On the Theory of Schur-Rings.

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Zusammenfassung. - Diese Arbeit versucht, die von Issai Schur [1] entdeckte und von Wielandt ([14], [15], [16], [17]) beträchtlich weiterentwickelte Methode zur Untersuchung von endlichen Permutationsgruppen zu einer Theorie der Schur-Ringe zu entfalten. Der Grundgedanke ist sehr einfach: Die Schur-Ringe werden nicht als eine spezielle Klasse von Ringen aufgefaßt, sondern als eine eigene mathematische Struktur. Nach unserer heutigen Ansicht fällt der Begriff der mathematischen Struktur weitgehend mit dem Begriff der Kategorie zusammen. Daher wird für die Schur-Ringe (genauer: für die Schur-Algebren) ein eigener Homomorphiebegriff (Definition 1.5) eingeführt, der eine Kategorie liefert (Theorem 1.6).

Ein weiterer Leitgedanke ist mit dem kategoriellen Grundgedanken sehr eng verknüpft. Die Theorie der Schur-Ringe wird als eine Verallgemeinerung der Theorie der endlichen Gruppen aufgefaßt und in diesem Sinne entwickelt. Dabei ist die Theorie der endlichen Gruppen vermittelt der Gruppenringe der endlichen Gruppen (die eine spezielle Teilkategorie der Kategorie aller Schur-Ringe sind) in die Theorie der Schur-Ringe eingefügt. Hierfür ist es wichtig, daß die Morphismen der Gruppenringe in der Kategorie der Schur-Ringe genau die von den Gruppenhomomorphismen induzierten Gruppenringhomomorphismen sind.

Die Einbettung der Theorie der endlichen Gruppen in die Theorie der Schur-Ringe vollzieht sich entlang dreier Entwicklungslinien.

Die erste ist eine verallgemeinerte Charakterentheorie ([2], [3], [5], [6], [7] und [8]), die die Theorie der (gewöhnlichen) Charaktere von endlichen Gruppen als Spezialfall enthält.

Die zweite ist die Verknüpfung der Struktur jedes Schur-Ringes T auf einer endlichen Gruppe G mit gewissen Klassen von Untergruppen von G . Es werden die Begriffe der T -Untergruppe (Abschnitt 3), des T -Normalteilers (Abschnitt 4), und der T -subnormalen Untergruppe (Abschnitt 8) eingeführt. Die T -Untergruppen bilden einen Teilverband des Verbandes aller Untergruppen von G (Theorem 3.4). Die T -Normalteiler sind genau die Kerne (Definition 6.1) der Homomorphismen der Schur-Algebren QT (Theoreme 6.2 und 6.3).

Der dritte und wohl zugleich der wichtigste Aspekt ist die Gültigkeit des Homomorphiesatzes (Theorem 6.2) und der Isomorphiesätze (Theoreme 7.1 und 7.2) für Schur-Algebren. Auf diese Sätze gründet sich der Vier-Untergruppen-Satz (Zassenhaus' Lemma; Theorem 9.1), der den Verfeinerungssatz für T -Subnormalketten (Theorem 9.2) und den Jordan-Hölder Satz für T -Kompositionsketten (Theorem 10.3) nach sich zieht.

Als die Theorie der Schur-Ringe ungefähr den soeben geschilderten Stand erreicht hatte, tauchte die Idee auf, diese Theorie auf beliebige Gruppen zu verallgemeinern ([9], [10], [11], [12], [13]). Das führte zum Begriff der Schur-Halbgruppe (Definition 1.9). Der zugehörige Homomorphiebegriff (Definition 1.11) liefert die Kategorie aller Schur-Halbgruppen (Theorem 1.12), die die Kategorie aller Gruppen als echte Teilkategorie enthält. Jedem Schur-Ring T über einer endlichen Gruppe G wird eine Schur-Halbgruppe $\mathbb{T}$ über G zugeordnet (Theorem 1.15). Jedem Homomorphismus φ einer Schur-Algebra QT über G wird ein Homomorphismus Φ von $\mathbb{T}$ zugeordnet (Theorem 1.16). Das Paar der Zuordnungen $QT \rightarrow \mathbb{T}$, $\varphi \rightarrow \Phi$ ist ein Funktor auf der Kategorie aller Schur-Algebren in die Kategorie aller Schur-Halbgruppen über endlichen Gruppen (Theorem 1.17).

The *method of Schur*, introduced into the theory of finite permutation groups in 1933 by ISSAI SCHUR's paper [1], consists in the investigation of a certain class of subrings of the group rings of finite groups which WIELANDT called *S-rings*, i.e. *Schur-rings* ([17], Definition 23.1). Compared with character theory, which still dominates many parts of the theory of finite permutation groups, it is much more elementary, and yet, in certain instances, it is no less successful than its powerful rival. Apart from SCHUR's original idea this is chiefly due to the work of WIELANDT ([14], [15], [16], and [17], Chapter IV).

Already in the early days of the *theory of Schur and Wielandt* underlying connections with character theory had been worked out and used in applications by WIELANDT ([16] and [17], Chapter V). But a new branch of this theory developed ([2] – [8]) when, by turning away from the elementary features of *S-rings*, attention was focused on the semi-simplicity of the *S-algebras* over the field $\mathbf{C}$ of complex numbers (a fact that was well known for a long time, cf. [15], p. 386, foot-note). Investigations in that direction led to the discovery that the coefficients of the irreducible representations of an *S-algebra* (over $\mathbf{C}$) on a finite group G can be expressed in terms of the irreducible representations of the group G itself [6]. From there it was only one further step to discover that every *S-ring* on a finite group G defines its own character theory on G [7]. That theory incorporates ordinary character theory as a special case by applying it to the group ring $\mathbf{Z}G$ qua *S-ring*. Though this generalized character theory still lacks any spectacular application (an application of minor importance is given in [4]) it has had one remarkable sequel so far. A generalization of normal subgroups arose from it [8]. Each *S-ring* T on a finite group G defines a class of *T-normal* subgroups of G which are characterized by those *S-rings* on G that are two-sided ideals of T . So the detour via the generalized character theory at least led to a fundamental notion of *S-ring* theory whose definition is free of character theoretical ingredients.

The concept of *T-normal* subgroup became the starting point for yet another development which embraces both the elementary and the character theoretical part of the theory of *S-rings*. It is the purpose of this paper to present this new approach to *S-ring* theory. The basic idea is quite simple. The *S-ring* is considered not as a special type of ring but as a mathematical structure in its own right. This means that a notion of *S-ring* homomorphism has to be produced so as to obtain a category. It turns out (cf. the remarks after Definition 1.3) that a slight modification is necessary. It is not the structure of an *S-ring* but the structure of an *S-algebra* over a field of characteristic zero (Definition 1.4; the field $\mathbf{Q}$ of rational numbers is sufficient) that has to be used in order to obtain a workable notion of homomorphism (Definition 1.5) and hence a category (Theorem 1.6). The kernels

of the homomorphisms of an S -algebra QT on a finite group G (Definition 6.1) are exactly the T -normal subgroups of G (Theorems 6.2 and 6.3).

We consider S -ring theory as a generalization of finite group theory where finite group theory is incorporated via the group rings qua S -rings. To realize this programme we introduce notions which link the S -ring structure with the structure of the underlying group. Apart from the generalized character theory already mentioned above this is primarily done by defining several classes of subgroups, namely the T -subgroups (Section 3), the T -normal subgroups (Section 4), and the T -subnormal subgroups (Section 8). Inter alia, the T -subgroups form a sublattice of the lattice of all subgroups of G (Theorem 3.4). Every homomorphism φ of an S -algebra QT on G into an S -algebra QS on a finite group F relates T -subgroups of G to S -subgroups of F (Section 5) and yields a lattice monomorphism of the lattice of all those T -subgroups of G containing the kernel of φ (Theorem 6.7).

Of the lattice theoretical properties of the T -normal subgroups we only know that the product of any two of them is again T -normal (Theorem 4.8), and of the T -subnormal subgroups we are able to prove that the intersection of any two of them is again T -subnormal (Theorem 8.5).

Apart from the generalized character theory and the various classes of subgroups it is the S -ring version of the *Homomorphism Theorem* (6.2) and of both *Isomorphism Theorems* (7.1 and 7.2) which supports the conception of S -ring theory as a generalization of finite group theory.

In consequence of these theorems the S -ring version of the *Four Subgroup Theorem* 9.1 (Zassenhaus' Lemma) can be proved which in turn implies the following results. As the factors of a T -subnormal chain we define certain S -rings on the subgroups of that chain which are connected with T . The *Refinement Theorem for T -subnormal chains* (9.2) and the *Jordan-Hölder Theorem for T -composition chains* (10.3) hold with S -algebra isomorphisms of the factors.

Once that state of the theory of S -rings was achieved the idea of generalizing the whole theory from finite to arbitrary groups broke through. The result was the concept of an S -semigroup on a group G ([9] and [10]). With its own notion of S -semigroup homomorphism (modelled after the S -algebra homomorphism) a category of S -semigroups is obtained which properly contains the category of all groups. The kernels of the homomorphisms of an S -semigroup T on G are the T -normal subgroups of G . With these concepts of homomorphism, kernel, and T -normal subgroup the S -semigroup versions of the Homomorphism Theorem, the Isomorphism Theorems, the Four Subgroup Theorem, the Jordan-Hölder Theorem for T -composition chains and other statements hold as proper generalizations of the relevant statements of group theory ([9], [10], [11], [12]; an even more recent development of these ideas can be found in [13]).

Every S -ring T on a finite group G is associated with an S -semigroup $\mathbf{T}$ on G (Theorem 1.15), every S -algebra homomorphism φ is associated with an S -semigroup homomorphism Φ (Theorem 1.16), and the pair of mappings $T \rightarrow \mathbf{T}$, $\varphi \rightarrow \Phi$ is a functor on the category of all S -rings into the category of all S -semigroups (Theorem 1.17).

1. - S -Rings, S -Algebras, S -Semigroups.

Let G be a group. For any set $X \subseteq G$ we write

$$X^{-1} := \{g^{-1} \mid g \in X\}.$$

If X is a finite set, then we form the element

$$\underline{X} := \sum_{g \in X} g$$

in the group ring $\mathbf{Z}G$.

DEFINITION 1.1. (WIELANDT [17], 23.1). - Let G be a finite group. A subring T of the group ring $\mathbf{Z}G$ is called an S -ring on G if there exists a set $\mathcal{T}$ of non-empty subsets of G such that the following hold.

- (1) $G = \bigcup_{\mathcal{T} \in \mathcal{T}} \mathcal{T}$.
- (2) $\mathcal{S} = \mathcal{T}$ or $\mathcal{S} \cap \mathcal{T} = \emptyset$ for all $\mathcal{S}, \mathcal{T} \in \mathcal{T}$.
- (3) $\mathcal{T}^{-1} \in \mathcal{T}$ for all $\mathcal{T} \in \mathcal{T}$.
- (4) $\tau = \sum_{\mathcal{T} \in \mathcal{T}} z_{\tau, \mathcal{T}} \underline{\mathcal{T}}$ with $z_{\tau, \mathcal{T}} \in \mathbf{Z}$ for all $\tau \in T$.
- (5) The ring T is generated by the elements $\underline{\mathcal{T}}$, $\mathcal{T} \in \mathcal{T}$ (which consequently are a $\mathbf{Z}$ -basis of T).

The set $\mathcal{T}$ is uniquely determined by the axioms (1)-(5). Therefore we call the elements of $\mathcal{T}$ the T -classes of G . The elements $\underline{\mathcal{T}}$, $\mathcal{T} \in \mathcal{T}$, of the group ring $\mathbf{Z}G$ are called the T -class sums of G .

The T -class $\mathcal{T}_1$ of G which contains the unit element $1 \in G$ is a subgroup of G , and

$$\tau \underline{\mathcal{T}_1} = \underline{\mathcal{T}_1} \tau = |\underline{\mathcal{T}_1}| \cdot \tau \quad \text{holds for all } \tau \in T$$

[15], (1.6). It follows that

$$\mathcal{T}_1 \mathcal{T} = \mathcal{T} \mathcal{T}_1 = \mathcal{T} \quad \text{for all } \mathcal{T} \in \mathcal{T}.$$

EXAMPLES 1.2. -

- 1) The *group ring* $\mathbb{Z}G$ of any finite group G is an S -ring on G with $\mathfrak{C} = G$.
- 2) The *centre* $\zeta(\mathbb{Z}G)$ of the group ring $\mathbb{Z}G$ is an S -ring on G with $\mathfrak{C} =$ the set of all conjugacy classes.
- 3) Let H be a subgroup of the finite group G . Then the subring $G//H$ of the group ring $\mathbb{Z}G$ which is generated by the double coset sums $\underline{HgH}$, $g \in G$, is an S -ring on G with $\mathfrak{C} = \{HgH | g \in G\}$. We call $G//H$ the *double coset S -ring* of G modulo H . With that notation we have $\mathbb{Z}G = G//1$.

Let F be a finite group, S an S -ring on F , and $\mathfrak{S}$ the set of all S -classes of F (i. e. $\mathfrak{S}$ plays the same rôle for S as $\mathfrak{C}$ does for T).

DEFINITION 1.3. - A mapping φ of T into S is called a homomorphism of the S -ring T on G into the S -ring S on F if the following hold.

- (1) $(\sigma\tau)^\varphi = \sigma^\varphi\tau^\varphi$ for all $\sigma, \tau \in T$.
- (2) $(\sigma + \tau)^\varphi = \sigma^\varphi + \tau^\varphi$ for all $\sigma, \tau \in T$.
- (3) For every T -class $\mathfrak{C}$ of G there exists an S -class $\mathfrak{S}$ of F such that

$$|\mathfrak{S}| \cdot \underline{\mathfrak{C}}^\varphi = |\mathfrak{C}| \cdot \underline{\mathfrak{S}} \quad \text{and} \quad |\mathfrak{S}^{-1}| \cdot (\underline{\mathfrak{C}}^{-1})^\varphi = |\mathfrak{C}^{-1}| \cdot \underline{\mathfrak{S}}^{-1}.$$

Properties 1.3 (1) and (2) of an S -ring homomorphism φ simply mean that φ is a ring homomorphism, and this does not need any further discussion. Property 1.3 (3) means the compatibility of φ with the “ S -structure” of the S -rings T and S , and the formulation we have chosen may need some motivation. We want our theory as close to group theory as possible. Every homomorphism α of the group G into the group F induces a homomorphism φ_α of the group ring $\mathbb{Z}G$ into the group ring $\mathbb{Z}F$ in the natural way. For any subgroup H of G , and any $g \in G$ the homomorphism α maps double coset onto double coset, namely

$$(HgH)^\alpha = H^\alpha g^\alpha H^\alpha,$$

but for φ_α we have the equation

$$|H^\alpha g^\alpha H^\alpha| \cdot (\underline{HgH})^{\varphi_\alpha} = |\underline{HgH}| \cdot \underline{H^\alpha g^\alpha H^\alpha}.$$

So if we want any group homomorphism α to induce an S -ring homomorphism of the double coset S -ring $G//H$ onto the double coset S -ring $G^\alpha//H^\alpha$ we have to choose the formulation 1.3 (3).

