On the Oscillation of Solutions of the Equation

$$[r(t)x^{(n-1)}(t)]' + \delta \sum_{i=1}^{m} p_i(t)\, \varphi\big(x[g_i(t)]\big) = 0 \quad (*)\ (**).$$

LU-SAN CHEN (Chung-Li, Taiwan-China)

Summary. – *In this paper we are dealing with the oscillatory and asymptotic behavior of n-th order* ($n > 1$) *retarded differential equations*

$$[r(t)x^{(n-1)}(t)]' + \delta \sum_{i=1}^{m} p_i(t)\varphi(x[g_i(t)]) = 0$$

which contain a damping term involving the $(n-1)$*-th derivative of the unknown function, where* $\delta = \pm 1$.

1. – Introduction.

In this paper we are dealing with the oscillatory and asymptotic behavior of n-th order ($n > 1$) retarded differential equations, which contain a damping term involving the $(n-1)$-th derivative of the unknown function. We now consider the following damped differential equations with retarded arguments

$$(*) \qquad [r(t)x^{(n-1)}(t)]' + \delta \sum_{i=1}^{m} p_i(t)\varphi\big(x[g_i(t)]\big) = 0$$

where $\delta = \pm 1$. The real valued functions r, p_i, g_i ($i = 1, 2, \dots, m$) and φ are supposed continuous and such that:

(i) *The functions* g_i, $i = 1, 2, \dots, m$ *are differentiable on* $[t_0, \infty)$ *and such that for every* $t \geq t_0$,

$$g_i(t) \leq t\,, \qquad g_i'(t) \geq 0\,, \qquad \lim_{t\to\infty} g_i(t) = \infty\,.$$

(ii) *The functions* p_i, $i = 1, 2, \dots, m$ *are nonnegative on* $[t_0, \infty)$ *and for some index* i_0, $1 \leq i_0 \leq m$, $p_{i_0}(t) > 0$ *for* $t \geq t_0$.

(iii) *The function* φ *is nondecreasing on* $(-\infty, \infty)$, $y \neq 0$ *implies* $y\varphi(y) > 0$ *and such that*

$$\int^{\infty} \frac{dy}{\varphi(y)} < \infty \qquad \textit{and} \qquad \int_{-\infty} \frac{dy}{\varphi(y)} < \infty\,.$$

(*) Entrata in Redazione il 18 febbraio 1976.
(**) This research was supported by the National Science Council.

Note: Condition (iii) implies that

$$\lim_{t\to\infty}\frac{\varphi(y)}{y}=\infty=\lim_{t\to-\infty}\frac{\varphi(y)}{y}. \tag{1}$$

(iv) *The function r is positive on* $[t_0, \infty)$.

For equation $(*)_{\delta=+1}$ we give some general condition results not only for the case where the condition

$$\int^{\infty}\frac{dt}{r(t)}=\infty \tag{2}$$

holds, but also for some cases in which this condition fails.

In what follows, we consider only such solutions of the equation $(*)$ which are defined for all large t. The oscillatory character is considered in the usual sense, *i.e.* a solution of the equation $(*)$ is called *oscillatory* if it has no last zero, otherwise it is called *nonoscillatory*.

Before stating the theorems, we give the following lemmas, the first two of which can be found in [1] and [2], therefore the proofs of which may be omitted.

LEMMA 1. – *Let $u(t)$ be a positive ν-times continuously differentiable function on $[a, \infty)$. If $u^{(\nu)}(t)$ is of constant sign and not identically zero for all large t, then there exist a $t_u \geq a$ and an integer l, $0 \leq l \leq \nu$ with $\nu + l$ odd if $u^{(\nu)}(t) \leq 0$, $\nu + l$ even if $u^{(\nu)}(t) \geq 0$ and such that for every $t \geq t_u$*

$$l > 0 \text{ implies } u^{(k)}(t) > 0 \qquad (k = 0, 1, \ldots, l-1)$$

and

$$l \leq \nu - 1 \text{ implies } (-1)^{l+k}u^{(k)}(t) > 0 \qquad (k = l, l+1, \ldots, \nu-1).$$

