surprising.

THEOREM IV. - It is impossible to obtain a distribution of complex spinors over the surface of a sphere such that the 6-vector concomitant describes the electromagnetic field of a classical electron at the centre of the sphere.

If it were possible, let the spinor distribution W be described as

$$W = W_1 + iW_2.$$

At any point the concomitants R_{ij} and Q_i of W are equal to the sums of the corresponding concomitants of W_1 and W_2 taken separately.

At any point P on the sphere, the concomitants of the real spinor W_1 gives a set of 3 mutually perpendicular vectors E , H and Q of equal magnitude. Corresponding to P make a small displacement $\vec{PP'}$ of magnitude ε times the component of H tangential to the sphere. This gives a mapping $P \rightarrow P'$ of the points of the sphere onto itself. From the fixed point theorem there is at least one point Q on the sphere which corresponds to itself.

At Q the component of H tangential to the sphere must be zero. The direction of H must be along the radius vector. The component of E along the radius vector is zero.

Since the concomitant H of W should be zero, the concomitant H of W_2 must be equal and opposite to that of W_1 , and is also along the radius vector. Hence the concomitant E of W_2 also has a zero component along the radius

vector. By addition the concomitant E of W has zero component along the radius vector, contrary to hypothesis.

The difficulty is a topological one. It is the difficulty of assigning phases to the otherwise predictable spinors, which shall be continuous over a complete sphere. The assignment of phases is possible with one isolated singularity. The same difficulty arises, but is only of pure mathematical interest, concerning the field of an isolated magnetic pole.

This topological barrier cannot be breached in four dimensions. It was this fact which led to the consideration of KALUZA'S 5-space. In 5-space, two distinct ways present themselves, for breaching the barrier. The first of these will now be described. Comparison with the KALUZA formalism shows that it could be associated with an isolated magnetic pole.

Assuming that the field is that of a magnetic pole, this together with the rest frame of the pole defines the spinors uniquely at every point apart from the phase. Suppose that a phase factor $e^{i\theta}$ is distributed uniformly round the fifth dimension. Then each point in 4-space corresponds to a circle in 5-space. It is only necessary to connect these circles together in such a way that the phase is continuous.

This space can be represented parametrically in terms of a parametric spinor

$$\tilde{W}_1 = [\alpha_1, \beta_1, \gamma_1, \delta_1],$$

by putting

$$x_1 = 2(\alpha_1\delta_1 - \beta_1\gamma_1), \quad x_2 = -2(\alpha_1\gamma_1 + \beta_1\delta_1), \quad x_3 = \alpha_1^2 + \beta_1^2 - \gamma_1^2 - \delta_1^2.$$

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2 = r,$$

$$x_1^2 + x_2^2 + x_3^2 = r^2$$

$$\frac{1}{2} dx_1 = \delta_1 d\alpha_1 - \gamma_1 d\beta_1 - \beta_1 d\gamma_1 + \alpha_1 d\delta_1$$

$$\frac{1}{2} dx_2 = -\gamma_1 d\alpha_1 - \delta_1 d\beta_1 - \alpha_1 d\gamma_1 - \beta_1 d\delta_1$$

$$\frac{1}{2} dx_3 = \alpha_1 d\alpha_1 + \beta_1 d\beta_1 - \gamma_1 d\gamma_1 - \delta_1 d\delta_1.$$

To these equations add, assuming that the period of the fifth dimension is $2\pi\sigma$,

$$\frac{r}{\sigma} dx_5 = -\beta_1 d\alpha_1 + \alpha_1 d\beta_1 - \delta_1 d\gamma_1 + \gamma_1 d\delta_1.$$

The field of an isolated magnetic pole could be given by

$$W = r^{-3/2}(1 + iX_0)W_1,$$

the operator X_0 corresponding exactly to rotation round the fifth dimension.

To determine the geometric significance of this space according to the KALUZA theory, the metric, which presumably could be expressed as

$$dx^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + \frac{\sigma^2}{r^2}(-\beta_1 d\alpha_1 + \alpha_1 d\beta_1 - \delta_1 d\gamma_1 + \gamma_1 d\delta_1)^2$$

is expressed in terms of new variables by the substitution

$$\begin{aligned}\alpha_1 &= \rho \cos \frac{1}{2}\theta \cos \frac{1}{2}(\lambda + \mu), & \gamma_1 &= \rho \sin \frac{1}{2}\theta \sin \frac{1}{2}(\mu - \lambda), \\ \beta_1 &= \rho \cos \frac{1}{2}\theta \sin \frac{1}{2}(\lambda + \mu), & \delta_1 &= \rho \sin \frac{1}{2}\theta \cos \frac{1}{2}(\mu - \lambda),\end{aligned}$$

where $r = \rho^2$. This gives

$$\begin{aligned}x_1 &= r \sin \theta \cos \mu, & x_2 &= -r \sin \theta \sin \mu, & x_3 &= r \cos \theta \\ & & -2dx_5/\sigma &= d\lambda + \cos \theta d\mu.\end{aligned}$$