But there is an aggravating deficiency with Definition 1.3.

It is true that the product of two S -ring homomorphisms again is an S -ring homomorphism in the sense of Definition 1.3. But what makes Definition 1.3 deficient is the fact that it does not yield enough S -ring homomorphisms. For any given S -rings T on G and S on F there should exist at least one homomorphism of T into S . This is not guaranteed by Definition 1.3, for 1.3 (3) can hold only if the positive integer $|\mathcal{S}|$ divides the positive integer $|\mathcal{T}|$.

However this deficiency can easily be remedied. If we consider the equations of 1.3 (3) in the group algebra $\mathbf{Q}G$ ($\mathbf{Q}$ the field of rational numbers) then they may have enough "solutions".

Now we are going to modify our concepts so as to obtain enough homomorphisms.

Let Ω always denote a subfield of the field $\mathbf{C}$ of complex numbers. In this section and in the Sections 3-10 it would suffice to take $\Omega = \mathbf{Q}$. In the next section we shall be concerned with $\Omega = \mathbf{C}$. Therefore we do not specify Ω . We could even substitute Ω by any other field of characteristic zero.

To generalize even further and take any field will cause some problems. But we do not want to investigate these questions in this paper.

DEFINITION 1.4. - *Let T be an S -ring on the finite group G . Then the subalgebra*

$$\Omega T := \left\{ \sum_{\mathcal{T} \in \mathcal{T}} a_{\mathcal{T}} \mathcal{T} \mid a_{\mathcal{T}} \in \Omega \right\}$$

of the group algebra ΩG which consists of the linear combinations of the T -class sums with coefficients from Ω is called an S -algebra on G over Ω .

Now we rephrase Definition 1.3 for S -algebras.

DEFINITION 1.5. - *Let T be an S -ring on the finite group G , and S an S -ring on the finite group F . A mapping φ of ΩT into ΩS is called a homomorphism of the S -algebra ΩT into the S -algebra ΩS if the following hold.*

- (1) $(\sigma\tau)^{\varphi} = \sigma^{\varphi}\tau^{\varphi}$ for all $\sigma, \tau \in \Omega T$.
- (2) $(\sigma + \tau)^{\varphi} = \sigma^{\varphi} + \tau^{\varphi}$ and $(a\tau)^{\varphi} = a\tau^{\varphi}$ for all $\sigma, \tau \in \Omega T$ and all $a \in \Omega$.
- (3) For every T -class $\mathcal{T}$ of G there exists an S -class $\mathcal{S}$ on F such that

$$\left(\frac{1}{|\mathcal{T}|} \cdot \mathcal{T} \right)^{\varphi} = \frac{1}{|\mathcal{S}|} \cdot \mathcal{S} \quad \text{and} \quad \left(\frac{1}{|\mathcal{T}^{-1}|} \cdot \mathcal{T}^{-1} \right)^{\varphi} = \frac{1}{|\mathcal{S}^{-1}|} \cdot \mathcal{S}^{-1}.$$

A homomorphism φ of an S -algebra ΩT into an S -algebra ΩS is called an *epimorphism*, a *monomorphism*, or an *isomorphism* if φ is a surjective, an injective, or a bijective mapping respectively. These definitions of epimor-

phism and monomorphism are naive ones. Whether they have the same meaning in the categorical sense will have to be discussed later

Definitions 1.4 and 1.5 now yield the desired result.

THEOREM 1.6. - *The class of all S -algebras on finite groups over Ω together with their homomorphisms (in the sense of Definition 1.5) form a category.*

PROOF. -

I. Let T be an S -ring on the finite group G , and S an S -ring on the finite group F . We denote by $\underline{S}_1$ the S -class of F containing the unit element of F . The linear extension onto ΩT of the mapping

$$\frac{1}{|\underline{T}|} \cdot \underline{T} \rightarrow \frac{1}{|\underline{S}_1|} \cdot \underline{S}_1 \text{ for all } \underline{T} \in \underline{T}$$

is a homomorphism of ΩT into ΩS in the sense of Definition 1.5. We call it the *zero homomorphism* of ΩT into ΩS . Therefore the set

$$\text{Hom}_\Omega(T, S)$$

of all homomorphisms of the S -algebra ΩT into the S -algebra ΩS is non-empty for any two S -rings T on G and S on F . If T' is an S -ring on G' , and S' is an S -ring on F' , then

$$(T, S) \neq (T', S') \text{ implies } \text{Hom}_\Omega(T, S) \cap \text{Hom}_\Omega(T', S') = \emptyset.$$

II. For each triple of S -rings

$$T \text{ on } G, \quad S \text{ on } F, \quad R \text{ on } E,$$

and each pair of homomorphisms

$$\varphi \in \text{Hom}_\Omega(T, S), \quad \psi \in \text{Hom}_\Omega(S, R)$$

their product as mappings is a homomorphism, i.e.

$$\varphi\psi \in \text{Hom}_\Omega(T, R),$$

and the multiplication of homomorphisms is associative.

III. The identity mapping $i_{\Omega T}$ of any S -algebra ΩT on G is a homomorphism of ΩT , and for every $\varphi \in \text{Hom}_\Omega(T, S)$ we have

$$i_{\Omega T}\varphi = \varphi = \varphi i_{\Omega S}.$$

I–III show that the properties of a category are satisfied, and our Theorem 1.6 is proved.

We mention two important subcategories of the category of all S -algebras over Ω . For any double coset S -ring $G//H$ we call the S -algebra $\Omega G//H := \Omega(G//H)$ the *double coset S -algebra* of G modulo H over Ω .

THEOREM 1.7. – *The class of all double coset S -algebras over Ω together with their homomorphisms (in the sense of Definition 1.5) form a proper subcategory of the category of all S -algebras over Ω .*

The proof of Theorem 1.7 is literally the same as that of Theorem 1.6. But we have to issue a warning. From the results of our later sections it is easy to deduce that the homomorphic image of a double coset S -algebra is a double coset S -algebra if a homomorphism is given *into a double coset S -algebra*. But there exist double coset S -algebras isomorphic (in the sense of Definition 1.5) to S -algebras which are not double coset S -algebras themselves. This means that the property of being a double coset S -algebra is not preserved by S -algebra isomorphisms. This fact follows from [17], Theorem 24.1. There the transitivity module of the stabilizer G_1 of a letter in a transitive permutation group G with a regular subgroup is an S -ring which — apart from trivial exceptions — is not a double coset S -ring, and yet its S -algebra over Ω is isomorphic to the double coset S -algebra $\Omega G//G_1$.

THEOREM 1.8. – *The class of all group algebras of finite groups over Ω (qua S -algebras over Ω) together with their homomorphisms (in the sense of Definition 1.5) form a proper subcategory of the category of all double coset S -algebras over Ω .*

The proof is clear. Let φ be a homomorphism of the group algebra ΩG qua S -algebra on G into the group algebra ΩF qua S -algebra on F . Then $\alpha: g \mapsto g^\varphi$ is a homomorphism of the group G into the group F . Conversely, if α is a homomorphism of the group G into the group F , then the linear extension α_Ω of α onto ΩG is a homomorphism in the sense of Definition 1.5 of ΩG as S -algebra on G into ΩF as S -algebra on F . Therefore $G \mapsto \Omega G$ is a one-to-one correspondence between the finite groups and their group algebras qua S -algebras, and $\alpha \mapsto \alpha_\Omega$ is a one-to-one correspondence between the group homomorphisms and the S -algebra homomorphisms of the group algebras of all finite groups over Ω . The pair of mappings $G \mapsto \Omega G$ and $\alpha \mapsto \alpha_\Omega$ is a functor of the category of all finite groups onto the category of all group algebras of finite groups over Ω as S -algebras. That way the theory of finite groups is embedded into the theory of S -algebras.

We started with S -rings, and went over to the richer structure of S -algebras over Ω . Now we go in the opposite direction, and go over to a poorer

structure namely to S -semigroups which have been modelled on the concept of S -rings. The poorer structure has wider applications. It is not bound to finite groups. So let G be an arbitrary group (not necessarily finite). The set

$$\bar{G} := \{X \mid \emptyset \neq X \subseteq G\}$$

is a semigroup with respect to the subset (or "complex") multiplication

$$(X, Y) \rightarrow XY := \{xy \mid x \in X \text{ and } y \in Y\}.$$

DEFINITION 1.9. ([10], p. 74). – A subsemigroup T of $\bar{G}$ is called an S -semigroup on G if it has a unit element and if there exists a set $\mathcal{T} \subseteq \bar{G}$ such that

$$(1) \quad G = \bigcup_{\mathcal{T} \in \mathcal{T}} \mathcal{T}.$$

$$(2) \quad \mathcal{S} = \mathcal{T} \text{ or } \mathcal{S} \cap \mathcal{T} = \emptyset \text{ for all } \mathcal{S}, \mathcal{T} \in \mathcal{T}.$$

$$(3) \quad \mathcal{T}^{-1} \in \mathcal{T} \text{ for all } \mathcal{T} \in \mathcal{T}.$$

$$(4) \quad X = \bigcup_{\substack{\mathcal{T} \in \mathcal{T} \\ \mathcal{T} \cap X \neq \emptyset}} \mathcal{T} \text{ for all } X \in T.$$

$$(5) \quad T \text{ is generated by } \mathcal{T}, \text{ that is every element of } T \text{ is the product of a finite number of elements of } \mathcal{T}.$$

The set $\mathcal{T}$ is uniquely determined by the axioms (1)-(5). Therefore we call the elements of $\mathcal{T}$ the T -classes of G .

EXAMPLES 1.10. –

1) Every group G is an S -semigroup on G with $\mathcal{T} = G$.

2) The semigroup (with respect to complex multiplication) that is generated by taking $\mathcal{T}$ to be the set of all conjugacy classes, is an S -semigroup on G .

3) Let H be a subgroup of the group G . Then the semigroup G/H (with respect to complex multiplication) which is generated by $\mathcal{T} = \{HgH \mid g \in G\}$ is an S -semigroup on G . We call G/H the *double coset S -semigroup* of G modulo H .

Let F be a group, Σ an S -semigroup on F , and $\mathcal{S}$ the set of all Σ -classes of F .

DEFINITION 1.11 ([10], Definition 2.1). – A mapping Φ of T into Σ is called a *homomorphism of the S -semigroup T on G into the S -semigroup Σ on F* if the following hold.

- (1) $(XY)^\Phi = X^\Phi Y^\Phi$ for all $X, Y \in T$.
- (2) $X^\Phi = \bigcup_{\substack{\mathcal{C} \in \mathfrak{T} \\ \mathcal{C} \subseteq X}} \mathcal{C}^\Phi$ for all $X \in T$.
- (3) For every T -class $\mathcal{C}$ of G there exists a Σ -class $\mathcal{S}$ of F such that
$$\mathcal{C}^\Phi = \mathcal{S} \quad \text{and} \quad (\mathcal{C}^{-1})^\Phi = \mathcal{S}^{-1}.$$

To speak of the mathematical structure of an S -semigroup means the following.

THEOREM 1.12. - *The class of all S -semigroups together with their homomorphisms (in the sense of Definition 1.11) form a category.*

PROOF. - For any two S -semigroups T on G and Σ on F the mapping

$$X \mapsto \mathcal{S}_1$$

of T into Σ which maps every element $X \in T$ onto the Σ -class $\mathcal{S}_1$ containing the unit element of F is a homomorphism of T into Σ in the sense of Definition 1.11 (note that $\mathcal{S}_1$ is a subgroup of F and also the unit element of Σ by [10], Lemma 1.2). We call it the *zero homomorphism* of T into Σ . Otherwise the proof of Theorem 1.12 is completely analogous to the proof of Theorem 1.6.

THEOREM 1.13. - *The class of all double coset S -semigroups together with their homomorphisms (in the sense of Definition 1.11) form a proper subcategory of the category of all S -semigroups.*

The remarks after Theorem 1.7 also apply analogously to the double coset S -semigroups and S -semigroups as to the double coset S -algebras and S -algebras.

THEOREM 1.14. - *The category of all groups is a proper subcategory of the category of all double coset S -semigroups.*

The group homomorphisms of a group G and the S -semigroup homomorphisms of G qua S -semigroup are one and the same. Therefore the whole of group theory is part of S -semigroup theory.

For the remainder of this section we study the relationship between S -rings and S -algebras on one side and S -semigroups on finite groups on the other.

THEOREM 1.15. - *Let G be a finite group, T an S -ring on G , and $\mathfrak{T}$ the set of all T -classes of G . Then the subsemigroup T of $\bar{G}$ which is generated by $\mathfrak{T}$ (i.e. every element of T is the product of a finite number of elements of $\mathfrak{T}$) is an S -semi-*

group on G with $\mathcal{T}$ as the set of all T -classes of G .

We call T the S -semigroup related to the S -ring T .

PROOF. 1.9(1)-(3) and 1.1(1)-(3) are identical, and 1.9(5) is satisfied by the definition of T . Under these conditions 1.9(4) is equivalent to the statement that for any $\mathcal{C}, \mathcal{C}' \in \mathcal{T}$ their complex product $\mathcal{C}\mathcal{C}'$ is the set theoretical union of elements of $\mathcal{T}$. By 1.1(4) we have

$$\mathcal{C}\mathcal{C}' = \sum_{\mathcal{C}'' \in \mathcal{T}} a_{\mathcal{C}, \mathcal{C}', \mathcal{C}''} \mathcal{C}'' \quad \text{with} \quad 0 \leq a_{\mathcal{C}, \mathcal{C}', \mathcal{C}''} \in \mathbb{Z}.$$

It is clear that $g \in \mathcal{C}\mathcal{C}'$ if and only if $a_{\mathcal{C}, \mathcal{C}', \mathcal{C}''} > 0$ for the unique $\mathcal{C}'' \in \mathcal{T}$ such that $g \in \mathcal{C}''$ holds. Hence

$$\mathcal{C}\mathcal{C}' = \bigcup_{\substack{\mathcal{C}'' \in \mathcal{T} \\ a_{\mathcal{C}, \mathcal{C}', \mathcal{C}''} > 0}} \mathcal{C}'',$$

and our theorem is proved. Moreover, our proof shows that the mapping

$$\tau = \sum_{\mathcal{C} \in \mathcal{T}} a_{\tau, \mathcal{C}} \mathcal{C} \mapsto \bigcup_{\substack{\mathcal{C} \in \mathcal{T} \\ a_{\tau, \mathcal{C}} > 0}} \mathcal{C}$$

is a semigroup homomorphism of that subsemigroup of the multiplicative semigroup of the ring T which is generated by the T -class sums $\mathcal{C}$, $\mathcal{C} \in \mathcal{T}$, onto T .