LEMMA 2. – *Let $u(t)$ be a $(\nu-1)$-times $(\nu > 1)$ continuously differentiable function on $[a, \infty)$. Let also $m(t)$ be a positive function on $[a, \infty)$ such that the function $m(t)u^{(\nu-1)}(t)$ is continuously differentiable on $[a, \infty)$. Suppose moreover that for every $t \geq a$ we have $u(t) > 0$, $\delta u^{(\nu-1)}(t) > 0$, $\delta\big(m(t)u^{(\nu-1)}(t)\big)' \leq 0$ and not identically zero for all large t, where $\delta = \pm 1$. Then there exists a constant $k > 0$ such that*

$$\frac{m(t)}{m^*(t)}\frac{|u^{(\nu-1)}(t)|}{u(t/2)}t^{\nu-1} \leq k$$

for all large t, where $m^*(t) = \max\limits_{t/2 \leq \theta \leq t} m(\theta)$.

LEMMA 3. – *Let equation $(*)_{\delta=+1}$ satisfy the conditions* (i)-(iv). *Then we have the following properties*:

(I) *If condition* (2) *holds, then for every nonoscillatory solution $x(t)$ of equation*

$(*)_{\delta=+1}$ *we have*

$$x(t)x^{(n-1)}(t) > 0 \qquad \text{for all large } t.$$

(II) *If for every* $T \geq t_0$

$$\int^{\infty} \frac{\sum_{i=1}^{m} \int_T^t p_i(\theta)\,d\theta}{r(t)}\,dt = \infty, \qquad t \geq T \tag{3}$$

then for every nonoscillatory solution $x(t)$ *of equation* $(*)_{\delta=+1}$ *with* $\lim_{t\to\infty} x(t) \neq 0$ *we have*

$$x(t)x^{(n-1)}(t) > 0 \qquad \text{for all large } t.$$

(III) *If*

$$\int^{\infty} \frac{dt}{r(t)} < \infty \qquad \text{and for some } \lambda > 1 \tag{4}$$

$$\sum_{i=1}^{m} \int^{\infty} g_i^{n-2}(t)\, p_i(t) \left(\int_t^{\infty} \frac{d\theta}{r(\theta)} \right)^{\lambda} dt = \infty$$

then for every nonoscillatory solution $x(t)$ *of equation* $(*)_{\delta=+1}$ *with* $\lim_{t\to\infty} x(t) \neq 0$ *we have*

$$x(t)x^{(n-1)}(t) > 0 \qquad \text{for all large } t.$$

PROOF. – Let $x(t)$ be a nonoscillatory solution of equation $(*)_{\delta=+1}$. Then without loss of generality $x(t) > 0$ for every $t \geq t_0$ and because of condition (i) there exists a $t_1 \geq t_0$ such that $x[g_i(t)] > 0$ for $t \geq t_1$ and $i = 1, 2, \ldots, m$. Thus, in all cases (I)-(III) we get

$$[r(t)x^{(n-1)}(t)]' = -\sum_{i=1}^{m} p_i(t)\varphi\big(x[g_i(t)]\big) < 0 \tag{5}$$

for every $t \geq t_1$: Moreover, since $p_{i_0}(t) > 0$ for all large t, the same holds for $[r(t)x^{(n-1)}(t)]'$ and consequently the function $r(t)x^{(n-1)}(t)$ is positive or negative for all large t. Thus since $r(t) > 0$ for every $t \geq t_0$, we must have $x^{(n-1)}(t) > 0$ or $x^{(n-1)}(t) < 0$ for all large t.

We shall prove that the assumption

$$x^{(n-1)}(t) < 0 \qquad \text{for all large } t$$

leads to a contradiction in all cases (I)-(III), provided that in cases (II) and (III) we have $\lim_{t\to\infty} x(t) \neq 0$. To do this we suppose that for some $t_2 \geq t_1$ we have $x^{(n-1)}(t) < 0$

for every $t \geqslant t_2$. Integrating (5) between $[t_2, t]$ we obtain

$$r(t)x^{(n-1)}(t) \leqslant r(t_2)x^{(n-1)}(t_2)$$

and consequently

$$-x^{(n-1)}(t) \geqslant -r(t_2)x^{(n-1)}(t_2)\frac{1}{r(t)}$$

for every $t \geqslant t_2$: Integrating again between $[t_2, t]$ we have

$$-x^{(n-2)}(t) + x^{(n-2)}(t_2) \geqslant -r(t_2)x^{(n-1)}(t_2)\int_{t_2}^{t}\frac{d\theta}{r(\theta)}$$

and consequently condition (2) implies

$$\lim_{t\to\infty} x^{(n-2)}(t) = -\infty$$

which contradicts the positivity of $x(t)$. This contradiction proves (I).