Comparison with § 2, putting $dx_5 = \frac{1}{2}\sigma(d\lambda + \omega\theta d\rho)$ gives

$$\gamma_{\mu 5} = \cos \theta$$

as the only non zero γ_{i5} . Equation (2.1), then gives

$$f_{\theta\mu} = -\sin \theta.$$

The total flux of the 6-vector f_{ik} is thus

$$-\sin \theta d\theta d\mu = -\frac{x_3}{r^3} dx_1 dx_2 - \frac{x_1}{r^3} dx_2 dx_3 - \frac{x_2}{r^3} dx_3 dx_1$$

which is the correct field for a magnetic pole.

This model has a topological singularity. If a small reducible circuit at one point in 3 space is opened out in the 3-space to envelop the pole and contract on the other side to a point in the 3-space, then at this point it would no longer be locally reducible, but would be wrapped once round the fifth dimension.

This result would seem to imply that in the model universe which will subsequently be described, isolated magnetic poles are associated with topological singularities. If a model is sought which is free from singularities, then isolated magnetic poles must be excluded automatically by these geometric considerations.

The second method of breaching the barrier will now be described briefly.

A distribution of spinors may clearly be found to give the field of a classical electron at the origin, if it is allowed to have a continuous line of singularities, with discontinuous phase, along the negative x_1 -axis. Let W_1 denote such a distribution. Similarly let W_2 denote a similar distribution with a line of singularities along the positive x_1 -axis.

Let $\theta = \cos^{-1}(x_1/r)$, and suppose that

$$\tilde{W}_1 T X_i X_j W_2 = 0.$$

Then

$$W = W_1 \cos \frac{\theta}{2} + W_2 \sin \frac{\theta}{2}$$

is singularity free, and has the correct field.

The wave-packet solution which will subsequently be obtained to represent an electron, conforms asymptotically to this result.

§ 9. - The geometrical model universe.

It is assumed that there is a distribution of complex basic spinors W in a 5-dimensional KALUZA space, and that these satisfy the fundamental 5-dimensional wave equation

$$X_i \frac{\partial}{\partial x_i} W = 0.$$

The extension of spinors to 5-dimensions requires the introduction of a fifth anticommuting matrix X_5 .

With the assumption that its square is $+1$, this fifth matrix is uniquely determined apart from sign, as

$$(9.1) \quad X_5 = i\gamma = iX_1X_2X_3X_0 = \begin{bmatrix} & & & i \\ & & i & \\ & -i & & \\ -i & & & \end{bmatrix}.$$

The five-dimensional wave equation is

$$X_i \frac{\partial}{\partial x_i} W = 0,$$

which is summed for $i = 1, 2, 3, 0, 5$. The sum $X_i \frac{\partial}{\partial x} W$ summed for these five values of i will be denoted by

$$\text{curl}_5 W.$$

If the summation is only with respect to $i = 1, 2, 3$ it will be denoted by

$$\text{curl}_3 W \quad \text{or} \quad \nabla W.$$

The equation $\text{curl}_5 W = 0$, in full, is

$$(9.2) \quad \begin{cases} \alpha_1 - \beta_2 - \delta_3 + \beta_0 + i\delta_5 = 0, \\ -\beta_1 - \alpha_2 - \gamma_3 - \alpha_0 + i\gamma_5 = 0, \\ \gamma_1 + \delta_2 - \beta_3 + \delta_0 - i\beta_5 = 0, \\ -\delta_1 + \gamma_2 - \alpha_3 - \gamma_0 - i\alpha_5 = 0, \end{cases}$$

Seeking first a semi-static solution, since

$$(9.3) \quad \text{curl}_5 W = \nabla W + \left(X_0 \frac{\partial}{\partial x_0} + X_5 \frac{\partial}{\partial x_5} \right) W = 0,$$

solutions will be sought for which

$$(9.4) \quad \nabla W = 0, \quad \left(X_0 \frac{\partial}{\partial x_0} + X_5 \frac{\partial}{\partial x_5} \right) W = 0.$$

From the second equation

$$\left(X_0 \frac{\partial}{\partial x_0} + X_5 \frac{\partial}{\partial x_5} \right)^2 W = \left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_5^2} \right) W = 0.$$

This implies that W is a sum of spinors W' W'' , which satisfy respectively

$$\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_5} \right) W' = 0, \quad \left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_5} \right) W'' = 0.$$

Such a separation can be effected by another method, when it applies to spinors which are not semi-static.