THEOREM 1.16. - Let φ be a homomorphism of the S -algebra ΩT on the finite group G into the S -algebra ΩS on the finite group F . Then there exists a unique homomorphism Φ of the S -semigroup T on G related to T into the S -semigroup Σ on F related to S such that

$$\left(\frac{1}{|\mathcal{C}|} \cdot \mathcal{C} \right)^{\varphi} = \left(\frac{1}{|\mathcal{C}^{\Phi}|} \cdot \mathcal{C}^{\Phi} \right) \quad \text{for all} \quad \mathcal{C} \in \mathcal{T}.$$

We call Φ the S -semigroup homomorphism related to the S -algebra homomorphism φ .

PROOF. - For every T -class $\mathcal{C}$ of G there exists (by Definition 1.5(3)) an S -class $\mathcal{S}$ of F such that

$$\left(\frac{1}{|\mathcal{C}|} \cdot \mathcal{C} \right)^{\varphi} = \frac{1}{|\mathcal{S}|} \cdot \mathcal{S}$$

and we set

$$\mathcal{C}^\Phi := \mathcal{S}.$$

We define

$$X^\Phi := \bigcup_{\substack{\mathcal{C} \in \mathcal{C} \\ \mathcal{C} \subseteq X}} \mathcal{C}^\Phi \quad \text{for all } X \in \mathbf{T}.$$

Thus $\Phi: X \rightarrow X^\Phi$ is a mapping which satisfies (2) and (3) of Definition 1.11.

Take any $X, Y \in \mathbf{T}$, and set

$$\xi = \sum_{\substack{\mathcal{C} \in \mathcal{C} \\ \mathcal{C} \subseteq X}} \frac{1}{|\mathcal{C}|} \cdot \mathcal{C}, \quad \eta = \sum_{\substack{\mathcal{C}' \in \mathcal{C} \\ \mathcal{C}' \subseteq Y}} \frac{1}{|\mathcal{C}'|} \cdot \mathcal{C}'.$$

With the equations

$$\left(\frac{1}{|\mathcal{C}|} \cdot \mathcal{C} \right) \left(\frac{1}{|\mathcal{C}'|} \cdot \mathcal{C}' \right) = \sum_{\substack{\mathcal{C}'' \in \mathcal{C} \\ \mathcal{C}'' \subseteq \mathcal{C}\mathcal{C}'}} a_{\mathcal{C}, \mathcal{C}', \mathcal{C}''} \frac{1}{|\mathcal{C}''|} \cdot \mathcal{C}'',$$

which hold by Definition 1.4, we obtain

$$\xi\eta = \sum_{\substack{\mathcal{C}'' \in \mathcal{C} \\ \mathcal{C}'' \subseteq X \\ \mathcal{C}'' \subseteq Y}} \left(\sum_{\mathcal{C}, \mathcal{C}' \in \mathcal{C}} a_{\mathcal{C}, \mathcal{C}', \mathcal{C}''} \right) \frac{1}{|\mathcal{C}''|} \cdot \mathcal{C}'' = \sum_{\substack{\mathcal{C}'' \in \mathcal{C} \\ \mathcal{C}'' \subseteq XY}} z_{\mathcal{C}''} \frac{1}{|\mathcal{C}''|} \cdot \mathcal{C}''$$

with positive rationals $z_{\mathcal{C}''}$. Since $\xi, \eta \in \Omega T$, the S -algebra homomorphism φ yields

$$\begin{aligned} & \left(\sum_{\substack{\mathcal{C} \in \mathcal{C} \\ \mathcal{C} \subseteq X}} \frac{1}{|\mathcal{C}|} \cdot \mathcal{C}^\Phi \right) \left(\sum_{\substack{\mathcal{C}' \in \mathcal{C} \\ \mathcal{C}' \subseteq Y}} \frac{1}{|\mathcal{C}'|} \cdot \mathcal{C}'^\Phi \right) = \xi^\varphi \eta^\varphi = (\xi\eta)^\varphi = \\ & = \sum_{\substack{\mathcal{C}'' \in \mathcal{C} \\ \mathcal{C}'' \subseteq XY}} z_{\mathcal{C}''} \frac{1}{|\mathcal{C}''|} \cdot \mathcal{C}''^\Phi \end{aligned}$$

It follows that $X^\Phi Y^\Phi = (XY)^\Phi$, and hence 1.11(1) is satisfied as well. Therefore Φ is a homomorphism of the S -semigroup $\mathbf{T}$ into the S -semigroup Σ . It certainly has the stated uniqueness property since every homomorphism of the S -semigroup $\mathbf{T}$ is uniquely determined by the images of the $\mathbf{T}$ -classes.

THEOREM 1.17. – *The pair of mappings*

$$\Omega T \rightarrow \mathbf{T} \quad (\text{defined by Theorem 1.15})$$

$$\varphi \rightarrow \Phi \quad (\text{defined by Theorem 1.16})$$

is a covariant functor on the category of all S -algebras on finite groups over Ω into the category of all S -semigroups on finite groups.

PROOF. - For $\varphi = i_{\Omega T} =$ the identity mapping of ΩT , we obtain $\Phi = i_T =$ the identity mapping of T . Let a triple of S -algebras

$$\Omega T \text{ on } G, \quad \Omega S \text{ on } F, \quad \Omega R \text{ on } E$$

be given with the related S -semigroups

$$T \text{ on } G, \quad \Sigma \text{ on } F, \quad P \text{ on } E,$$

and let

$$\varphi: \Omega T \rightarrow \Omega S, \quad \psi: \Omega S \rightarrow \Omega R$$

be a pair of S -algebra homomorphisms with the related S -semigroup homomorphisms

$$\Phi: T \rightarrow \Sigma, \quad \Psi: \Sigma \rightarrow P.$$

Then

$$\frac{1}{|(\mathfrak{C}^\Phi)^\Psi|} \cdot (\mathfrak{C}^\Phi)^\Psi = \left(\frac{1}{|\mathfrak{C}^\Phi|} \cdot \mathfrak{C}^\Phi \right)^\psi = \left(\frac{1}{|\mathfrak{C}|} \cdot \mathfrak{C} \right)^{\varphi\psi}$$

holds for every $\mathfrak{C} \in \mathfrak{T}$. Therefore the S -semigroup homomorphism

$$\Phi\Psi: T \rightarrow P$$

is related to the S -algebra homomorphism

$$\varphi\psi: \Omega T \rightarrow \Omega R,$$

and our theorem is proved..

If we apply Theorem 1.15 to the double coset S -algebras, then the S -semigroup related to a double coset S -algebra $\Omega G//H$ obviously is the double coset S -semigroup G/H .

THEOREM 1.18. - *The pair of mappings*

$$\Omega G//H \rightarrow G/H \quad (\text{defined by Theorem 1.15})$$

$$\varphi \rightarrow \Phi \quad (\text{defined by Theorem 1.16})$$

is a covariant functor on the category of all double coset S -algebras on finite groups over Ω onto the category of all double coset S -semigroups on finite groups.

If we apply the functor of Theorem 1.17 to the group algebras of finite groups over Ω qua S -algebras, then we obtain the inverse functor $\Omega G \mapsto G$, $\alpha_\Omega \mapsto \alpha$ of the functor mentioned after Theorem 1.8.

An important question remains unanswered by this paper. Is the functor of Theorem 1.17 a functor *onto* the category of all S -semigroups on finite groups? We re-word that question as

PROBLEM 1.19. – *Let T be an S -semigroup on a finite group G with $\mathcal{T}$ as the set of all T -classes of G . Is the ring T generated by all T -class sums $\mathcal{C}$, $\mathcal{C} \in \mathcal{T}$, an S -ring on G , i.e. is the product of any two T -class sums always a linear combination of T -class sums with (necessarily) non-negative integral coefficients? In other words, given a decomposition $\mathcal{T}$ of a finite group G such that $\mathcal{T}$ is invariant by taking inverses (i.e. 1.1(1)–(3) hold), does*

$$\mathcal{C}\mathcal{C}' = \bigcup_{\substack{\mathcal{C}'' \in \mathcal{T} \\ \mathcal{C}'' \cap \mathcal{C}\mathcal{C}' \neq \emptyset}} \mathcal{C}'' \quad \text{for all } \mathcal{C}, \mathcal{C}' \in \mathcal{T}$$

always imply

$$\mathcal{C}\mathcal{C}' = \sum_{\mathcal{C}'' \in \mathcal{T}} a_{\mathcal{C}, \mathcal{C}', \mathcal{C}''} \mathcal{C}'' \quad \text{with } 0 \leq a_{\mathcal{C}, \mathcal{C}', \mathcal{C}''} \in \mathbf{Z} \quad \text{for all } \mathcal{C}, \mathcal{C}' \in \mathcal{T}?$$

2. – T -Characters.

From now on we keep the following notations throughout this paper.

G is a finite group.

T is an S -ring on G .

T is the S -semigroup on G related to T (Theorem 1.15).

$\mathcal{T} = \{\mathcal{C}_1, \dots, \mathcal{C}_t\}$ is the set of T -classes (= T -classes) of G .

$t_i = |\mathcal{C}_i|$ the number of elements in the i -th T -class of G .

$\tau_i = \mathcal{C}_i$ ($i = 1, \dots, t$) the T -class sums of G .

Under the assumption of 1.1(1)–(3), (5) condition (4) of the definition of an S -ring is equivalent to the statement that all products

$$\tau_i \tau_j = \sum_{k=1}^t a_{ijk} \tau_k \quad (i, j = 1, \dots, t)$$

of T -class sums are linear combinations of the T -class sums with (necessarily) non-negative integers a_{ijk} as coefficients. For every T -class $\mathcal{C}_i$ there exists a T -class $\mathcal{C}_j$ such that $\mathcal{C}_i^{-1} = \mathcal{C}_j$ (1.1(3)). We write i^* for that j , i.e.

$$\mathcal{C}_{i^*} := \mathcal{C}_i^{-1} \quad (i = 1, \dots, t).$$

For every

$$\xi = \underline{X}, \quad \emptyset \neq X \subseteq G,$$

the *average* of X is defined in the group algebra $\mathbf{Q}G$ as

$$\widehat{\xi} := \frac{1}{|X|} \cdot \underline{X}.$$

If H is a subgroup of G , then the subgroup average

$$\varepsilon_H := \widehat{H}$$

is an idempotent of the group algebra $\mathbf{Q}G$.

In $\mathbf{Q}T$ we have the equations

$$\left. \begin{aligned} \widehat{\tau_i \tau_j} &= \sum_{k=1}^t \widehat{a_{ijk}} \widehat{\tau_k} \\ \widehat{a_{ijk}} &:= \frac{a_{ijk} t_k}{t_i t_j} \end{aligned} \right\} \quad (i, j, k = 1, \dots, t)$$

for the T -class averages $\widehat{\tau_i} = \frac{1}{t_i} \cdot \widehat{\tau_i}$ ($i = 1, \dots, t$).

We choose the notation such that $\widehat{\tau_1}$ is the T -class containing the unit element of G . Then $\widehat{\tau_1}$ is the unit element of $\mathbf{Q}T$, and also of $\mathbf{C}T$.

For any element

$$\gamma = \sum_{g \in G} c_g g \quad (c_g \in \mathbf{C})$$

of the group algebra $\mathbf{C}G$ we form the element

$$\gamma^* = \sum_{g \in G} \overline{c_g} g^{-1}$$

with $\overline{c_g}$ as the conjugate of c_g in $\mathbf{C}$. The mapping

$$\gamma \mapsto \gamma^*$$

is an involutory anti-automorphism of $\mathbf{C}G$ onto itself, i.e.

$$\gamma^{**} = \gamma, \quad (\gamma + \delta)^* = \gamma^* + \delta^*, \quad (\gamma\delta)^* = \delta^*\gamma^* \quad \text{for all } \gamma, \delta \in \mathbf{C}G.$$

Every S -ring T on G , and also every S -algebra $\mathbf{Q}T$ or $\mathbf{C}T$ is invariant

under the mapping $\gamma \rightarrow \gamma^*$. In particular, we have the equations

$$\tau_i^* = \tau_{i^*} = \overline{\mathfrak{C}_i^{-1}},$$

$$\widehat{\tau_i^*} = \widehat{\tau_{i^*}} = \frac{1}{|\mathfrak{C}_i^{-1}|} \cdot \overline{\mathfrak{C}_i^{-1}}.$$

In this section we merely give a fundamental definition which we shall use later. Its usefulness is based on the following statement.

THEOREM 2.1 (WIELANDT [15], p. 386, foot-note). – *Let Ω be a subfield of $\mathbf{C}$ which is invariant under the mapping $c \rightarrow \bar{c}$ (= the conjugate of c). Then ΩT is a semi-simple Ω -algebra.*

PROOF. – ΩT is invariant under the mapping $\gamma \rightarrow \gamma^*$ of $\mathbf{C}G$ onto itself. Therefore for every $\tau \in \Omega T$ the equation $\tau(\Omega T)\tau = 0$ implies $\tau\tau^*(\tau\tau^*)^* = 0$, hence $\tau\tau^* = 0$, and $\tau = 0$. It follows that ΩT is semi-simple.

In particular, $\mathbf{C}T$ is a semi-simple $\mathbf{C}$ -algebra, and as such its representation theory is well known.

DEFINITION 2.2. – *Let D be a representation of $\mathbf{C}T$ qua $\mathbf{C}$ -algebra by matrices over the field $\mathbf{C}$. The complex valued function*

$$\chi: g \rightarrow g\chi := \text{trace } D(\widehat{\tau_i}) \quad \text{for } g \in \mathfrak{C}_i,$$

on G is called the T -character of G related to D .

The T -character χ is called irreducible if D is irreducible.

For what is already known about T -character theory we refer to [3], [5], [6] and [7]. In this paper we concentrate on the elementary parts of S -ring theory and on the information about the structure of finite groups which can be obtained this way. In the next section we introduce the first concept that links S -ring theory with group theory.

3. – T -Subgroups.

A subgroup H of G is called a T -subgroup of G ([10], Definition 1.3) if

$$H = \bigcup_{H \cap \mathfrak{C}_i \neq \emptyset} \mathfrak{C}_i,$$

i.e. if H is a union of T -classes. The meaning of the T -subgroups for the S -ring T on G is shown by the following.