To prove (II) we remark that the assumption $\lim\limits_{t\to\infty} x(t) \neq 0$ implies the existence of a constant $k_1 > 0$ such that

$$\varphi(x[g_i(t)]) \geqslant k_1, \qquad i = 1, 2, \ldots, m \text{ for every } t \geqslant t_2 .$$

Thus, from equation $(*)_{\delta=+1}$ leads to the following inequality

$$\text{(6)} \qquad [r(t)x^{(n-1)}(t)]' + \sum_{i=1}^{m} p_i(t)k_1 \leqslant 0 \qquad \text{for every } t \geqslant t_2 .$$

Integrating (6) between $[t_2, t]$ we obtain

$$r(t)x^{(n-1)}(t) - r(t_2)x^{(n-1)}(t_2) + k_1\sum_{i=1}^{m}\int_{t_2}^{t} p_i(\theta)\,d\theta \leqslant 0 ,$$

and consequently

$$-x^{(n-1)}(t) \geqslant k_1 \frac{\sum_{i=1}^{m}\int_{t_2}^{t} p_i(\theta)\,d\theta}{r(t)} \qquad \text{for every } t \geqslant t_2 .$$

Using this inequality and condition (3) we obtain again the contradiction $\lim\limits_{t\to\infty} x^{(n-2)}(t) = -\infty$.

To prove (III) we rewrite equation $(*)_{\delta=+1}$ as follows:

$$(7) \qquad [r(t)\,x^{(n-1)}(t)]' + \sum_{i=1}^{m} p_i(t)\,\frac{\varphi(x[g_i(t)])}{x[g_i(t)/2]}\,x\left[\frac{g_i(t)}{2}\right] = 0\,, \qquad t \geqslant t_3$$

and we remark that (1) and $\lim_{t\to\infty} x(t) \neq 0$ imply the existence of some $t_3 \geqslant t_2$ and of a constant $k_2 > 0$ such that

$$\frac{\varphi(x[g_i(t)])}{x[g_i(t)/2]} > k_2\,, \qquad i = 1, 2, \ldots, m \text{ for every } t \geqslant t_3\,.$$

From above inequality, (7) leads to

$$(8) \qquad [r(t)\,x^{(n-1)}(t)]' + kk_2 \sum_{i=1}^{m} p_i(t)\,x\left[\frac{g_i(t)}{2}\right] \leqslant 0\,, \qquad t \geqslant t_3\,.$$

Applying Lemma 2 with $\nu = n-1$ and $m(t) = 1$, from (8) we derive the following

$$(9) \qquad [r(t)\,x^{(n-1)}(t)]' + kk_2 \sum_{i=1}^{m} p_i(t)\,g_i^{n-2}(t)\,x^{(n-2)}(t) \leqslant 0\,, \qquad t \geqslant t_3\,.$$

By (9), $x^{(n-2)}(t)$ is obviously a positive solution of the following equation

$$(10) \qquad [r(t)\,y']' + \frac{kk_2 \sum_{i=1}^{m} g_i^{n-2}(t)\,p_i(t)\,x^{(n-2)}(t) + \mu(t)}{x^{(n-2)}(t)}\; y = 0\,, \qquad t \geqslant t_3$$

where

$$\mu(t) = -\,[r(t)\,x^{(n-1)}(t)]' - kk_2 \sum_{i=1}^{m} g_i^{n-2}(t)\,p_i(t)\,x^{(n-2)}(t)\,, \qquad t \geqslant t_3\,.$$