Put

$$W' = \frac{1}{2}(1 + X_0 X_5) W,$$

$$W'' = \frac{1}{2}(1 - X_0 X_5) W.$$

Then

$$(9.5) \quad X_0 W' = -X_5 W', \quad X_0 W'' = X_5 W''.$$

Thus

$$(9.6) \quad \begin{cases} \left(X_0 \frac{\partial}{\partial x_0} + X_5 \frac{\partial}{\partial x_5} \right) W' = X_5 \left(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_0} \right) W'' \\ \left(X_0 \frac{\partial}{\partial x_0} + X_5 \frac{\partial}{\partial x_5} \right) W'' = X_5 \left(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_0} \right) W'. \end{cases}$$

There are reasons, however, why equation (9.4) cannot be satisfied everywhere. For example, the time rate varies with the gravitational potential, as will, consequently, the value of the operator $\frac{\partial}{\partial x_0}$, whereas $\frac{\partial}{\partial x_5}$ will remain constant.

It is an experimental fact [4] that the time rate is slower in a gravitational field. Assuming a rest frame let x'_0 denote the time in the absence of gravity, and let x_0 denote the time at a point with gravitational potential $V < 1$. Then

$$V dx'_0 = dx_0,$$

$$\frac{\partial}{\partial x'_0} = V \frac{\partial}{\partial x_0}.$$

The wave packet must keep a constant frequency according to the universal time x'_0 , if the different parts are not to become out of phase. If (9.3) is satisfied at a point where $V = E$, then

$$0 = \left(X_0 \frac{\partial}{\partial x_0} + X_5 \frac{\partial}{\partial x_5} \right) W = \left(\frac{1}{E} X_0 \frac{\partial}{\partial x'_0} + X_5 \frac{\partial}{\partial x_5} \right) W,$$

so that

$$\frac{1}{E} X_0 \frac{\partial}{\partial x_0} = -X_5 \frac{\partial}{\partial x_5}, \quad \frac{\partial}{\partial x'_0} = E X_0 X_5 \frac{\partial}{\partial x_5}.$$

At a point with potential V ,

$$(9.7) \quad \frac{\partial}{\partial x_0} = \frac{1}{V} \frac{\partial}{\partial x'_0} = \frac{E}{V} X_0 X_5 \frac{\partial}{\partial x_5}.$$

Operating on equation (9.3) with the operator $(1 \pm X_0 X_5)$, since $X_0 X_5$

commutes with X_1, X_2, X_3 , but anticommutes with X_0 and X_5 , this gives

$$\nabla W' + \left(X \frac{\partial}{\partial x_0} + X_5 \frac{\partial}{\partial x_5} \right) W'' = 0,$$

$$\nabla W'' + \left(X_0 \frac{\partial}{\partial x_0} + X_5 \frac{\partial}{\partial x_5} \right) W' = 0.$$

Substitution from (9.6) and (9.7) gives

$$\nabla W' + X_5 \left(1 + \frac{E}{V} \right) \frac{\partial}{\partial x_5} W'' = 0,$$

$$\nabla W'' + X_5 \left(1 - \frac{E}{V} \right) \frac{\partial}{\partial x_5} W' = 0.$$

Ignoring the factor $1/V$ which is nearly equal to unity, these equations may be written

$$(9.8) \quad \begin{cases} \nabla W' + X_5(E + V) \frac{\partial}{\partial x_5} W'' = 0 \\ \nabla W'' + X_5(-E + V) \frac{\partial}{\partial x_5} W' = 0 \end{cases}$$

Assuming that W includes a factor e^{ix_5} , so that the operator $\frac{\partial}{\partial x_5}$ may be identified with i , this becomes equivalent to DIRAC'S ⁽⁶⁾ equations for the relativistic electron

$$(9.9) \quad \begin{cases} \nabla W' + iX_5(E + V) W'' = 0, \\ \nabla W'' + iX_5(-E + V) W' = 0. \end{cases}$$

It must be noted here, however, that W could include, instead, a factor e^{-ix_5} . In this case equations (13.9) will remain valid if one small alteration is made, namely that the sign of W'' is changed. It will be seen later that a combination of spinor distributions of both types are necessary for a description of the electron.

These equations have been obtained for a gravitational field. But clearly, for an electron, the electromagnetic field is more important. KALUZA'S metric (2.1)

$$ds^2 = (dx^5 + \gamma_{i5} dx^i)^2 + g_{ik} dx^i dx^k,$$

⁽⁶⁾ P. A. M. DIRAC [1] Chapter XII.

would require a modification of the expression for $\text{curl}_5 W$. For a static electron this would give

$$\text{curl}_3 W + X_0 \frac{\partial}{\partial x_0} W + X_5 \left(\frac{\partial}{\partial x_5} + \gamma_{50} \frac{\partial}{\partial x_0} \right) W = 0.$$

Putting

$$\frac{\partial W}{\partial x_0} = iEW,$$

which incidentally identifies the energy with the frequency, and

$$\left(\frac{\partial}{\partial x_5} + \gamma_{50} \frac{\partial}{\partial x_0} \right) W = iVW,$$

substitution leads in the same way to (9.8) and (9.9).