THEOREM 3.1. – *Let H be a subgroup of G . The following statements are equivalent.*

- (1) H is a T -subgroup of G .
- (2) $\underline{H} \in T$.
- (3) $\mathcal{C}_1 \leq H$ and $H\mathcal{C}_iH$, $\mathcal{C}_i \in \mathcal{T}$, are the S -classes of an S -ring S on G contained in T .
- (4) $T \cap G//H$ is an S -ring on G , and $T \cap H//1$ is an S -ring on H .

Note that, in an obvious notation, under the above conditions

$$T \cap G//H = \sum_{i=1}^t \underline{Z \cdot H\mathcal{C}_iH}$$

$$T \cap H//1 = \sum_{\mathcal{C}_i \subseteq H} \underline{Z \cdot \tau_i}.$$

PROOF. -

- I. The equivalence of (1) and (2) is trivial.
- II. Assume that (2) holds. Then $\mathcal{C}_1 \leq H$. Suppose that

$$H\mathcal{C}_iH \cap H\mathcal{C}_jH \neq \emptyset$$

Take any $g \in H\mathcal{C}_iH \cap H\mathcal{C}_jH$. Then there exist $u, v, w, x \in H$, $g_i \in \mathcal{C}_i$, $g_j \in \mathcal{C}_j$ such that

$$g = ug_iv = wg_jx.$$

Therefore

$$g_i = u^{-1}wg_jxv^{-1} \in H\mathcal{C}_jH.$$

But $H\mathcal{C}_jH$ is a union of T -classes since $\underline{H\tau_jH} \in T$. Therefore $\mathcal{C}_i \subseteq H\mathcal{C}_jH$, and hence $H\mathcal{C}_iH \subseteq H\mathcal{C}_jH$. Similarly $H\mathcal{C}_jH \subseteq H\mathcal{C}_iH$, and we obtain

$$H\mathcal{C}_iH = H\mathcal{C}_jH.$$

The set $\mathcal{S} = \{H\mathcal{C}_iH \mid i = 1, \dots, t\}$ yields a decomposition of G into non-empty, subsets which satisfies 1.1(1) and (2), and also (3) because of

$$(H\mathcal{C}_iH)^{-1} = H^{-1}\mathcal{C}_i^{-1}H^{-1} = H\mathcal{C}_{i^*}H \in \mathcal{S} \quad \text{for all } i = 1, \dots, t.$$

For the element

$$\underline{H\tau_iH} = \sum_{g \in G} c_g g \in T$$

we have $c_g \neq 0$ if and only if $g \in H\mathcal{C}_iH$. By [17], Proposition 22.2, we obtain

$\underline{H\mathfrak{C}_iH} \in T$ for all $i = 1, \dots, t$ and also

$$(\underline{H\mathfrak{C}_iH})(\underline{H\mathfrak{C}_jH}) = \sum_{k=1}^t b_{ijk} \tau_k \in T \quad \text{for all } i, j = 1, \dots, t.$$

These products are elements of T with the property that they are invariant by left and right multiplication with every element $h \in H$. Therefore

$$b_{ijk} = b_{ijl} \quad \text{if } H\mathfrak{C}_kH = H\mathfrak{C}_lH,$$

and each of these products is a linear combination of the elements $\underline{H\mathfrak{C}_kH}$ ($k = 1, \dots, t$). It follows that

$$S = \sum_{i=1}^t \underline{Z \cdot H\mathfrak{C}_iH}$$

is an S -ring on G contained in T with the $H\mathfrak{C}_iH$ ($i = 1, \dots, t$) as the S -classes of G . We have proved (3).

III. Assume that (3) holds. Then $S \subseteq T \cap G//H$. Take any $\tau \in T \cap G//H$.

$$\tau = \sum_{i=1}^t z_i \tau_i \quad \text{because } \tau \in T.$$

$\tau = h\tau = \tau h$ for all $h \in H$ because $\tau \in G//H$. Therefore

$$z_i = z_j \quad \text{if } H\mathfrak{C}_iH = H\mathfrak{C}_jH,$$

and it follows that $\tau \in S$. Hence $T \cap G//H = S$ is an S -ring on G . Moreover $H = H\mathfrak{C}_1H$ is an S -class of G , and therefore

$$H = \bigcup_{\mathfrak{C}_i \cap H \neq \emptyset} \mathfrak{C}_i.$$

Obviously those T -classes which are contained in H are the R -classes of an S -ring R on H , and $R = T \cap H//1$. Thus we have proved (4).

IV. Assume that (4) holds. Every S -ring on H contains $\underline{H}$. Therefore $\underline{H} \in T \cap H//1 \subseteq T$, and (1) and (2) hold.

The following remark may sometimes be useful.

LEMMA 3.2. — *Let H be a T -subgroup of G . Then*

$$\widehat{\varepsilon_H \tau_i \varepsilon_H} = \widehat{H\mathfrak{C}_iH} \quad \text{for all } i = 1, \dots, t.$$

PROOF. — By Theorem 3.1 $T \cap G//H$ is an S -ring on G , and the sets

$H\mathcal{C}_iH$ ($i = 1, \dots, t$) are the $T \cap G // H$ -classes of G . It is easy to see that $\underline{H\tau_iH} \in T \cap G // H$. Therefore $\underline{H\tau_iH}$ is a linear combination of the $T \cap G // H$ -class sums, and

$$\underline{H\tau_iH} = n_i \cdot \underline{H\mathcal{C}_iH}$$

must hold with a positive integer n_i . To calculate n_i we count the elements which occur on both sides of the above equation with their multiplicities and obtain

$$|H|^2 \cdot t_i = n_i |H\mathcal{C}_iH|,$$

and hence

$$\widehat{\varepsilon_H \tau_i \varepsilon_H} = \widehat{H\mathcal{C}_iH}.$$

DEFINITION 3.3. – *A subgroup H of G is called a T -subgroup of G if 3.1(1)–(4) hold.*

Thus the terms *T -subgroup* and *T -subgroup* (with T the S -semigroup on G related to T) are synonymous. Therefore we can use all results on T -subgroups proved in [10].

THEOREM 3.4. – *Let H and K be T -subgroups of G . Then the intersection $H \cap K$ and the subgroup $\langle H, K \rangle$ generated by H and K are T -subgroups, i.e. the T -subgroups of G form a sublattice of the lattice of all subgroups of G .*

PROOF. – [10], Theorem 1.5.

4. – T -Normal Subgroups.

As in Section 3 we start with the relevant notion of S -semigroup theory. A subgroup K of G is called *T -normal* ([10], Definition 1.9) if K is a T -subgroup such that

$$K\mathcal{C}_i = \mathcal{C}_iK \quad \text{holds for all } i = 1, \dots, t.$$

This definition of T -normality is not sufficient for T -normality. At least we do not know yet whether it is sufficient or not. The proper concept of T -normality is based on the following statement.

THEOREM 4.1. – *Let S and T be S -rings on G . Let $\mathcal{S}_1$ be the S -class of G containing the unit element of G (which implies that $\mathcal{S}_1$ is a subgroup of G). Then the following statements are equivalent.*

- (1) $\underline{\mathcal{S}_1}$ is in the centre of T and $S = T \cap G // \mathcal{S}_1$.

- (2) S is a two-sided ideal of T .
 (3) $S \subseteq T$ and every S -character is a T -character of G .

If these conditions are satisfied then

$$\Omega S = (\Omega T)\underline{\mathfrak{S}}_1 = \underline{\mathfrak{S}}_1(\Omega T) = \underline{\mathfrak{S}}_1(\Omega T)\underline{\mathfrak{S}}_1$$

holds.

PROOF. I. – Assume that (1) holds. $\varepsilon = \frac{1}{|\underline{\mathfrak{S}}_1|} \cdot \underline{\mathfrak{S}}_1$ is a central idempotent of $\mathbf{Q}T$. Therefore $(\mathbf{Q}T)\varepsilon$ is a two-sided ideal of $\mathbf{Q}T$. But ε is also the unit element of $\mathbf{Q}S$, and hence

$$\mathbf{Q}(T \cap G // \mathfrak{S}_1) = \mathbf{Q}S = (\mathbf{Q}S)\varepsilon \subseteq (\mathbf{Q}T)\varepsilon = \varepsilon(\mathbf{Q}T)\varepsilon \subseteq \mathbf{Q}(T \cap G // \mathfrak{S}_1).$$

It follows that $\mathbf{Q}S = \mathbf{Q}(T \cap G // \mathfrak{S}_1)$ is a two-sided ideal of $\mathbf{Q}T$, and

$$TST \subseteq (\mathbf{Q}T)(\mathbf{Q}S)(\mathbf{Q}T) = \mathbf{Q}S.$$

Since every element of TST is a linear combination of the T -class sums with rational integers as coefficients, and lies in $\mathbf{Q}S$, it must be contained in S . Therefore S is a two-sided ideal of T .

II. Assume that (2) holds. Let χ be an irreducible S -character of G (Definition 2.2). Then

$$\eta = \frac{1}{|G|} \sum_{g \in G} z\chi(g^{-1})g$$

is a primitive central idempotent of $\mathbf{C}S$ with z a positive rational integer ([6], p. 218 or [7], p. 351). η also is a primitive central idempotent of $\mathbf{C}T$ since $\mathbf{C}S$ is a two-sided ideal of the semi-simple $\mathbf{C}$ -algebra $\mathbf{C}T$. Therefore χ , as the coefficient function of a primitive central idempotent of $\mathbf{C}T$, is an irreducible T -character of G (again by [6], p. 218 or [7], p. 351). Every S -character of G is a linear combination of irreducible S -characters because $\mathbf{C}S$ is a semi-simple $\mathbf{C}$ -algebra. Therefore every S -character is also a T -character.

III. Assume that (3) holds. Let χ be an irreducible S -character of G . Then, as in II,

$$\eta = \frac{1}{|G|} \sum_{g \in G} z\chi(g^{-1})g$$

is a central idempotent of $\mathbf{C}S$. Since χ is a T -character, and since $\mathbf{C}T$ is

a semi-simple $\mathbf{C}$ -algebra,

$$\chi = \sum_{\rho} a_{\rho} \chi_{\rho}$$

is a linear combination of certain irreducible T -characters χ_{ρ} with positive integers a_{ρ} as coefficients. Each element

$$\eta_{\rho} = \frac{1}{|G|} \sum_{g \in G} z_{\rho} \chi_{\rho}(g^{-1}) g$$

is a primitive central idempotent of $\mathbf{C}T$, and

$$\eta = \sum_{\rho} \frac{z a_{\rho}}{z_{\rho}} \cdot \eta_{\rho}$$

is a linear combination of them. Since the idempotents η_{ρ} are pairwise orthogonal, we obtain

$$\eta = \eta^2 = \sum_{\rho} \left(\frac{z a_{\rho}}{z_{\rho}} \right)^2 \eta_{\rho}$$

which implies

$$\frac{z a_{\rho}}{z_{\rho}} = 1,$$

and hence

$$\eta = \sum_{\rho} \eta_{\rho}.$$

Furthermore

$$(\mathbf{C}S)\eta \subseteq (\mathbf{C}T)\eta \subseteq \sum_{\rho} (\mathbf{C}T)\eta_{\rho}.$$

Let F be a representation of $\mathbf{C}S$ such that χ is the S -character of G afforded by F , and let F_{ρ} be an irreducible representation of $\mathbf{C}T$ such that χ_{ρ} is the T -character of G afforded by F_{ρ} . Then

$$\chi(1) = \text{trace } F(\varepsilon_{\mathbb{S}_1}) = \text{degree } F, \text{ since } \varepsilon_{\mathbb{S}_1} \text{ is the unit element of } \mathbf{C}S.$$

$$\chi_{\rho}(1) = \text{trace } F_{\rho}(\varepsilon_{\mathbb{T}_1}) = \text{degree } F_{\rho}, \text{ since } \varepsilon_{\mathbb{T}_1} \text{ is the unit element of } \mathbf{C}T.$$

$$\left(\sum_{\rho} a_{\rho} \chi_{\rho}(1) \right)^2 = \chi(1)^2 = \dim_{\mathbf{C}} (\mathbf{C}S)\eta \leq \sum_{\rho} \dim_{\mathbf{C}} (\mathbf{C}T)\eta_{\rho} = \sum_{\rho} \chi_{\rho}(1)^2.$$

This inequality can hold only for $\chi = \chi_{\rho}$, and hence χ is an irreducible T -character of G , and $\eta = \eta_{\rho}$ is a primitive central idempotent of $\mathbf{C}T$. Conse-

quently $(CT)\eta$ is a minimal two-sided ideal of CT , as $(CS)\eta$ is a minimal two-sided ideal of CS . But

$$\dim_{\mathcal{C}}(CS)\eta = \chi(1)^2 = \chi_{\rho}(1)^2 = \dim_{\mathcal{C}}(CT)\eta,$$

and $S \subseteq T$ by assumption. Therefore

$$(CS)\eta = (CT)\eta$$

is a minimal two-sided ideal of CT . As a semi-simple $\mathcal{C}$ -algebra CS is the direct sum of its minimal two-sided ideals, and therefore CS is a two-sided ideal of CT , and its unit element $\varepsilon_{\mathcal{S}_1}$ is a central idempotent of CT . It follows that $\mathcal{S}_1$ is in the centre of T . Furthermore

$$CS = (CT)\varepsilon_{\mathcal{S}_1} = \mathcal{C}(T \cap G // \mathcal{S}_1),$$

and therefore

$$S = T \cap G // \mathcal{S}_1.$$

Hence (1) holds, and our theorem is proved.

DEFINITION 4.2. - Let S and T be S -rings on G . Then S is called T -normal, if 4.1(1)-(3) hold.

Not every S -ring S on G which is contained in an S -ring T on G is T -normal. S need not even be equal to an S -ring $T \cap G // H$ where H is a T -subgroup of G (Theorem 3.1(4)). This is shown by the following example.

Let G be any primitive but not doubly transitive finite permutation group. Let $\mathcal{C}_1$ be the stabilizer of one letter, and take $T = G // \mathcal{C}_1$. Then $t > 2$. Let S be the S -ring on G with the S -classes $\mathcal{C}_1$ and $\bigcup_{i=2}^t \mathcal{C}_i$. Then there exists no T -subgroup H of G such that $S = T \cap G // H$, since there is no subgroup properly between $\mathcal{C}_1$ and G .

Therefore those S -rings S on G which have the property that $S = T \cap G // H$ for a subgroup H of G are a proper subclass of the class of all S -rings on G contained in T . They are in one-to-one correspondence with the T -subgroups of G . The class of all T -normal S -rings is a proper subclass of them. We look at those T -subgroups now which are related to the T -normal S -rings.

THEOREM 4.3. - Let K be a T -subgroup of G . Then the following statements are equivalent.

- (1) $\underline{K}$ is in the centre of T .

(2) $T \cap G//K$ is a two-sided ideal of T .

(3) Every $T \cap G//K$ -character is a T -character.

The proof follows from Theorem 4.1 for $S = T \cap G//K$.