Since, from (9),

$$\mu(t) \geqslant 0 \qquad \text{for every } t \geqslant t_3$$

the functions $r(t)$ and

$$kk_2 \sum_{i=1}^{m} p_i(t)\,g_i^{n-2}(t) + \frac{\mu(t)}{x^{(n-2)}(t)}$$

are obviously subject to the conditions

$$\int^{\infty} \frac{dt}{r(t)} < \infty$$

and for some $\lambda > 1$

$$\int^{\infty} \left(kk_2 \sum_{i=1}^{m} p_i(t)\,g_i^{n-2}(t) + \frac{\mu(t)}{x^{(n-2)}(t)}\right)\left(\int^{\infty} \frac{dt}{r(t)}\right)^{\lambda} dt = \infty\,.$$

Thus, applying a result due to MOORE [7] we conclude that all solutions of (10) are oscillatory. But this is a contradiction, since $x^{(n-2)}(t)$ is a nonoscillatory solution of (10). This contradiction proves (III).

2. – Main results.

THEOREM 1. – *In addition to conditions* (i)-(iv) *assume that for every* $T \geqslant t_0$

$$\sum_{i=1}^{m} \int^{\infty} p_i(t) \left(\sum_{i=1}^{m} \int_{T}^{t} \frac{g_i^{n-2}(\theta)\, g_i'(\theta)}{r^*[g_i(\theta)]}\, d\theta \right) dt = \infty \tag{11}$$

where $r^*(t) = \max\limits_{t/2 \leqslant \theta \leqslant t} r(\theta)$.

Then:

(α) *Under condition* (2) *every solution of equation* $(*)_{\delta=+1}$ *is for* n *even oscillatory and* n *odd either oscillatory or tending monotonically to zero as* $t \to \infty$ *together with its first* $n-2$ *derivatives.*

(β) *Under conditions* (3) *or* (4) *every solution of equation* $(*)_{\delta=+1}$ *is either oscillatory or tending monotonically to zero as* $t \to \infty$ *together with its first* $n-2$ *derivatives.*

Note: In the case where the function $r(t)$ is nondecreasing, condition (11) can be replaced by

$$\int^{\infty} \frac{\sum_{i=1}^{m} g_i^{n-1}(t)\, p_i(t)}{r(t)}\, dt = \infty\,. \tag{11*}$$

PROOF. – Let $x(t)$ be a nonoscillatory solution of equation $(*)_{\delta=+1}$ with $\lim\limits_{t\to\infty} x(t) \neq 0$. Then, without loss of generality, that $t_1 \geqslant t_0$ is chosen so that $x[g_i(t)] > 0$, $i = 1, 2, \dots, m$ for every $t \geqslant t_1$. In view of equation $(*)_{\delta=+1}$ and conditions (ii), (iii) we have $[r(t)x^{(n-1)}(t)]' \leqslant 0$ for every $t \geqslant t_1$, where this function is not identically zero for all large t. Under one of the conditions (2), (3) and (4) we get, from Lemma 3,

$$x^{(n-1)}(t) > 0 \quad \text{for all large } t$$

without loss of generality we assume that

$$x^{(n-1)}(t) > 0 \quad \text{for every } t \leqslant t_1\,.$$

By Lemma 1 there exists a $t_2 \geqslant t_1$ such that $x'(t) > 0$ or $x'(t) < 0$ for every $t \geqslant t_2$ and consequently we have to examine the following cases:

Case 1) $x'(t) > 0$ on $[t_2, \infty)$.

Let

(12) $$z(t) = -[r(t)x^{(n-1)}(t)]\sum_{i=1}^{m}\int_{t_2}^{t}\frac{g_i^{n-2}(\theta)g_i'(\theta)}{r^*[g_i(\theta)]\varphi(x[g_i(\theta)])}d\theta, \qquad t\geq t_2$$

we obviously have

(13) $$z(t)\leq 0 \quad \text{for every } t\geq t_2 .$$

From (12) for every $t\geq t_2$, we obtain

$$z'(t) = -[r(t)x^{(n-1)}(t)]'\sum_{i=1}^{m}\int_{t_2}^{t}\frac{g_i^{n-2}(\theta)g_i'(\theta)}{r^*[g_i(\theta)]\varphi([x[g_i(\theta)])}d\theta - \sum_{i=1}^{m}\frac{r(t)x^{(n-1)}(t)g_i^{n-2}(t)g_i'(t)}{r^*[g_i(t)]\varphi(x[g_i(t)])}$$