At this point it may be pertinent to compare the significance of (9.9) with the significance in the DIRAC theory. Firstly DIRAC'S equations refer to a statistical aggregate of possible states, the mathematical structure which is not attempted here. Equation (9.9) refers to a unique state. However, one would expect that the differential equations of a single state would carry over to a statistical aggregate of states.

Secondly there is a difference in interpretation. DIRAC referred to «negative energy» states. Replacing energy by the almost equivalent «frequency», this changes to a «negative frequency» state. As free, rather than forced oscillations, W'' would correspond to waves propagated in a reverse direction round the fifth dimensional cylinder. The description «negative frequency» would therefore be pertinent. But in the KALUZA theory geodesics traversing the fifth dimension in the reverse direction are the paths of positively charged instead of negatively charged particles. Thus the positron arises spontaneously without the necessity for any «theory of holes».

§ 10. - The structure of the electron.

The semi-static solutions of $\text{curl}_5 W = 0$ contain terms of four types, involving respectively factors of the form

$$e^{i(x_0+x_5)}, \quad e^{-i(x_0+x_5)}, \quad e^{i(x_0-x_5)}, \quad e^{i(x_5-x_0)}.$$

The first two are propagated in one direction, the last two in the opposite direction round the fifth dimensional cylinder. It may be assumed therefore, that the first two correspond to an electron and the last two to a positron.

As free oscillations the waves

$$W' = \frac{1}{2}(1 + X_0 X_5) W$$

would be of the first two types, while

$$W'' = \frac{1}{2}(1 - X_0 X_5) W$$

would be of the last two types.

However, W' and W'' are coupled together by equations (9.9), and in the case of the electron would have to involve factors $e^{i(x_0+x_5)}$ or $e^{-i(x_0+x_5)}$, thus W'' is a forced oscillation, being propagated in a reverse direction to that of its free oscillation.

The fact that there are two distinct types of wave for the electron, involving respectively the factors $e^{i(x_0+x_5)}$ and $e^{-i(x_0+x_5)}$ allows a breach in the topological barrier.

As seen in § 10, a distribution of spinors may be found which gives the classical electromagnetic field of an electron at the origin but has a line of singularities along the negative x_1 -axis. Denote this distribution by W_1 . There is a similar distribution W_2 which has a line of singularities along the positive x_1 -axis. Then if $\theta = \cos^{-1}(x_1/r)$, the distribution

$$W = W_1 \cos \frac{\theta}{2} e^{i(x_0+x_5)} + W_2 \sin \frac{\theta}{2} e^{-i(x_0+x_5)}$$

also gives the classical electromagnetic field of an electron and is without singularities, except at the origin.

The solution of $\text{curl}_5 W = 0$ which will now be found conforms asymptotically to this concept. From the manner in which it is obtained it would seem to be the simplest stationary wave-packet solution.

Taking equations (9.9) with $E = 1$ and $V = 1 - \frac{1}{r}$, since $E + V$ is approximately 2, these equations becomes

$$(10.1) \quad \begin{cases} \nabla W' + 2iX_5 W'' = 0 \\ \nabla W'' + \frac{i}{r} X_5 W' = 0 \end{cases}$$

If

$$W' = \frac{1}{2}(1 + X_0 X_5) W'$$

is any solution of

$$(10.2) \quad \nabla W' + \frac{2}{r} W' = 0,$$

then putting

$$W'' = \frac{1}{2} i X_5 \nabla W',$$

the pair of spinors W' , W'' will satisfy (10.1).

If W is a spinor function of position and R is a function of r , then

$$(10.3) \quad \nabla(RW) = R\nabla W + R'\mathfrak{D}W,$$

where $R' = dR/dr$, and

$$(10.4) \quad \mathfrak{D} = x_i X_i / r, \quad (i = 1, 2, 3),$$

an operator which satisfies

$$(10.5) \quad \mathfrak{D}^2 = 1.$$

Let V_1 denote a constant spinor, which will in the first case be given by

$$\tilde{V}_1 = [1, 0, 0, 0]$$

Then if

$$V_2 = \mathfrak{D}V_1,$$

$$\tilde{V}_2 = [x_1/r, -x_2/r, 0, -x_3/r]$$

and

$$\mathfrak{D}V_1 = V_2, \quad \mathfrak{D}V_2 = V_1, \quad \nabla V_1 = 0, \quad \nabla V_2 = \frac{2}{r} V_1.$$

Put

$$W_0 = R(V_1 + V_2),$$

where R is a function of r satisfying

$$(10.6) \quad R'' + \frac{2}{r} R' + \frac{2}{r} R = 0.$$

Then

$$\nabla W_0 = \frac{2R}{r} V_1 + R'(V_1 + V_2)$$

$$\begin{aligned}
\nabla^2 W_0 &= \left(\frac{2R'}{r} - \frac{2R}{r^2} \right) V_2 + \frac{2R'}{r} V_1 + R''(V_1 + V_2) = \\
&= \left(R'' + \frac{2R'}{r} \right) (V_1 + V_2) - \frac{2R}{r^2} V_2 \\
\left(\nabla^2 + \frac{2}{r} \right) W &= \frac{-2R}{r^2} V_2.
\end{aligned}$$

Thus W_0 satisfies (10.2) to the order of approximation being considered. A more exact solution could be obtained by adding a multiple of V_2 multiplied by a series of negative powers of r beginning with r^{-2} . There would be no point in this, however, unless (10.1) were replaced by a more exact equation.