DEFINITION 4.4. – A subgroup K of G is called T -normal, if 4.3(1)–(3) hold.

Note that every T -normal subgroup is a T -subgroup of G by 4.3(1). Note also that $\mathbf{Z}G$ -normality is normality in the ordinary sense.

PROPOSITION 4.5. – Let K be a subgroup of G . Then the following statements are equivalent.

(1) K is a T -normal subgroup of G .

(2) $T \cap G//K$ is a T -normal S -ring on G .

We state relationships between various notions of normality.

PROPOSITIONS 4.6. – Let S and T be S -rings on G , and assume $S \subseteq T$. Then every T -normal S -subgroup of G is S -normal. In particular every $G//\mathfrak{C}_1$ -normal T -subgroup, and also every normal T -subgroup, is T -normal.

The proof is obvious from Theorem 4.3(1).

THEOREM 4.7. – Let H be a T -subgroup of G , and K a T -normal subgroup of G . Then

(1) $HK = KH$ is a T -subgroup of G .

(2) HK is a $T \cap G//H$ -normal subgroup of G .

4.7(2) is due to ALAN CAMINA.

PROOF. –

I. Since $\underline{H} \in T$, and $\underline{K}$ is in the centre of T , we obtain $\underline{HK} = \underline{KH} \in T$, hence $HK = KH$. Moreover $\underline{HK} = |H \cap K| \cdot \underline{HK}$. Therefore $\underline{HK} \in T$ ([17], Proposition 22.1), and HK is a T -subgroup of G .

II. For every $\tau \in T \cap G//H$ we have

$$\underline{H}\tau = |H| \cdot \tau = \tau \underline{H} \quad \text{because } \tau \in G//H,$$

$$\underline{K}\tau = \tau \underline{K} \quad \text{because } \tau \in T \text{ and because of the } T\text{-normality of } K.$$

In $\mathbf{Q}T$ the equation

$$\underline{HK} = \frac{1}{|H \cap K|} \cdot \underline{HK}$$

and the following computation hold.

$$\underline{HK}\tau = \left(\frac{1}{|\underline{H} \cap \underline{K}|} \cdot \underline{HK} \right) \tau = \tau \left(\frac{1}{|\underline{H} \cap \underline{K}|} \cdot \underline{HK} \right) = \tau \underline{HK}.$$

Therefore $\underline{HK}$ is in the centre of $T \cap G // H$, and hence $\underline{HK}$ is $T \cap G // H$ -normal.

THEOREM 4.8. - *Let H and K be T -normal subgroups of G . Then $\underline{HK}$ is a T -normal subgroup of G .*

PROOF. - $\underline{HK}$ is a subgroup of G by Theorem 4.7(1), and $\underline{HK} = \frac{1}{|\underline{H} \cap \underline{K}|} \cdot \underline{HK}$ is in the centre of T since $\underline{H}$ and $\underline{K}$ are in the centre of T .

THEOREM 4.9. - *Let H be a T -normal subgroup, and K a subgroup of G containing H . Then K is T -normal if and only if K is $T \cap G // H$ -normal.*

PROOF. -

I. Let K be T -normal. Then K is $T \cap G // H$ -normal by Theorem 4.7(2).

II. Let K be $T \cap G // H$ -normal. Using the T -normality of H and working in $\mathbf{Q}T$ we obtain for all $\tau \in T$

$$\underline{K}\tau = \frac{1}{|\underline{H}|^2} \cdot \underline{KH}^2\tau = \frac{1}{|\underline{H}|^2} \cdot \underline{K(H\tau H)} = \frac{1}{|\underline{H}|^2} \cdot (\underline{H\tau H})\underline{K} = \tau \left(\frac{1}{|\underline{H}|^2} \cdot \underline{H^2K} \right) = \tau \underline{K}.$$

Therefore $\underline{K}$ is in the centre of T , and hence K is T -normal.

We generalize the notion of simplicity of groups to S -rings.

DEFINITION 4.10. - *Let T be an S -ring on G different from the trivial S -ring $G // G = \{z \cdot \underline{G} \mid z \in \mathbf{Z}\}$. Then T is called simple, and G is called T -simple, if there does not exist any T -normal subgroup different from $\mathcal{T}_1$ and G .*

The Homomorphism Theorem 6.2 will show that 4.10 is the correct definition of simplicity in the sense that T is simple if and only if the S -algebra ΩT has no homomorphisms other than monomorphisms and zero homomorphisms.

We point to a special case of simplicity of S -rings which has already appeared in the literature. According to WIELANDT ([17], Definition 23.2) an S -ring $T \neq G // G$ is called *primitive*, if there does not exist any T -subgroup different from $\mathcal{T}_1$ and G . Every primitive S -ring is simple, but there are simple S -rings which are not primitive (take $T = \mathbf{Z}G$ for any non-abelian finite simple group G).

We started this section by quoting the corresponding notion of T -normality in S -semigroup theory, but so far we have not referred to it again. We will do so now. We repeat that T denotes the S -semigroup on G related to T (cf. Theorem 1.15).

LEMMA 4.11.- *Every T -normal subgroup of G is T -normal.*

PROBLEM 4.12. - *Is every T -normal subgroup K of G also T -normal in G , i.e. does*

$$\underline{K} \in T \text{ and } K\mathfrak{C}_i = \mathfrak{C}_i K \text{ for all } i = 1, \dots, t$$

always imply that

$$\underline{K}\tau_i = \tau_i \underline{K} \text{ for all } i = 1, \dots, t?$$

We can solve this problem for a class of S -rings, namely for the double coset S -rings. We prove a more general theorem, and, for lucidity's sake, we repeat some statements for double coset S -rings which are already proved for arbitrary S -rings. Refer also to [8], Theorem 2.3 and Definition 3.1. We give 6 equivalent statements, two purely group theoretical, two ring theoretical, and two character theoretical.

1_H^G denotes the (ordinary) character of G induced by the 1-character 1_H of the subgroup H of G , and similarly with 1_K^G for a subgroup K of G instead of H . As usual $(\cdot, \cdot)_G$ means the inner product of characters.

THEOREM 4.13. - *Let H and K be subgroups of G . Then the following statements are equivalent.*

- (1) $H \leq K$ and $HgK = KgH$ for all $g \in G$.
- (2) $K = H(K \cap K^g)$ for all $g \in G$.
- (3) $\underline{K}$ is in the $\mathfrak{C}$ -centre of the double coset S -ring $G//H$.
- (4) The double coset S -ring $G//K$ is a two-sided ideal of $G//H$.
- (5) Every $G//K$ -character is a $G//H$ -character of G .
- (6) $H \leq K$ and $(1_H^G, 1_K^G)_G = (1_K^G, 1_K^G)_G$.

The equivalence of (1) and (6) was overlooked by the author and was pointed out to him by J. L. ALPERIN during the Algebra Symposium 1966/67 at the University of Warwick, Coventry, England.

PROOF. - The equivalence of (1) and (2) holds for arbitrary groups and was proved in [10], Theorem 1.15.

Under the additional assumption $H \leq K$ the equivalence of (3), (4),

(5) is a special case of Theorem 4.3 for $T = G//H$. Also under the additional assumption $H \leq K$ the equivalence of (3), (4), (5), (6) has been proved in [8], Theorem 2.3. Yet the assumption $H \leq K$ is superfluous. (3) and (4) certainly imply $H \leq K$. That (5) also implies $H \leq K$ can be seen as follows.

Our proof 4.1, III that 4.1(3) implies 4.1(1) used the assumption $S \subseteq T$ at only two points. Without that assumption we have shown that every primitive central idempotent of $\mathbf{CS}$ is also a central idempotent of $\mathbf{CT}$. Since the unit element ϵ_{S_1} of $\mathbf{CS}$ is the sum of all primitive central idempotents of $\mathbf{CS}$ (by the semi-simplicity of $\mathbf{CS}$), ϵ_{S_1} is a central idempotent of $\mathbf{CT}$. If we apply that remark to $S = G//K$ (where ϵ_K is the unit element of $\mathbf{CG//K}$) and to $T = G//H$, it follows that $\underline{K}$ is in the centre of $G//H$, hence $H \leq K$ and $G//K \subseteq G//H$. Therefore the equivalence of (3), (4), (5), (6) holds.

(1) implies

$$(1') \quad HgK = KgK \quad \text{for all } g \in G.$$

Conversely, (1') implies

$$H \subseteq HK = H1K = K1K = K$$

and

$$HgK = KgK = (Kg^{-1}K)^{-1} = (Hg^{-1}K)^{-1} = KgH \quad \text{for all } g \in G.$$

Hence (1) and (1') are equivalent.

It is well known that $(1_H^G, 1_K^G)_G$ is equal to the number of double cosets HgK , $g \in G$, and $(1_K^G, 1_K^G)_G$ is equal to the number of double cosets KgK , $g \in G$. Therefore (1') and (6) are equivalent, and our theorem is proved.

The double coset S -semigroup G/H is the S -semigroup related by Theorem 1.15 to the double coset S -ring $G//H$. A subgroup K of G is a G/H -subgroup if and only if $H \leq K$. The G/H -classes of G are the double cosets HgH , $g \in G$. Hence a subgroup K of G is G/H -normal if and only if

$$H \leq K \quad \text{and} \quad K(HgH) = (HgH)K \quad \text{for all } g \in G.$$

But $H \leq K$ implies $HgK = (HgH)K$ and $KgH = K(HgH)$. Therefore as a consequence of Theorem 4.13 we obtain

THEOREM 4.14. - *Let K be a subgroup of G . Then K is $G//H$ -normal if and only if K is G/H -normal.*

Thus the question 4.12 is answered affirmatively for double coset S -rings. Therefore all statements on G/H -normal subgroups automatically hold for finite groups with $G//H$ -normality as well. We refer the reader to [10], 1.16 - 1.21.

5. - Homomorphisms.

To the notations introduced at the beginning of Section 2 we add the following.

F is a finite group.

S is an S -ring on F .

Σ is the S -semigroup on F related to S (Theorem 1.15).

$\mathcal{S} = \{\mathcal{S}_1, \dots, \mathcal{S}_s\}$ is the set of all S -classes ($= \Sigma$ -classes) of F .

$s_j = |\mathcal{S}_j|$ the number of elements in the j -th S -class of F .

$\sigma_j = \underline{\mathcal{S}}_j$ ($j = 1, \dots, s$) the S -class sums of F .

Ω is a subfield of $\mathbf{C}$ which is invariant under conjugacy $c \rightarrow \bar{c}$.

φ is a homomorphism of the S -algebra ΩT on G into the S -algebra ΩS on F .

Φ is the homomorphism related to φ of the S -semigroup T on G related to T into the S -semigroup Σ on F related to S (Theorem 1.16).

$\bar{T}$ is the set of all non-empty subsets of G which are unions of T -classes ($= T$ -classes).

$\bar{\Sigma}$ is the set of all non-empty subsets of F which are unions of Σ -classes ($= S$ -classes).

Φ can be uniquely extended to a mapping $\bar{\Phi}$ of $\bar{T}$ into $\bar{\Sigma}$ ([10], Proposition 2.2) by

$$X^{\bar{\Phi}} := \bigcup_{\mathcal{C}_i \subseteq X} \mathcal{C}_i^{\Phi} \quad \text{for all } X \in \bar{T}.$$

Let H be a T -subgroup of G . Note that H need not be an element of T , but we have $H \in \bar{T}$. By [10], Proposition 2.4(1), $H^{\bar{\Phi}}$ is a Σ -subgroup, and hence it is an S -subgroup of F .

PROPOSITION 5.1. -

$$\varepsilon_H^{\Phi} = \varepsilon_{H^{\bar{\Phi}}}$$

i.e. φ maps every subgroup average of a T -subgroup of G onto a subgroup average of an S -subgroup of F .

PROOF. - Because of $H \in \bar{T}$ we have

$$\varepsilon_H = \frac{1}{|H|} \sum_{\mathcal{C}_i \subseteq H} \tau_i = \frac{1}{|H|} \sum_{\mathcal{C}_i \subseteq H} t_i \widehat{\tau_i},$$

$$\varepsilon_H^\circ = \frac{1}{|H|} \sum_{\mathcal{C}_i \subseteq H} t_i \widehat{\tau_i^\circ}.$$

From

$$(\varepsilon_H^\circ)^2 = (\varepsilon_H^2)^\circ = \varepsilon_H^\circ \quad \text{and} \quad \widehat{\tau_i^\circ} = \frac{1}{|\mathcal{C}_i^\Phi|} \cdot \mathcal{C}_i^\Phi \quad (1.16)$$

we obtain $H^\Phi H^\Phi \subseteq H^\Phi$, and hence (by the finiteness of F) we have another proof of the fact that H^Φ is an S -subgroup of F . For every T -class average τ_i such that $\mathcal{C}_i \subseteq H$ the equation

$$\varepsilon_H \widehat{\tau_i} = \widehat{\tau_i} \varepsilon_H = \varepsilon_H$$

holds. The analogue holds for the S -class averages of F contained in H^Φ which, by Definition 1.5(3) and 1.16, implies

$$\varepsilon_{H^\Phi} \widehat{\tau_i^\circ} = \widehat{\tau_i^\circ} \varepsilon_{H^\Phi} = \varepsilon_{H^\Phi} \quad \text{for all } \mathcal{C}_i \subseteq H.$$

Therefore

$$\varepsilon_H^\circ \varepsilon_{H^\Phi} = \varepsilon_{H^\Phi} \varepsilon_H^\circ = \varepsilon_{H^\Phi}.$$

On the other hand

$$\varepsilon_{H^\Phi} = \frac{1}{|H^\Phi|} \sum_{\mathcal{S}_j \subseteq H^\Phi} s_j \widehat{\sigma_j}.$$

For every $\mathcal{S}_j \subseteq H^\Phi$ there exists $\mathcal{C}_i \subseteq H$ such that

$$\widehat{\sigma_j} = \widehat{\tau_i^\circ}.$$

Hence we obtain

$$\widehat{\sigma_j} \varepsilon_H^\circ = \widehat{\tau_i^\circ} \varepsilon_H^\circ = (\widehat{\tau_i} \varepsilon_H)^\circ = \varepsilon_H^\circ, \quad \varepsilon_H^\circ = \varepsilon_{H^\Phi} \varepsilon_H^\circ = \varepsilon_{H^\Phi},$$

and Proposition 5.1 is proved.