$$= \sum_{i=1}^{m}p_i(t)\varphi(x[g_i(t)])\sum_{i=1}^{m}\int_{t_2}^{t}\frac{g_i^{n-2}(\theta)g_i'(\theta)}{r^*[g_i(\theta)]\varphi(x[g_i(t)])}d\theta$$

$$-\sum_{i=1}^{m}\frac{g_i^{n-2}(t)}{r^*[g_i(t)]}\cdot\frac{r(t)x^{(n-1)}(t)}{x'[g_i(t)/2]}\cdot\frac{x'[g_i(t)/2]g_i'(t)}{\varphi(x[g_i(t)])}.$$

Since the functions φ and $x(t)$ are nondecreasing and the function $r(t)x^{(n-1)}(t)$ is nonincreasing, we get

$$z'(t)\geq\sum_{i=1}^{m}p_i(t)\sum_{i=1}^{m}\int_{t_2}^{t}\frac{g_i^{n-2}(\theta)g_i'(\theta)}{r^*[g_i(\theta)]}d\theta - 2\sum_{i=1}^{m}\frac{x^{(n-1)}[g_i(t)]}{x'[g_i(t)]}\cdot\frac{r[g_i(t)]}{r^*[g_i(t)]}g_i^{n-2}(t)\cdot\frac{(x[g_i(t)/2])'}{\varphi(x[g_i(t)/2])}$$

for every $t\geq t_2$: Thus applying Lemma 2 with $u = x'$, $m = r$, $\nu = n-1$ and $g_i(t)$ in place of t, we have

$$z'(t)\geq\sum_{i=1}^{m}p_i(t)\sum_{i=1}^{m}\int_{t_2}^{t}\frac{g_i^{n-2}(\theta)g_i'(\theta)}{r^*[g_i(\theta)]}d\theta - 2k\sum_{i=1}^{m}\frac{(x[g_i(t)/2])'}{(x[g_i(t)/2])}$$

for every $t\geq t_3$, where $t_3\geq t_2$ is chosen property. By above inequality, integration between $[t_3, t]$ and taking into account conditions (iii) and (11) we obtain $\lim_{t\to\infty} z(t) = \infty$, which contradicts (13).

Case 2). $x'(t) < 0$ on $[t_2, \infty)$.

Let

(14) $$\tilde{z}(t) = -[r(t)x^{(n-1)}(t)]\sum_{i=1}^{m}\int_{t_2}^{t}\frac{g_i^{n-2}(\theta)g_i'(\theta)}{r^*[g_i(\theta)]}d\theta, \qquad t\geq t_2$$

we obviously have

$$\tilde{z}(t) \leqslant 0 \quad \text{for every } t \geqslant t_2 . \tag{15}$$

From (14) for every $t \geqslant t_2$, we obtain

$$\begin{aligned}
\tilde{z}'(t) &= -[r(t)x^{(n-1)}(t)]' \sum_{i=1}^{m} \int_{t_2}^{t} \frac{g_i^{n-2}(\theta)g_i'(\theta)}{r^*[g_i(\theta)]} d\theta - \sum_{i=1}^{m} \frac{r(t)x^{(n-1)}(t)}{r^*[g_i(t)]} g_i^{n-2}(t)g_i'(t) \\
&= \sum_{i=1}^{m} p_i(t)\varphi(x[g_i(t)]) \sum_{i=1}^{m} \int_{t_2}^{t} \frac{g_i^{n-2}(\theta)g_i'(\theta)}{r^*[g_i(\theta)]} d\theta - \sum_{i=1}^{m} \frac{r(t)x^{(n-1)}(t)}{r^*[g_i(t)]} g_i^{n-2}(t)g_i'(t) \\
&\geqslant \sum_{i=1}^{m} p_i(t)\varphi(x[g_i(t)]) \sum_{i=1}^{m} \int_{t_2}^{t} \frac{g_i^{n-2}(\theta)g_i'(\theta)}{r^*[g_i(\theta)]} d\theta \\
&\qquad + 2\sum_{i=1}^{m} \frac{x^{(n-1)}[g_i(t)]}{|x'[g_i(t)/2]|} \cdot \frac{r[g_i(t)]}{r^*[g_i(t)]} g_i^{n-2}(t) \left(x\left[\frac{g_i(t)}{2}\right]\right)' .
\end{aligned}$$