Put

$$\begin{cases} W_1' = \frac{1}{2}(1 + X_0 X_5) W_0 \\ W_1'' = \frac{1}{2} i X_5 \nabla W_1' \end{cases}$$

The spinor distribution so defined satisfies asymptotically equations (10.1), but it would not by itself satisfy (9.9), since it would not yield the correct electromagnetic field for the potential $1 - 1/r$ which was assumed in (10.1).

A second spinor distribution is obtained in exactly the same way, except that the fixed spinor V_1 is given by

$$\tilde{V}_1 = [0, 1, 0, 0].$$

Correspondingly

$$\tilde{V}_2 = [-x_2/r_1 - x_1/r, -x_3/r, 0].$$

The spinor distributions defined in exactly the same way from these are denoted by W_2' and W_2'' .

The resulting solution of

$$\text{curl}_5 W = 0$$

is then given by

$$(10.7) \quad W = (W_1' + W_1'')e^{i(x_0+x_5)} + (W_2' + W_2'')e^{-i(x_0+x_5)}.$$

It remains only to show that this distribution gives asymptotically the field of the electron which would in turn justify the use of the potential $1 - 1/r$ in (14.1). The terms W'' are small as compared with W' , and for a first approximation may be ignored.

The solution of

$$R'' + \frac{2}{r}R' + \frac{2}{r}R = 0$$

is asymptotically of the form

$$e^{ik\sqrt{r}}/r.$$

The factor $e^{ik\sqrt{r}}$ does not affect the concomitants of the spinor. The concomitants of W are the sums of the corresponding concomitants of the two parts since the cross products when integrated with respect to x_5 give zero. Of the 6-vector concomitants, the introduction of the factor $(1 + X_0X_5)$ has the effect of annihilating the magnetic terms, but just doubles the electric terms. The field is thus a multiple of the sum of the electric components of the 6-vector concomitants of

$$\left[\frac{1}{r} + \frac{x_1}{r^2}, \frac{-x_2}{r^2}, 0, \frac{-x_3}{r^2} \right] \quad \text{and} \quad \left[\frac{-x_2}{r^2}, \frac{1}{r} + \frac{-x_1}{r^2}, \frac{-x_3}{r^2}, 0 \right].$$

This gives

$$\begin{aligned} E_1 &= \frac{-(r+x_1)^2 + x_2^2 + x_3^2}{r^4} + \frac{x_2^2 + (r-x_1)^2 - x_3^2}{r^4} = \frac{-4x_1}{r^3}, \\ E_2 &= \frac{-2rx_2 - 2x_1x_2}{r^4} + \frac{-2rx_2 + 2x_1x_2}{r^4} = \frac{-4x_2}{r^3}, \\ E &= \frac{-2x_3 - 2x_1x_3}{r^4} + \frac{-2rx_3 + 2x_1x_3}{r^4} = \frac{-4x_3}{r^3}. \end{aligned}$$

The field is thus, asymptotically the correct field for the classical electron.

However, a difficulty arises with the short range fields if R is taken as a complex solution of the equation

$$R'' + 2R'/r + 2R/r = 0.$$

On evaluating the cross products

$$\tilde{W}_1' \phi W_1'' \quad \text{and} \quad \tilde{W}_2' \phi W_2''$$

there arise non-zero terms corresponding to a radial momentum flow Q_i , which can be inward or outward according to the sign of i which is chosen for the solution. No such difficulty arises if R is taken as the real solution of the differential equation. This solution is asymptotically of the form

$$\sin(k\sqrt{r} + \varepsilon)/r.$$

The corresponding electric field is not then equal, asymptotically, to that of the classical electron, but involves an additional factor $\sin^2(k\sqrt{r+\epsilon})/r$. There would not appear to be any experimental reason why this modified electron field should not prove to be correct. There is, in fact, strong theoretical corroboration for this modification.

Assuming that, near the centre of an electron

$$E^2 - V^2 = k^2,$$

the equations give

$$[\nabla^2 + k^2]W = 0.$$

For V a constant, spinor solutions are given by

$$W_0 = Ve^{ikr}/r \quad \text{and} \quad W_0 = V \sin kr/r.$$

The former solution has a singularity at the centre, the latter solution is singularity free. Seeking singularity free solutions the latter must be chosen. In order to make the asymptotic solution a continuation of the solution at the centre, it is clear that the factor $\sin(k\sqrt{r+\epsilon})/r$ should be preferred to $e^{i(k\sqrt{r+\epsilon})/r}$.