Moreover the proof yields some numerical relations which we deduce from the equation

$$\frac{1}{|H^\Phi|} \sum_{\mathcal{S}_j \subseteq H^\Phi} s_j \widehat{\sigma_j} = \varepsilon_{H^\Phi} = \varepsilon_H^\circ = \frac{1}{|H|} \sum_{\mathcal{C}_i \subseteq H} t_i \widehat{\tau_i^\circ},$$

and which we state as

$$\text{PROPOSITION 5.2. - } |H| \cdot s_j = |H^\Phi| \cdot \sum_{\substack{\mathcal{C}_i \subseteq H \\ \mathcal{C}_i^\Phi = \mathcal{S}_j}} t_i \quad \text{for all } \mathcal{S}_j \subseteq H^\Phi.$$

COROLLARY 5.3. - *If φ is an automorphism of ΩT , then*

$$\mathfrak{C}_i^\Phi = \mathfrak{C}_j \text{ implies } t_i = t_j.$$

With every T -subgroup H of G there are associated (Theorem 3.1(4))

the S -ring $T \cap G // H$ on G ,

the S -ring $T \cap H // 1$ on H .

Let us study the effect of φ on these S -rings.

PROPOSITION 5.4. - *$S \cap G^\Phi // H^\Phi$ is an S -ring on G^Φ , and the restriction of φ to $\Omega(T \cap G // H)$ yields an epimorphism of the S -algebra $\Omega(T \cap G // H)$ on G onto the S -algebra $\Omega(S \cap G^\Phi // H^\Phi)$ on G^Φ .*

PROOF. - By Theorem 3.1(4) $S \cap G^\Phi // 1$ is an S -ring on G^Φ since G^Φ is an S -subgroup of F . A subgroup of G^Φ is an S -subgroup of F if and only if it is an $S \cap G^\Phi // 1$ -subgroup of G^Φ . Therefore $S \cap G^\Phi // H^\Phi = (S \cap G^\Phi // 1) \cap \cap G^\Phi // H^\Phi$ is an S -ring on G^Φ . The images of the $T \cap G // H$ -class averages (Lemma 3.2) are

$$(\varepsilon_H \widehat{\tau_i} \varepsilon_H)^\varphi = \varepsilon_H^\varphi \widehat{\tau_i^\varphi} \varepsilon_H^\varphi = \varepsilon_{H^\Phi} \widehat{\tau_i^\varphi} \varepsilon_{H^\Phi}$$

which are $S \cap G^\Phi // H^\Phi$ -class averages, again by Lemma 3.2. Since

$$(\varepsilon_H \widehat{\tau_i} \varepsilon_H)^{\varphi*} = (\varepsilon_H \widehat{\tau_i^*} \varepsilon_H)^\varphi = \varepsilon_{H^\Phi} \widehat{(\tau_i^\varphi)^*} \varepsilon_{H^\Phi} = (\varepsilon_{H^\Phi} \widehat{\tau_i^\varphi} \varepsilon_{H^\Phi})^*$$

holds, the restriction of φ to $\Omega(T \cap G // H)$ yields an epimorphism of the S -algebra $\Omega(T \cap G // H)$ on G onto the S -algebra $\Omega(S \cap G^\Phi // H^\Phi)$ on G^Φ .

PROPOSITION 5.5. - *$S \cap H^\Phi // 1$ is an S -ring on H^Φ , and the restriction of φ to $\Omega(T \cap H // 1)$ yields an epimorphism of the S -algebra $\Omega(T \cap H // 1)$ on H onto the S -algebra $\Omega(S \cap H^\Phi // 1)$ on H^Φ .*

The proof is obvious.

COROLLARY 5.6. -

$$(\Omega T)^\varphi = \text{Im } \varphi = \Omega(S \cap G^\Phi // 1)$$

is an S -algebra on G^Φ , i.e. every homomorphic image of an S -algebra is an S -algebra.

PROPOSITION 5.7. – *Let K be a T -normal subgroup of G . Then $K^{\bar{\phi}}$ is an $S \cap G^{\bar{\phi}}//1$ -normal subgroup of $G^{\bar{\phi}}$.*

PROOF. – ε_K is in the centre of ΩT . Therefore $\varepsilon_{K^{\bar{\phi}}} = \varepsilon_K^{\bar{\phi}}$ is in the centre of $\text{Im } \varphi = \Omega(S \cap G^{\bar{\phi}}//1)$, and hence $K^{\bar{\phi}}$ is an $(S \cap G^{\bar{\phi}}//1)$ -normal subgroup of $G^{\bar{\phi}}$.

Let L be an S -subgroup of F contained in $G^{\bar{\phi}}$. Then the inverse image

$$L^{\bar{\phi}^{-1}} := \bigcup_{\mathcal{C}_i^{\bar{\phi}} \subseteq L} \mathcal{C}_i$$

is a T -subgroup of G ([10], Proposition 2.5(1)), and hence it is a T -subgroup of G . Moreover $(L^{\bar{\phi}^{-1}})^{\bar{\phi}} = L$ holds ([10], Proposition 2.5(2)).

We continue our discussions of homomorphisms, but we need a new concept, and therefore we start a new section.

6. – The Homomorphism Theorem.

If α is a homomorphism of the group G , then $\text{Ker } \alpha$ is a normal subgroup of G , the cosets $g\text{Ker } \alpha$, $g \in G$, are the sets of those elements of G which have the same images under α , and $\text{Im } \alpha$ is isomorphic to the factor group $G/\text{Ker } \alpha$. Our next results can be considered as a generalization of these group theoretical facts for finite groups, which we obtain from specializing $T = \mathbf{Z}G$. We keep all notations introduced earlier in this paper.

DEFINITION 6.1. – *Let φ be a homomorphism of the S -algebra ΩT on G into the S -algebra ΩS on F . Then*

$$\text{Ker } \varphi := \bigcup_{\widehat{\tau_i^{\bar{\phi}}} = \widehat{\sigma_1}} \mathcal{C}_i$$

(where $\widehat{\sigma_1}$ is the unit element of ΩS) is called the kernel of φ .

Note that $\text{Ker } \varphi = \mathcal{S}_1^{\bar{\phi}^{-1}}$, where $\mathcal{S}_1$ is the S -class of F containing the unit element of F . Moreover $\mathcal{S}_1$ is an S -subgroup of F contained in $G^{\bar{\phi}}$, and therefore $\text{Ker } \varphi$ is a T -subgroup of G . So the kernel of an S -algebra homomorphism of ΩT is not a substructure of the S -algebra ΩT , but a subgroup of G .

THE HOMOMORPHISM THEOREM 6.2. – *Let φ be a homomorphism of the S -algebra ΩT on G into the S -algebra ΩS on F , and set $K = \text{Ker } \varphi$. Then*

- (1) $\widehat{\tau_i^{\bar{\phi}}} = \widehat{\tau_j^{\bar{\phi}}}$ if and only if $\mathcal{C}_i K = \mathcal{C}_j K$.
- (2) K is a T -normal subgroup of G , and hence the S -ring $T \cap G//K$ on

G (which has the sets $\mathcal{C}_i K = K \mathcal{C}_i K$ ($i = 1, \dots, t$) as the $(T \cap G // K)$ -classes of G) is a two-sided ideal of T .

- (3) The restriction of φ to $\Omega(T \cap G // K)$ yields an isomorphism of the S -algebra $\Omega(T \cap G // K)$ on G onto the S -algebra $\text{Im } \varphi = \Omega(S \cap G^{\bar{\varphi}} // 1)$ on $G^{\bar{\varphi}}$.

PROOF I. — K is a T -subgroup of G , and $\varepsilon_K^{\bar{\varphi}} = \widehat{\sigma}_1$ holds by Proposition 5.1 and Definition 6.1.

Assume that $\tau_i^{\bar{\varphi}} = \tau_j^{\bar{\varphi}}$. Then, using $(\tau_j^{\bar{\varphi}})^* = (\tau_i^{\bar{\varphi}})^{\bar{\varphi}}$ (Definition 1.5(3)), $\widehat{\sigma}_1$ appears in

$$(\tau_i^{\bar{\varphi}})^* \tau_i^{\bar{\varphi}} = (\tau_j^{\bar{\varphi}})^* \tau_i^{\bar{\varphi}} = (\tau_j^{\bar{\varphi}})^{\bar{\varphi}} \tau_i^{\bar{\varphi}} = (\tau_j^* \tau_i)^{\bar{\varphi}} = \sum_{k=1}^t \widehat{a}_{j*ik} \tau_k^{\bar{\varphi}}$$

with a positive coefficient. Therefore there exists $\mathcal{C}_i \subseteq \mathcal{C}_j^{-1} \mathcal{C}_i$ such that $\tau_i^{\bar{\varphi}} = \widehat{\sigma}_1$, that is $\mathcal{C}_i \subseteq K$. Take any $g \in \mathcal{C}_i$. Then there exist $x \in \mathcal{C}_i$ and $y \in \mathcal{C}_j$ such that $g = y^{-1}x$, hence

$$x = yg \in \mathcal{C}_j \mathcal{C}_i \subseteq \mathcal{C}_j K.$$

But $\mathcal{C}_j K$ is a union of T -classes, because $\tau_j K \in T$. Therefore $\mathcal{C}_i \subseteq \mathcal{C}_j K$, and hence $\mathcal{C}_i K \subseteq \mathcal{C}_j K$. Interchanging i and j we also obtain $\mathcal{C}_j K \subseteq \mathcal{C}_i K$, and so

$$\tau_i^{\bar{\varphi}} = \tau_j^{\bar{\varphi}} \text{ implies } \mathcal{C}_i K = \mathcal{C}_j K.$$

Since

$$\begin{aligned} (\tau_i \varepsilon_K)^{\bar{\varphi}} &= \tau_i^{\bar{\varphi}} \varepsilon_K^{\bar{\varphi}} = \tau_i^{\bar{\varphi}} \widehat{\sigma}_1 = \tau_i^{\bar{\varphi}} \\ \mathcal{C}_k &\subseteq \mathcal{C}_i K \text{ implies } \tau_k^{\bar{\varphi}} = \tau_i^{\bar{\varphi}}, \end{aligned}$$

and therefore

$$\tau_i^{\bar{\varphi}} = \tau_j^{\bar{\varphi}} \text{ if and only if } \mathcal{C}_i K = \mathcal{C}_j K.$$

Thus we have proved (1). Another proof of (1) can be obtained from Theorem 1.16 and [10], Theorem 2.7. Furthermore, from the foregoing it is clear that

$$\tau_i^{\bar{\varphi}} = \tau_j^{\bar{\varphi}} \text{ if and only if } \varepsilon_K \tau_i \varepsilon_K = \varepsilon_K \tau_j \varepsilon_K.$$

Thus, using Lemma 3.2, (3) holds as well.

II. Let J be the set of all $\tau \in T$ such that $\tau^{\bar{\varphi}} = 0$, that is the kernel of φ qua algebra homomorphism. Each T -class average

$$\widehat{\tau}_i = \varepsilon_K \widehat{\tau}_i \varepsilon_K + (\widehat{\tau}_i - \varepsilon_K \widehat{\tau}_i \varepsilon_K)$$

can be decomposed into the $(T \cap G // K)$ -class average $\varepsilon_K \widehat{\tau_i} \varepsilon_K$ (Lemma 3.2) and the element $\widehat{\tau_i} - \varepsilon_K \widehat{\tau_i} \varepsilon_K$ of J . Therefore

$$T = (T \cap G // K) + J,$$

and from (3) we obtain

$$(T \cap G // K) \cap J = 0.$$

If the element

$$\tau = \sum_{i=1}^t x_i \widehat{\tau_i} = \sum_{i=1}^t x_i \varepsilon_K \widehat{\tau_i} \varepsilon_K + \sum_{i=1}^t x_i (\widehat{\tau_i} - \varepsilon_K \widehat{\tau_i} \varepsilon_K), \quad x_i \in \Omega,$$

of T is contained in J , then

$$\sum_{i=1}^t x_i \varepsilon_K \widehat{\tau_i} \varepsilon_K = 0$$

since φ induces an isomorphism of $T \cap G // K$ onto $\text{Im } \varphi$. Therefore J is generated, as an Ω -vector space, by the elements

$$\widehat{\tau_i} - \varepsilon_K \widehat{\tau_i} \varepsilon_K \quad (i = 1, \dots, t).$$

Let η be the unit element of the two-sided ideal J of the semi-simple Ω -algebra ΩT (Theorem 2.1). Then η is a central idempotent of ΩT , and we have also

$$\eta = \sum_{i=1}^t y_i (\widehat{\tau_i} - \varepsilon_K \widehat{\tau_i} \varepsilon_K), \quad y_i \in \Omega.$$

Therefore

$$\varepsilon_K \eta = \eta \varepsilon_K = \eta \varepsilon_K^2 = \varepsilon_K \eta \varepsilon_K = 0,$$

and, since $\varepsilon_{\mathcal{C}_i} - \varepsilon_K = \varepsilon_{\mathcal{C}_i} - \varepsilon_K \varepsilon_{\mathcal{C}_i} \varepsilon_K \in J$, this means that

$$\varepsilon_{\mathcal{C}_i} - \varepsilon_K = (\varepsilon_{\mathcal{C}_i} - \varepsilon_K) \eta = \varepsilon_{\mathcal{C}_i} \eta = \eta.$$

It follows that $\varepsilon_K = \varepsilon_{\mathcal{C}_i} - \eta$ is in the centre of ΩT , and therefore $\underline{K}$ is in the centre of T . We have proved that K is a T -normal subgroup of G , and therefore $\mathcal{C}_i K = K \mathcal{C}_i = K \mathcal{C}_i K$ for all $i = 1, \dots, t$.

THE CANONIC EPIMORPHISM THEOREM 6.3. — *Let K be a T -normal subgroup of G . Then*

- (1) $\varphi_K: \tau \mapsto \tau \varepsilon_K$ is an epimorphism of the S -algebra ΩT on G onto the T -normal S -algebra $\Omega(T \cap G // K)$ on G .

(2) $\text{Ker } \varphi_K = K$.

We call φ_K the *canonic epimorphism* or the *projection* of ΩT onto $\Omega(T \cap G // K)$, and we consider the T -normal S -ring $T \cap G // K$ as the *factor S -ring* of T modulo K . In order to keep in accordance with the notation in group theory, we write $T/(T \cap K // 1) := T \cap G // K$ for the factor S -ring of T modulo K .

PROOF. — ε_K is a central idempotent of ΩT . Therefore φ_K is an epimorphism of the Ω -algebra ΩT onto the Ω -algebra $\Omega(T \cap G // K)$. By Lemma 3.2 the image

$$\widehat{\tau_i}^{\varphi_K} = \widehat{\tau_i} \varepsilon_K = \varepsilon_K \tau_i \varepsilon_K$$

of any T -class average is a $(T \cap G // K)$ -class average. Hence 1.5(3) holds for φ_K , and φ_K is a homomorphism in the sense of Definition 1.5. Our statement (2) follows from the fact that ε_K is the $\widehat{\sigma_1}$ of $\text{Im } \varphi_K = \Omega(T \cap G // K)$.