Moreover, since $\lim_{t\to\infty} x(t) \neq 0$, there exists a positive constant η such that

$$\varphi(x[g_i(t)]) \geqslant \eta , \qquad i = 1, 2, \ldots, m \text{ for every } t \geqslant t_2 .$$

Thus, by applying Lemma 2 with $u = |x'|$, $m = r$, $\nu = n-1$ and $g_i(t)$ in place of t, we finally obtain

$$\tilde{z}'(t) \geqslant \eta \sum_{i=1}^{m} p_i(t) \sum_{i=1}^{m} \int_{t_2}^{t} \frac{g_i^{n-2}(\theta)g_i'(\theta)}{r^*[g_i(\theta)]} d\theta + 2k \sum_{i=1}^{m} \left(x\left[\frac{g_i(t)}{2}\right]\right)'$$

for every $t \geqslant t_3$, where $t_3 \geqslant t_2$ is chosen property, by condition (11) and the fact that the solution $x(t)$ is bounded, this inequality leads to $\lim_{t\to\infty} \tilde{z}(t) = \infty$, which contradicts (15).

We have proved by that for every nonoscillatory solution $x(t)$ of equation $(*)_{\delta=+1}$ with $\lim_{t\to\infty} x(t) = 0$ and consequently $x(t)x'(t) < 0$ for all large t. If condition (2) is satisfied, then $x(t)x^{(n-1)}(t) > 0$ for all large t and consequently n must be odd. Moreover, as it is easy to see, $\lim_{t\to\infty} x(t) = 0$ implies that $\lim_{t\to\infty} x^{(j)}(t) = 0$ for $j = 1, 2, \ldots, n-2$.

THEOREM 2. – *In addition to conditions* (i)-(iv), (2) *and* (11) *assume that for every* $\xi \neq 0$

$$\sum_{i=1}^{m} \int^{\infty} p_i(t)\varphi\left(\frac{\xi g_i^{n-2}(t)}{r^*[g_i(t)]}\right) dt = \pm\infty . \tag{16}$$

Then every solution $x(t)$ of equation $(*)_{\delta=-1}$ satisfies exactly one of the following:

(α) *$x(t)$ is oscillatory,*

(β) *$x(t)$ and its first $n-2$ derivatives tend monotonically to zero as $t\to\infty$,*

(γ) It holds

$$\lim_{t\to\infty} r(t)x^{(n-1)}(t)=\infty \qquad \textit{and} \qquad \lim_{t\to\infty} x^{(j)}(t)=\infty\,, \quad j=0,1,\dots,n-2$$

or

$$\lim_{t\to\infty} r(t)x^{(n-1)}(t)=-\infty \qquad \textit{and} \qquad \lim_{t\to\infty} x^{(j)}(t)=-\infty\,, \quad j=0,1,\dots,n-2\,.$$

Moreover, (β) occurs only in the cases of even n.

PROOF. – Let $x(t)$ be a nonoscillatory solution of equation $(*)_{\delta=-1}$ with $\lim_{t\to\infty} x(t)\neq 0$. As in the proof of Theorem 1, we may (and do) assume, without loss of generality that for $t_1\geqslant t_0$ it holds

$$x[g_i(t)]>0\,, \quad i=1,2,\dots,m \text{ for every } t\geqslant t_1\,. \tag{17}$$

Using equation $(*)_{\delta=-1}$, conditions (11) and (ii), it is easy to see that for $t_2\geqslant t_1$ we have $x^{(n-1)}(t)>0$ or $x^{(n-1)}(t)<0$ on $[t_2,\infty)$. Thus, we have the following cases:

Case 1) $x^{(n-1)}(t)>0$ on $[t_2,\infty)$.