Assuming R to be a real function of r ,

$$W_0 = R(V_1 + V_2), \quad \nabla W_0 + 2RV_1/r + R(V_1 + V_2),$$

$W'\psi W''$ involves such factors as

$$(2R^2/r + RR')\tilde{\bar{V}}_1\psi iX_5V_1,$$

$$(2R^2/r + 2RR')\tilde{\bar{V}}_1\psi iX_5V_2,$$

$$RR'\tilde{\bar{V}}_2\psi iX_5V_2.$$

The following are the only non-zero concomitants. There is a concomitant Q_i with components

$$(2R^2/r^2 + 2RR'/r)[0, x_3, -x_2],$$

which clearly corresponds to a mechanical spin. There is a magnetic field composed of two parts of which the first has components

$$(2R^2/r + RR')[-1, 0, 0]$$

and the second

$$RR'/r[-x_1^2 + x_2^2 + x_3^2, -2x_1x_2, -2x_1x_3]$$

which clearly corresponds to a magnetic spin.

§ 11. - Relation between space curvature and concomitants.

The main hypothesis of the model is that the space curvatures are determined by the concomitants of the spinor distribution.

The KALUZA curvature

$$f_{ik} = \frac{\partial \gamma_{k_5}}{\partial x^i} - \frac{\partial \gamma_{i_5}}{\partial x^k}$$

is determined by the concomitant

$$\tilde{W} T X_i X_k W.$$

Since each point in 4-space corresponds to a ring of points in 5-space, the electromagnetic field may be taken as the mean value of this concomitant along this ring.

The connection with 4-dimensional curvature is not so simple. EINSTEIN equates the EINSTEIN tensor

$$G_{ij} = R_{ij} - \frac{1}{2g_{ij}} R,$$

where R_{ij} and R are the contracted curvature tensors, with the energy-momentum-stress tensor. For an energy-momentum distribution this is $Q_i Q_j$. For an electromagnetic field it is

$$Q_i Q_j + P_i P_j - \frac{1}{4} g_{ij} (P_k P^k + Q_k Q^k) = Q_i Q_j + P_i P_j - \frac{1}{4} g_{ij} (I^2 + J^2).$$

For arbitrary numerical λ , μ , ν , it would seem that the equation

$$G_{ij} = \lambda Q_i Q_j + \mu P_i P_j + \nu g_{ij} (I^2 + J^2)$$

would be sufficiently close to EINSTEIN'S assumption to conform generally with his theory. Certainly the term $\lambda Q_i Q_j$ would ensure the correct gravitational effect. For the sake of stability of large masses there must be included also stress terms. Unfortunately EINSTEIN'S Theory is not here an adequate guide. His equations lead to the instability of a sufficiently large mass so that his theory implies that it would collapse into a closed universe of its own. This constitutes a *reductio ad absurdum* for the theory in its present form.

It is thought that the generalised form given here could be adapted to replace this instability by stability.

§ 12. - Stationary orbits.

If an electron describes an orbit about a nucleus in such a way that the period of rotation is a multiple of the period of oscillation, then the fields of the electron in the same position but at different times, would be in phase, and no modification would be necessary. If, on the other hand, the ratio of the periods were not integral there would be interference because of the difference in phase. The oscillations of the spinor field would be forced into a frequency which was not the natural frequency, but was a multiple of the frequency of revolution. The situation is essentially similar to that of an electron with a gravitational field, as described in obtaining DIRAC'S Relativistic Electron equations as described in § 13, and this would induce forced oscillations of type W'' . The interaction of W' and W'' would introduce concomitants of type H_i and Q_i , ($i = 1, 2, 3$). The magnetic field would deflect the electron from its orbit into another orbit, until it finds one with a period which is a multiple of the period of oscillation. Thus the model conforms to the general principles of Quantum Theory.

§ 13. - Frequency of spectral lines.

The potential energy of the electron might be defined (see § 9) as the potential at points for which

$$\left(X_0 \frac{\partial}{\partial x_0} + X_5 \frac{\partial}{\partial x_5} \right) W = 0.$$

This makes the frequency of a stationary electron exactly equal to its potential energy. For a moving electron the total energy, potential and kinetic, is still equal to the frequency. To show this, it is only necessary to make a LORENTZ transformation from the frame for which the electron is at rest. Under a LORENTZ transformation wave numbers and frequency transform in exactly the same way as momentum and energy. Thus the identity of energy and frequency remains after the transformation.