Theorems 6.2 and 6.3 show that the T -normal subgroups of G are exactly the kernels of the homomorphisms of the S -algebra ΩT .

We draw a couple of conclusions from the Homomorphism Theorem 6.2. We start with rephrasing Proposition 5.2.

THEOREM 6.4. — *Let H be a T -subgroup of G , and φ a homomorphism of the S -algebra ΩT on G into the S -algebra ΩS on F . Then*

$$|H| \cdot |\mathcal{S}_j| = |H^{\bar{\varphi}}| \cdot |\mathcal{T}_i(H \cap \text{Ker } \varphi)| \quad \text{for any } \mathcal{T}_i \subseteq H \text{ such that } \mathcal{T}_i^{\bar{\varphi}} = \mathcal{S}_j.$$

PROOF. — We set $K = \text{Ker } \varphi$ and assume that $\mathcal{T}_i^{\bar{\varphi}} = \mathcal{T}_j^{\bar{\varphi}}$. Then $\mathcal{T}_i K = \mathcal{T}_j K$ (Theorem 6.2(1)) and for every $g_i \in \mathcal{T}_i$ there exist $g_j \in \mathcal{T}_j$ and $k \in K$ such that $g_i = g_j k$. If we further assume that $\mathcal{T}_i, \mathcal{T}_j \subseteq H$, then $k \in H \cap K$. Therefore $\mathcal{T}_i \subseteq \mathcal{T}_j(H \cap K)$. Conversely, $\mathcal{T}_k \subseteq \mathcal{T}_j(H \cap K)$ implies $\mathcal{T}_k^{\bar{\varphi}} = \mathcal{T}_j^{\bar{\varphi}}$. It follows that $\mathcal{T}_j(H \cap K)$ is the union of all those T -classes $\mathcal{T}_i \subseteq H$ for which $\mathcal{T}_i^{\bar{\varphi}} = \mathcal{T}_j^{\bar{\varphi}}$ holds, and our theorem follows from Proposition 5.2.

As a special case of Theorem 6.4 we take the S -class $\mathcal{S}_1$ of F which contains the unit element of F . Then $\mathcal{S}_1$ is a subgroup of F contained in $H^{\bar{\varphi}}$. For the T -class $\mathcal{T}_1$ of G which contains the unit element of G , we have $\mathcal{T}_1^{\bar{\varphi}} = \mathcal{S}_1$ and $\mathcal{T}_1 \subseteq H \cap \text{Ker } \varphi$. Thus we have proved

THEOREM 6.5. — *Let H be a T -subgroup of G , and φ a homomorphism of the S -algebra ΩT on G into the S -algebra ΩS on F . Then*

$$|H : H \cap \text{Ker } \varphi| = |H^{\bar{\varphi}} : \mathcal{S}_1|.$$

In particular

$$|G : \text{Ker } \varphi| = |G^{\bar{\Phi}} : \mathfrak{S}_1|.$$

THEOREM 6.6. — *Let φ be a homomorphism of the S -algebra ΩT on G into the S -algebra ΩS on F , and L an $(S \cap G^{\bar{\Phi}}/1)$ -normal subgroup of $G^{\bar{\Phi}}$. Then $L^{\bar{\Phi}^{-1}}$ is a T -normal subgroup of G .*

PROOF. — By the $(S \cap G^{\bar{\Phi}}/1)$ -normality of L we have $\Omega(S \cap G^{\bar{\Phi}}/L)$ as a two-sided ideal of $\text{Im } \varphi = \Omega(S \cap G^{\bar{\Phi}}/1)$. The set J of all $\tau \in \Omega T$ such that $\tau^{\bar{\Phi}} \in \Omega(S \cap G^{\bar{\Phi}}/L)$ is a two-sided ideal of ΩT . We set $K = \text{Ker } \varphi$. The two-sided ideal of those elements $\tau \in \Omega T$ such that $\tau^{\bar{\Phi}} = 0$ (that is the kernel of φ qua algebra homomorphism) is equal to $(\Omega T)(\varepsilon_{\mathfrak{C}_1} - \varepsilon_K)$ (cf. part II of the proof of 6.2) and is contained in J . Let η be the unit element of J . We set $H = L^{\bar{\Phi}^{-1}}$. Then we have also $\varepsilon_H^{\bar{\Phi}} = \varepsilon_L$. Therefore $\eta - \varepsilon_H \in (\Omega T)(\varepsilon_{\mathfrak{C}_1} - \varepsilon_K)$. Certainly $\mathfrak{C}_1 \leq K \leq H$ holds since $\mathfrak{C}_i^{\bar{\Phi}} = \mathfrak{S}_i \subseteq L$ for all $\mathfrak{C}_i \subseteq K$. Hence

$$\varepsilon_H(\varepsilon_{\mathfrak{C}_1} - \varepsilon_K) = 0,$$

$$\eta - \varepsilon_H = (\eta - \varepsilon_H)(\varepsilon_{\mathfrak{C}_1} - \varepsilon_K) = \eta(\varepsilon_{\mathfrak{C}_1} - \varepsilon_K) = \varepsilon_{\mathfrak{C}_1} - \varepsilon_K.$$

Since all of η , $\varepsilon_{\mathfrak{C}_1}$, ε_K are in the centre of ΩT , so is ε_H , and therefore H is T -normal.

As in ordinary group theory the correspondence between T -subgroups of G and the S -subgroups in $G^{\bar{\Phi}}$ becomes a one-to-one correspondence if we restrict ourselves to those T -subgroups of G which contain the T -normal subgroup $\text{Ker } \varphi$. We denote by $\langle X, Y \rangle$ the subgroup of G generated by the subgroups X and Y of G .

THEOREM 6.7. — *Let φ be a homomorphism of the S -algebra ΩT on G into the S -algebra ΩS on F . Let X and Y be T -subgroups of G containing $\text{Ker } \varphi$. Then*

- (1) $(X^{\bar{\Phi}})^{\bar{\Phi}^{-1}} = X$.
- (2) $(X \cap Y)^{\bar{\Phi}} = X^{\bar{\Phi}} \cap Y^{\bar{\Phi}}$ and $\langle X, Y \rangle^{\bar{\Phi}} = \langle X^{\bar{\Phi}}, Y^{\bar{\Phi}} \rangle$.
- (3) X is a T -normal subgroup of G if and only if $X^{\bar{\Phi}}$ is an $(S \cap G^{\bar{\Phi}}/1)$ -normal subgroup of $G^{\bar{\Phi}}$.

Therefore, taking $(L^{\bar{\Phi}^{-1}})^{\bar{\Phi}} = L$ for all S -subgroups $L \leq G^{\bar{\Phi}}$ into account, ([10], Proposition 2.5(2)), the mapping $X \rightarrow X^{\bar{\Phi}}$ is an isomorphism of the lattice of all T -subgroups of G containing $\text{Ker } \varphi$ onto the lattice of all S -subgroups

of $G^{\bar{\varphi}}$, which brings into one-to-one correspondence the T -normal subgroups of G containing $\text{Ker } \varphi$ with the $(S \cap G^{\bar{\varphi}}/1)$ -normal subgroups of $G^{\bar{\varphi}}$.

PROOF. — (1) and (2) merely restate [10], Theorem 2.10, because of Theorem 1.16 and Definition 3.3. Statement (3) follows from Proposition 5.7, Theorem 6.6, statement (1), and from $(L^{\bar{\varphi}})^{-1} = L$ for all S -subgroups $L \leq G^{\bar{\varphi}}$.

A homomorphism of a group is a monomorphism if and only if its kernel is the unit subgroup. The analogue holds for S -algebra homomorphisms as is shown in the following.

PROPOSITION 6.8. — *Let φ be a homomorphism of the S -algebra ΩT on G into the S -algebra ΩS on F . Then φ is a monomorphism if and only if $\text{Ker } \varphi = \mathfrak{C}_1$.*

PROOF. — If φ is a monomorphism, then $\widehat{\tau}_1$ is the only T -class average mapped onto $\widehat{\sigma}_1$, hence $\text{Ker } \varphi = \mathfrak{C}_1$.

If $\text{Ker } \varphi = \mathfrak{C}_1$, then $\widehat{\tau}_i \neq \widehat{\tau}_j$ for $i \neq j$ because of Theorem 6.2 (1). Since the $\widehat{\tau}_i$ ($i = 1, \dots, t$) are all of the $\text{Im } \varphi = (S \cap G^{\bar{\varphi}}/1)$ -class averages of $G^{\bar{\varphi}}$, they are an Ω -basis of $\text{Im } \varphi$, and hence φ is a monomorphism.

We have defined the concept of S -algebra monomorphism in a naive way as an injective homomorphism (Section 1 after Definition 1.5). In the category of all S -algebras over Ω a homomorphism φ of an S -algebra ΩT on G into an S -algebra ΩS on F is called *monic*, if for every pair ψ_1, ψ_2 of homomorphisms of any S -algebra ΩR on any finite group E into ΩT the equation

$$\psi_1 \varphi = \psi_2 \varphi \quad \text{implies} \quad \psi_1 = \psi_2.$$

As an application of the foregoing results we prove

THEOREM 6.9. — *Let φ be a homomorphism of the S -algebra ΩT on G into the S -algebra ΩS on F . Then φ is a monomorphism if and only if φ is monic in the category of all S -algebras over Ω .*

PROOF. — I. Let φ be a monomorphism. Then the injectivity of φ as a mapping immediately yields that φ is monic.

II. Assume that φ is monic. Set $K = \text{Ker } \varphi$. Then, since K is a T -subgroup, $R = T \cap K/1$ is an S -ring on K contained in T (Theorem 3.1(4) and Definition 3.3). Let ψ_1 be the injection of ΩR into ΩT , and ψ_2 the zero homomorphism of ΩR into ΩT (cf. the proof of Theorem 1.6). Then $\psi_1 \varphi = \psi_2 \varphi$ holds since φ induces the zero homomorphism of ΩR into ΩS . It follows that $\psi_1 = \psi_2$ because φ is monic. Therefore K has only one R -class, namely $\mathfrak{C}_1$. Hence $K = \mathfrak{C}_1$, and φ is a monomorphism by Proposition 6.8.

7. – The Isomorphism Theorems.

We want to emphasize that S -ring theory can be considered as a generalization of finite group theory. In this context the Isomorphism Theorems play an important rôle. They have already been stated for double coset S -rings in [8], Theorems 3.10 and 3.11.

THE FIRST ISOMORPHISM THEOREM 7.1. – *Let K be a T -normal subgroup of G , and L a T -subgroup of G containing K . Then*

- (1) *L is T -normal if and only if L is $(T \cap G // K)$ -normal.*
- (2) *Assume that L is T -normal. Then*
 - (a) *$T/(T \cap K // 1) := T \cap G // K$ is the factor S -ring of the S -ring T on G modulo the T -normal subgroup K .*
 - (b) *$(T \cap G // K)/((T \cap G // K) \cap L // 1) := (T \cap G // K) \cap G // L = T \cap G // L$ is the factor S -ring of the S -ring $T \cap G // K$ on G modulo the $(T \cap G // K)$ -normal subgroup L .*
 - (c) *$T/(T \cap L // 1) := T \cap G // L$ is the factor S -ring of the S -ring T on G modulo the T -normal subgroup L of G .*

The iterated factor S -ring (b) (qua factor S -ring of the factor S -ring (a)) is equal to the factor S -ring (c).

7.1(1) repeats Theorem 4.9. The isomorphism statement of 7.1(2) is even an equality, and a tautology. The reason for this peculiarity is the fact that for S -rings the factor structure also is a substructure.

THE SECOND ISOMORPHISM THEOREM 7.2. – *Let H be a T -subgroup of G , and K a T -normal subgroup of G . Then*

- (1) *$H \cap K$ is a $(T \cap H // 1)$ -normal subgroup of H .*
- (2) *K is a $(T \cap HK // 1)$ -normal subgroup of HK .*
- (3) *The S -algebra $\Omega(T \cap (H // H \cap K))$ on H is isomorphic to the S -algebra $\Omega(T \cap HK // K)$ on HK .*

Note that $T \cap (H // H \cap K)$ is the factor S -ring of the S -ring $T \cap H // 1$ on H modulo the $(T \cap H // 1)$ -normal subgroup $H \cap K$, and $T \cap HK // K$ is the factor S -ring of the S -ring $T \cap HK // 1$ on HK modulo the $(T \cap HK // 1)$ -normal subgroup K .

PROOF. – Take the projection φ_K of the S -algebra ΩT on G onto the factor S -algebra $\Omega(T \cap G // K)$ on G (Theorem 6.3(1)). φ_K induces an epimorphism φ of the S -algebra $\Omega(T \cap H // 1)$ on H onto the S -algebra $\Omega(T \cap HK // K)$ on

HK . Then $H \cap K = \text{Ker } \varphi$ is a $(T \cap H // 1)$ -normal subgroup of H (Theorem 6.2(2)), and the S -algebra $\Omega(T \cap (H // H \cap K))$ on H is isomorphic to the S -algebra $\text{Im } \varphi = \Omega(T \cap HK // K)$ on HK (Theorem 6.2(3)). K is in the centre of T , since K is a T -normal subgroup of G . Therefore K is also in the centre of $T \cap HK // 1$, and hence K is a $(T \cap HK // 1)$ -normal subgroup of HK .

All group algebras of finite groups over Ω are objects of the category of all S -algebras over Ω , and all group homomorphisms (qua homomorphisms of the related group algebras) are morphisms of that category. Yet the group algebras and the epimorphisms of S -algebras onto group algebras are embedded in the category of all S -algebras over Ω in a remarkable way.

THEOREM 7.3. — *Let φ be an epimorphism of the S -algebra ΩT on G onto the group algebra ΩF of F (qua S -algebra on F). Then*

- (1) *$\text{Ker } \varphi$ is a normal subgroup of G .*
- (2) *The group $G / \text{Ker } \varphi$ is isomorphic to the group F .*
- (3) *$\alpha: g \mapsto \widehat{\tau}_i^\circ$ for $g \in \mathfrak{C}_i$, is an epimorphism of the group G onto the group F , and $\text{Ker } \alpha = \text{Ker } \varphi$ holds.*

PROOF. — For any $x, y \in G$ there exist $i, j, n \in \{1, \dots, t\}$ such that

$$x \in \mathfrak{C}_i, \quad y \in \mathfrak{C}_j, \quad xy \in \mathfrak{C}_n.$$

Then

$$x^z y^z = \widehat{\tau}_i^\circ \widehat{\tau}_j^\circ = (\widehat{\tau}_i \widehat{\tau}_j)^\circ = \sum_{k=1}^t \widehat{a}_{ijk} \widehat{\tau}_k^\circ$$

is a single element $f \in F$ (Definition 1.5(3)), which implies

$$\widehat{\tau}_k^\circ = f \text{ for all } k \text{ such that } \widehat{a}_{ijk} > 0, \text{ and } \sum_{k=1}^t \widehat{a}_{ijk} = 1.$$

Since $\widehat{a}_{ijn} > 0$, it follows that $f = \widehat{\tau}_n^\circ = (xy)^z$ and therefore

$$(xy)^z = x^z y^z \text{ for all } x, y \in G.$$

α is an epimorphism of the group G onto the group F . For $g \in \mathfrak{C}_i$ we have

$$g^z = 1 \text{ if and only if } \mathfrak{C}_i \subseteq \text{Ker } \varphi.$$

Therefore $\text{Ker } \alpha = \text{Ker } \varphi$ is a normal subgroup of G . Our theorem is proved.