By $[r(t)x^{(n-1)}(t)]'\geqslant 0$, $t\geqslant t_2$, we get

$$r(t)x^{(n-1)}(t)\geqslant r(t_2)x^{(n-1)}(t_2)$$

and consequently

$$x^{(n-1)}(t)\geqslant r(t_2)x^{(n-1)}(t_2)\frac{1}{r(t)} \quad \text{for every } t\geqslant t_2$$

from condition (2), implies that $\lim_{t\to\infty} x^{(n-2)}(t)=\infty$ and hence

$$\lim_{t\to\infty} x^{(j)}(t)=\infty \qquad \text{for } j=0,1,\dots,n-2\,.$$

Taking $t_3\geqslant t_2$ such that

$$x^{(j)}(t)>0\,, \quad t\geqslant t_3 \text{ for } j=0,1,\dots,n-2\,. \tag{18}$$

Applying Taylar's theorem to the function $x(t)$ we obtain

$$x(t)=\sum_{k=0}^{n-2}\frac{x^{(k)}(t/2)}{k!}(t/2)^k+\frac{x^{(n-1)}(t^*)}{(n-1)!}\frac{t^{(n-1)}}{2^{(n-1)}}\,, \tag{19}$$

where t^* is a point between $t/2$ and t, and every $t\geqslant 2t_3=t_4$.

From (18) and (19) it follows that for $t \geqslant t_4$

$$x(t) \geqslant \frac{t^{n-1}}{2^{n-1}(n-1)!} \, \frac{x^{(n-1)}(t^*)\, r(t^*)}{r(t^*)} \geqslant \frac{x^{(n-1)}(t_3)\, r(t_3)}{2^{n-1}(n-1)!} \, \frac{t^{n-1}}{r^*(t)}$$

and consequently there exists some $t_5 \geqslant t_4$ such that

$$x[g_i(t)] \geqslant \xi \frac{g_i^{n-1}(t)}{r^*[g_i(t)]}, \qquad i = 1, 2, \ldots, m, \text{ for every } t \geqslant t, \tag{20}$$

where $\xi = \big(x^{(n-1)}(t_3) r(t_3)/2^{n-1}(n-1)!\big)$. Integrating equation $(*)_{\delta=-1}$ between $[t_5, t]$ and using (20) and condition (16) we get $\lim\limits_{t\to\infty} r(t) x^{(n-1)}(t) = \infty$. Hence the solution $x(t)$ satisfies (γ).

Case 2) $x^{(n-1)}(t) < 0$ on $[t_2, \infty)$.

By considering the functions $z_* = -z$ and $\tilde{z}_* = -\tilde{z}$ respectively in place of the functions z and $\tilde{z}$ of the proof of Theorem 1 and using Lemma 2, we obtain the desired contraditions. The proof of the theorem is now obvious.

REFERENCES

[1] I. KIGURADZE, *On the oscillation of solutions of the equation* $d^m u/dt^m + \alpha(t)|u|^n \operatorname{sgn} u = 0$, Mat. Sb., **65** (1964), pp. 172-187 (Russian).

[2] I. KIGURADZE, *The problem of oscillation of solutions of nonlinear differential equations*, Differential'nye Uravnenija, **1** (1965), pp. 995-1006 (Russian).

[3] LU-SAN CHEN - CHEH-CHIH YEH, *Nonoscillation generating delay terms in odd order differential equations*, Rend. Accad. Sci. Fis. Mat. Napoli (to appear).

[4] R. S. DAHIYA, *Nonoscillation generating delay terms in even order differential equations*, Hiroshima Math. J., **5** (1975), pp. 385-394.

[5] V. ŠEVELO - N. VARĖCH, *On the oscillation of solutions of the equation* $\big(r(t)x^{(n-1)}(t)\big)' + p(t) f\big(x(\tau(t))\big) = 0$, Ukrain. Mat. Ž., **25** (1973), pp. 707-714 (Russian).

[6] Y. G. SFICAS, *On the oscillatory and asymptotic behavior of damped differential equations with retarded argument*, University of Ioannina, Technical report no. 41, September 1975.

[7] R. MOORE, *The behavior of solutions of a linear differential equation of second order*, Pacific J. Math., **5** (1955), pp. 125-145.