If an electron makes a transition from one stationary orbit to another there will be a change from a wave W_1 involving a factor $e^{iE_1x_0}$ to a wave W_2 involving a factor $e^{iE_2x_0}$. For simplicity suppose that the wave during the transition is of the form

$$W = (1 - k) W_1 + k W_2,$$

where k increases from 0 to 1.

Concomitants of the form $\tilde{W}_1 \psi \tilde{W}_1$ and $W_2 \psi W_2$ correspond to stationary orbits and do not yield electromagnetic radiation. But there are also terms

of the form

$$(k - k^2)(\tilde{W}_1\psi W_2 + \tilde{W}_2\psi W_1).$$

The term $\tilde{W}_1\psi W_2$ involves a factor $e^{i(E_2-E_1)x_0}$, while $\tilde{W}_2\psi W_1$ involves a factor $e^{i(E_1-E_2)x_0}$. Taken together these yield a factor $\cos(E_1 - E_2)x_0$. This is exactly the correct frequency rule for spectral lines.

§ 14. - Superposition and resonance.

If the linear differential equation involves the spinor, and the electromagnetic field is of degree 2 in the spinor, then the superposition of two solutions of the differential equation would not normally add the corresponding electromagnetic fields.

Let

$$W = W_1 + W_2$$

denote the superposition of two solutions. Concomitants of W are then of the form

$$\begin{aligned}\tilde{W}\psi W &= \tilde{W}_1\psi W_1 + \tilde{W}_2\psi W_2 \\ &+ \tilde{W}_1\psi W_2 + \tilde{W}_2\psi W_1.\end{aligned}$$

If W_1 involves a factor e^{inx_0} and W_2 a factor e^{imx_0} then $\tilde{W}_1\psi W_2$ and $\tilde{W}_2\psi W_1$ involve factors $e^{\pm i(m-n)x_0}$ this implies that unless $m=n$, both terms oscillate about a zero mean. Ignoring these rapid oscillations, it follows that the concomitants of W are the sums of the corresponding concomitants of W_1 and W_2 respectively.

Thus ignoring the rapidly oscillating terms, the electromagnetic fields are effectively additive, unless there is resonance. When there is resonance it might be expected that exceptional results would follow.

If two electrons were exactly in resonance, it might therefore be expected that there would be a strong interaction. The only case for which two electrons could be kept in exact resonance is when they occupy the same stationary orbit about a nucleus. The resulting strong interaction would no doubt eject at least one of the electrons from its orbit. This would account for the PAULI exclusion principle.

§ 15. - Quantization of light.

In the consideration so far given to spinor waves no account was taken of space curvature. When this is taken into account it will be seen that the light can be quantitized. This results not at all from the electromagnetic field which seems to have a completely neutral effect.

Considering waves propagated in the x_2 -directions, if there is a vector potential v_i , equations (9.2) give

$$\begin{aligned}\alpha_1 - \delta_3 + iv_1\delta_1 + iv_3\delta_3 &= 0, \\ -\delta_1 - \alpha_3 - iv_1\alpha_1 - iv_3\alpha_3 &= 0, \\ -\alpha_2 - \alpha_0 &= 0, \\ \delta_2 + \delta_0 &= 0.\end{aligned}$$

The only bounded solution is given by $\alpha_1 = \alpha_3 = \delta_1 = \delta_3 = 0$, just as if the vector potential terms were omitted.

The quantization is entirely due to the gravitational effect of the quantum. This may be investigated by the examination of complicated mathematical equations, but the result is most easily explained as follows.

Clearly a wave packet may be formed by the superposition of a set of monochromatic waves of very nearly equal frequency and direction. But in flat space-time a wave packet expands and dissipates.

The effect of gravitational field is to slow down the time rate at a point of higher gravitational potential. Thus relative to universal time a wave packet would act as a condensing lens which tended to concentrate the waves. If the total energy of the wave packet were such that the lens effect exactly counteracted the tendency of the wave packet to expand and dissipate then it would be stable and would be propagated without relative change.

It can be shown that the total energy of such a wave packet is independent of the size and is exactly proportional to the frequency.

Firstly, given a solution of the differentiation equation in the form of a steady wave packet of frequency ν and energy E , a LORENTZ transformation will yield another solution in which both the frequency and the energy are multiplied by the same factor. Ignoring the corresponding FITZGERALD contraction the wave packet will be of the same size and shape.

Secondly a more remarkable result follows for wave packets of different size.

Plot a wave packet in one space onto another space so that $x_i \rightarrow x'_i = \lambda^2 x_i$, with $W' = \lambda^{-1} W$. This gives $E'_i = \lambda^{-2} E_i$. The vector and gravitational potentials are the same in both spaces, so that the differential equation is unaltered. An element of volume involves an extra factor λ^6 , while the energy density involves a factor λ^{-4} . Thus the total energy is multiplied by λ^2 , exactly the same as the change in frequency.

Much more detailed calculation, however, would be needed to determine whether the ratio of energy to frequency was exactly the same as for an electron.