8. – *T*-Subnormal Subgroups.

DEFINITION 8.1. – *A subgroup L of G is called a T -subnormal subgroup of G , if there exists a chain*

$$G = L_0 \geq L_1 \geq \dots \geq L_r = L$$

of T -subgroups of G such that L_i is a $(T \cap L_{i-1} // 1)$ -normal subgroup of L_{i-1} for all $i = 1, \dots, r$. Such a chain of subgroups is called a T -subnormal chain of G , and the S -rings $T \cap L_{i-1} // L_i$ ($i = 1, \dots, r$) are called its factors. Any T -subnormal chain from G to the T -normal subgroup $\mathfrak{T}_1$ is called a T -subnormal series of G .

For the S -semigroup T related to the S -ring T (Theorem 1.15) the notion of T -subnormal subgroup can be introduced by substituting T -normality in Definition 8.1 ([11], Definition 1.1.) A theory of T -subnormal subgroups has been initiated in [11]. We shall note the differences between T -subnormality and T -subnormality.

PROPOSITION 8.2. – *Every T -subnormal subgroup of G is T -subnormal.*

PROOF. – Lemma 4.11.

Whenever T -normality coincides with T -normality, then the T -subnormal and the T -subnormal subgroups are one and the same. For instance, a subgroup K of G is $G // H$ -subnormal if and only if it is G/H -subnormal (Theorem 4.14). Yet, even if Problem 4.12 were answered in the affirmative, that is if T -subnormal and T -subnormal subgroups were always the same, there would still remain a difference between T -subnormal chains and T -subnormal chains, because the factors of a T -subnormal chain are S -rings whereas the factors of a T -subnormal chain are S -semigroups ([11], Definition 1.1). Therefore the theory of T -subnormal chains is not made superfluous by the theory of T -subnormal chains.

Even if Problem 1.19 could be answered in the affirmative, that is if every S -semigroup on a finite group were related to an S -ring in the sense of Theorem 1.15, this would not be sufficient to deduce the theory of T -normality from that of T -normality nor vice versa. If an even stronger statement than the affirmative answer of Problem 1.19 were true, namely that not only is every S -semigroup on a finite group related to an S -ring in the sense of Theorem 1.15, but also that every S -semigroup homomorphism is related to an S -algebra homomorphism in the sense of Theorem 1.16, then the theory of T -subnormality could be deduced from the theory of T -subnormality by the functor of Theorem 1.17. And only if the functor of Theorem 1.17 had an inverse could the theory of T -subnormality be deduced from the theory of T -subnormality.

These considerations may justify the remainder of this paper which is analogously written to [11].

LEMMA 8.3. — *Let K be a T -subnormal subgroup of G , and let L be a $(T \cap K // 1)$ -subnormal subgroup of K . Then L is a T -subnormal subgroup of G .*

PROOF. — Let $K = L_0 \geq L_1 \geq \dots \geq L_r = L$ be a $(T \cap K // 1)$ -subnormal chain of K . Then L_i is a $((T \cap K // 1) \cap L_{i-1} // 1)$ -normal subgroup of L_{i-1} . Since $(T \cap K // 1) \cap L_{i-1} // 1 = T \cap L_{i-1} // 1$, this chain extends every T -subnormal chain from G to K to a T -subnormal chain from G to L .

THEOREM 8.4. — *Let H be a T -subgroup of G , and let L be a T -subnormal subgroup of G . Then $H \cap L$ is a $(T \cap H // 1)$ -subnormal subgroup of H .*

PROOF. — Take any T -subnormal chain from G to L as in Definition 8.1. $H = H \cap G = H \cap L_0$ is $(T \cap H // 1)$ -normal, and hence $(T \cap H // 1)$ -subnormal. Assume that $H \cap L_{i-1}$ is already shown to be $(T \cap H // 1)$ -subnormal. Since L_i is a $(T \cap L_{i-1} // 1)$ -normal subgroup of L_{i-1} by assumption, the Second Isomorphism Theorem 7.2 yields $H \cap L_i = (H \cap L_{i-1}) \cap L_i$ as a $(T \cap (H \cap L_{i-1}) // 1)$ -normal subgroup of $H \cap L_{i-1}$. Therefore $H \cap L_i$ is a $(T \cap H // 1)$ -subnormal subgroup of H by Lemma 8.3. That proves our theorem.

THEOREM 8.5. — *Let K and L be T -subnormal subgroups of G . Then $K \cap L$ is a T -subnormal subgroup of G .*

PROOF. — $K \cap L$ is a $(T \cap K // 1)$ -subnormal subgroup of K by Theorem 8.4. Therefore $K \cap L$ is a T -subnormal subgroup of G by Lemma 8.3.

Some further statements on the special case of $G // H$ -subnormal ($= G/H$ -subnormal) subgroups can be made. We refer the reader to [11], Theorem 1.5, Corollary 1.6, and Theorem 1.7.

9. — The Four Subgroup Theorem.

In the last section we have dealt exclusively with the T -subnormal subgroups. Now we turn to the factors of the T -subnormal chains. Our aim is to prove a Jordan–Hölder Theorem for T -composition chains. As in ordinary group theory a basis for these investigations is Zassenhaus' Lemma whose S -ring version we are now going to formulate and to prove. It is almost entirely dependent on the validity of the Isomorphism Theorems for S -algebras.

THE FOUR SUBGROUP THEOREM 9.1. — *Let K_0 , K , L_0 , L be T -subgroups of G . Assume that K_0 is a $(T \cap K // 1)$ -normal subgroup of K , and L_0 a $(T \cap L // 1)$ -normal subgroup of L . Then*

- (1) $(K \cap L_0)K_0$ is a $(T \cap (K \cap L)K_0 // 1)$ -normal subgroup of $(K \cap L)K_0$.
- (2) $(K_0 \cap L)L_0$ is a $(T \cap (K \cap L)L_0 // 1)$ -normal subgroup of $(K \cap L)L_0$.
- (3) The S -algebra

$$\Omega(T \cap (K \cap L)K_0 // (K \cap L_0)K_0) \text{ on } (K \cap L)K_0$$

is isomorphic to the S -algebra

$$\Omega(T \cap (K \cap L)L_0 // (K_0 \cap L)L_0) \text{ on } (K \cap L)L_0.$$

$$(4) \quad |(K \cap L)K_0 : (K \cap L_0)K_0| = |(K \cap L)L_0 : (K_0 \cap L)L_0|.$$

PROOF. — $K \cap L_0$, $K \cap L$, K_0 are T -subgroups of G contained in K (Theorem 3.4). Therefore they are $(T \cap K // 1)$ -subgroups of K as well. Every $(T \cap K // 1)$ -normal subgroup of K is permutable with every $(T \cap K // 1)$ -subgroup of K (Theorem 4.7(1)). It follows that $(K \cap L_0)K_0$ and $(K \cap L)K_0$ are indeed $(T \cap K // 1)$ -subgroups of K (Theorem 3.4). By the same arguments $(K_0 \cap L)L_0$ and $(K \cap L)L_0$ are $(T \cap L // 1)$ -subgroups of L .

$K_0 \cap L = (K \cap L) \cap K_0$ is a $(T \cap (K \cap L) // 1)$ -normal subgroup of $K \cap L$, and $\varphi: \tau \rightarrow \tau \varepsilon_{K_0}$ is an isomorphism of the S -algebra $\Omega(T \cap (K \cap L) // (K_0 \cap L))$ on $K \cap L$ onto the S -algebra $\Omega(T \cap (K \cap L)K_0 // K_0)$ on $(K \cap L)K_0$ by the Second Isomorphism Theorem 7.2.

Also $K \cap L_0$ is a $(T \cap (K \cap L) // 1)$ -normal subgroup of $K \cap L$, and hence $(K_0 \cap L)(K \cap L_0)$ is a $(T \cap (K \cap L) // 1)$ -normal subgroup of $K \cap L$ (Theorem 4.8). $(K_0 \cap L)(K \cap L_0)$ is also a $(T \cap (K \cap L) // (K_0 \cap L))$ -normal subgroup of $K \cap L$. Therefore $((K_0 \cap L)(K \cap L_0))^{\bar{\varphi}} = (K_0 \cap L)(K \cap L_0)K_0 = (K \cap L_0)K_0$ is a $(T \cap (K \cap L)K_0 // K_0)$ -normal subgroup of $(K \cap L)K_0$ (Proposition 5.7). By the First Isomorphism Theorem 7.1 $(K \cap L_0)K_0$ is also a $(T \cap (K \cap L)K_0 // 1)$ -normal subgroup of $(K \cap L)K_0$. Thus we have proved (1), and (2) follows by symmetry.

The isomorphism φ maps the $(T \cap (K \cap L) // (K_0 \cap L))$ -normal S -algebra (i.e. the two sided ideal, cf. Theorem 4.1 and Definition 4.2) $\Omega(T \cap (K \cap L) // (K_0 \cap L)(K \cap L_0))$ isomorphically onto the $(T \cap (K \cap L)K_0 // K_0)$ -normal S -algebra $\Omega(T \cap (K \cap L)K_0 // (K \cap L_0)K_0)$. In the same way the S -algebra $\Omega(T \cap (K \cap L) // (K_0 \cap L)(K \cap L_0))$ is proved to be isomorphic to the S -algebra $\Omega(T \cap (K \cap L)L_0 // (K_0 \cap L)L_0)$, and hence (3) holds. (4) follows from

$$|(K \cap L)K_0 : (K \cap L_0)K_0| = |(K \cap L) : (K_0 \cap L)(K \cap L_0)| = |(K \cap L)L_0 : (K_0 \cap L)L_0|$$

by an elementary computation.

The first consequence of the Four Subgroup Theorem for the investigation of the factors of T -subnormal chains is the following.

THE REFINEMENT THEOREM FOR T -SUBNORMAL CHAINS 9.2. - Assume that

$$(i) \quad G = K_0 \geq K_1 \geq \dots \geq K_r = L$$

$$(ii) \quad G = L_0 \geq L_1 \geq \dots \geq L_s = L$$

are two T -subnormal chains from G to the T -subnormal subgroup L . Set

$$K_{i,j} = K_i(K_{i-1} \cap L_j) \quad \text{and} \quad L_{j,i} = L_j(L_{j-1} \cap K_i)$$

for $i = 1, \dots, r$ and $j = 1, \dots, s$. Then

$$(1) \quad K_{i-1} = K_{i,0} \geq K_{i,1} \geq \dots \geq K_{i,s} = K_i$$

is a $(T \cap K_{i-1}/1)$ -subnormal chain from K_{i-1} to K_i for each $i = 1, \dots, r$.

$$(2) \quad L_{j-1} = L_{j,0} \geq L_{j,1} \geq \dots \geq L_{j,r} = L_j$$

is a $(T \cap L_{j-1}/1)$ -subnormal chain from L_{j-1} to L_j for each $j = 1, \dots, s$.

(3) The S -algebras

$$\Omega(T \cap K_{i,j-1}/K_{i,j}) \quad \text{and} \quad \Omega(T \cap L_{j,i-1}/L_{j,i})$$

are isomorphic for all $i = 1, \dots, r$ and all $j = 1, \dots, s$.

$$(4) \quad |K_{i,j-1}:K_{i,j}| = |L_{j,i-1}:L_{j,i}|$$

for all $i = 1, \dots, r$ and all $j = 1, \dots, s$.

Joining the chains (1), respectively (2), together, we obtain refinements of the T -subnormal chains (i) and (ii) for which

$$T \cap K_{i,j-1}/K_{i,j} \leftrightarrow T \cap L_{j,i-1}/L_{j,i}$$

is a one-to-one correspondence of their factors such that corresponding factors have isomorphic S -algebras over Ω , and their related subgroups are of equal index.

PROOF. - The subgroups $K_i \leq K_{i-1}$ and $L_j \leq L_{j-1}$ satisfy the hypotheses of the Four Subgroup Theorem which immediately yields our statements.

10. - The Jordan-Hölder Theorem for T -Composition Chains.

DEFINITION 10.1 - A T -subnormal chain

$$G = L_0 > L_1 > \dots > L_r = L$$

is called a T -composition chain from G to the T -subnormal subgroup L , and its factors are called its composition factors, if L_i is a maximal $(T \cap L_{i-1}/1)$ -

normal subgroup of L_{i-1} , that is if there does not exist any $(T \cap L_{i-1}/1)$ -normal subgroup of L_{i-1} properly between L_{i-1} and L_i , for all $i = 1, \dots, r$. Any T -composition chain from G to the T -normal subgroup $\mathfrak{C}_1$ of G is called a T -composition series of G .

PROPOSITION 10.2. — *A T -subnormal chain*

$$G = L_0 > L_1 > \dots > L_r = L$$

is a T -composition chain if and only if L_{i-1} is $(T \cap L_{i-1}/L_i)$ -simple, that is if and only if its factors $T \cap L_{i-1}/L_i$ are simple for $i = 1, \dots, r$.

PROOF. — Suppose that H is a T -subgroup of G such that $L_{i-1} \geq H \geq L_i$. Since L_i is $(T \cap L_{i-1}/1)$ -normal by assumption, H is $(T \cap L_{i-1}/1)$ -normal if and only if H is $(T \cap L_{i-1}/L_i)$ -normal (Theorem 4.9), and our Proposition follows from Definition 4.10.

THE JORDAN-HÖLDER THEOREM FOR T -COMPOSITION CHAINS 10.3. — *If*

$$G = K_0 > K_1 > \dots > K_r = L$$

$$G = L_0 > L_1 > \dots > L_s = L$$

are two T -composition chains from G to the T -subnormal subgroup L , then

$$(1) \quad r = s.$$

(2) *There exists a permutation π of $\{1, \dots, r\}$ such that the S -algebras*

$$\Omega(T \cap K_{i-1}/K_i) \quad \text{and} \quad \Omega(T \cap L_{\pi(i)-1}/L_{\pi(i)})$$

are isomorphic and

$$|K_{i-1}:K_i| = |L_{\pi(i)-1}:L_{\pi(i)}|$$

holds for all $i = 1, \dots, r$.

The proof immediately follows from the Refinement Theorem 9.2.

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