§ 16. - Maxwell's equations and the classical electromagnetic theory.

For a given electromagnetic field R_{ij} there are two 4-vector derivatives which may be termed the electric current e_i and the magnetic current m_i . MAXWELL'S equations are equivalent to the vanishing of e_i and m_i in freespace.

To obtain the connection with this spinor theory the components $\text{div } E = e_0$ and $\text{div } H = m_0$ will be calculated and from the result the complete electric and magnetic currents may be obtained from the principle of LORENTZ invariance.

Using equations (3.4) with $W_1 = \bar{W}$, and substituting from (13.2) it follows that

$$e_0 = i(\bar{\alpha}\delta_5 - \bar{\delta}\alpha_5 + \bar{\beta}\gamma_5 - \bar{\gamma}\beta_5) + (\bar{\alpha}\beta_0 - \bar{\beta}\alpha_0 + \bar{\gamma}\delta_0 - \bar{\delta}\gamma_0) + \text{complex conjugate}.$$

Representing the first term, with complex conjugate, by e'_0 and the second term by e''_0 , it is clear that the electric current may be separated into two parts.

For the purpose of LORENTZ invariance the first part e'_0 may be compared to the concomitant P_i in (3.4). Thus e'_i is represented by the 4-vector with components

$$i(-\bar{\alpha}\gamma_5 + \bar{\gamma}\alpha_5 - \bar{\beta}\delta_5 + \bar{\delta}\beta_5) + \text{complex conjugate},$$

$$i(-\bar{\alpha}\delta_5 + \bar{\delta}\alpha_5 + \bar{\beta}\gamma_5 - \bar{\gamma}\beta_5) + \text{complex conjugate},$$

$$i(\bar{\alpha}\beta_5 - \bar{\beta}\alpha_5 - \bar{\gamma}\delta_5 + \bar{\delta}\gamma_5) + \text{complex conjugate},$$

$$i(\bar{\alpha}\delta_5 - \bar{\delta}\alpha_5 + \bar{\beta}\gamma_5 - \bar{\gamma}\beta_5) + \text{complex conjugate}.$$

This differs from zero significantly, only in the neighbourhood of a charged particle, and thus conforms with classical theory.

The term e''_i is given by

$$\bar{\alpha}\beta_i - \bar{\beta}\alpha_i + \bar{\gamma}\delta_i - \bar{\delta}\gamma_i + \text{complex conjugate}.$$

This represents a rotation of the spin vector W in spin space. For the interaction of two distinct spinor fields it will oscillate with small intensity about a zero mean, and could safely be neglected. The only other possibility is that a stable structure can exist which would maintain a continuous rotation of this type in spin-space. Such a structure would represent a charged particle, and would be consistent with classical theory.

In the same way the magnetic current is expressible as a sum of two

parts, of which m'_i is given by

$$i(\bar{\alpha}\beta_5 + \bar{\beta}\alpha_5 + \bar{\gamma}\delta_5 + \bar{\delta}\gamma_5) + \text{complex conjugate},$$

$$i(\bar{\alpha}\alpha_5 - \bar{\beta}\beta_5 - \bar{\gamma}\gamma_5 + \bar{\delta}\delta_5) + \text{complex conjugate},$$

$$i(\bar{\alpha}\gamma_5 + \bar{\gamma}\alpha_5 - \bar{\beta}\delta_5 - \bar{\delta}\beta_5) + \text{complex conjugate},$$

$$i(\bar{\alpha}\alpha_5 + \bar{\beta}\beta_5 + \bar{\gamma}\gamma_5 + \bar{\delta}\delta_5) + \text{complex conjugate}.$$

This only differs from zero in the neighbourhood of a charged particle, but rather remarkably, it does not exclude the possibility of a magnetic pole. The exclusion of the magnetic pole is clearly dictated by geometric considerations since the previous work indicates that it cannot exist without a topological singularity.

Spinor terms involving e^{ix_5} or e^{-ix_5} contribute respectively positive and negative magnetic density. The geometric impossibility of a magnetic pole ensures that the terms in e^{ix_5} and in e^{-ix_5} are balanced. This accounts for the two separate parts which arise in the model of the electron, and it explains incidentally how these two parts are kept together. Each by itself would have a magnetic pole and the two parts are kept together by strong magnetic forces.

The terms m''_i is given by

$$\bar{\alpha}\gamma_i - \bar{\gamma}\alpha_i - \bar{\beta}\delta_i + \bar{\delta}\beta_i + \text{complex conjugate}.$$

This again represents a rotation of the spin vector in spin space. There can be no particle which consistently maintains such a rotation and it is clear that it can only occur in the interaction between two particles, when it will oscillate with small amplitude about a zero mean.

Thus the theory indicates that MAXWELL's equations will be valid to a sufficient degree of accuracy.

The divergencies of the vectors e'_i , e''_i , m'_i , m''_i are all zero.

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