Asymptotic Behavior of Nonoscillatory Solutions of Nonlinear Differential Equations with Forcing Term (*).

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Sunto. – *In questo lavoro si studiano il comportamento asintotico delle soluzioni non oscillatorie di una classe di equazioni differenziali di ordine superiore al secondo. In particolare, si dánno condizioni sufficienti perchè tutte le soluzioni, non oscillatorie e limitate, dell'equazione considerata, abbiano per limite lo zero quando la variabile indipendente tende all'infinito.*

1. – Introduction.

We shall be concerned with the asymptotic behavior of nonoscillatory solutions of the differential equation

$$[r(t)\,y^{(n-m)}(t)]^{(m)} + a(t)\,f\big(y(g(t))\big) = b(t)\,, \tag{A}$$

where n, m are positive integers such that $n \geqq 2$, $0 < m < n$, $a(t)$, $b(t)$, $g(t)$, $r(t)$ are real-valued continuous functions on $[0, \infty)$, and $f(y)$ is a real-valued continuous function on $(-\infty, \infty)$. We also assume that $r(t) > 0$, $\lim\limits_{t\to\infty} g(t) = \infty$, and $yf(y) > 0$ for $y \neq 0$.

In what follows we shall confine our discussion to those solutions $y(t)$ of (A) which exist on some half-line $[T_y, \infty)$, $T_y > 0$, and satisfy

$$\sup\{|y(t)|: t_0 \leqq t < \infty\} > 0$$

for every $t_0 \in [T_y, \infty)$. Such a solution is said to be oscillatory if it has an unbounded set of zeros in its domain of existence. Otherwise, the solution is said to be nonoscillatory.

In the oscillation theory of nonlinear ordinary differential equations one of the important problems is to find sufficient conditions which guarantee that all (bounded) nonoscillatory solutions of a given equation with forcing term tend to zero as the independent variable goes to infinity. Since the work of HAMMETT [4] this problem has been the subject of numerous investigations; see, for example, ATKINSON [1].

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GRAEF and SPIKES [2], GRIMMER [3], KARTSATOS [5], KUSANO and ONOSE [6], LONDEN [7], SINGH [9], and SINGH and DAHIYA [10].

The purpose of this paper is to proceed further in this direction to provide conditions under which all bounded nonoscillatory solutions of (A) approach zero as $t \to \infty$. Our results include a generalization of a theorem of SINGH [9] for the fourth order retarded equation $y^{(4)}(t) + a(t)y(g(t)) = b(t)$.

2. – Main results.

We need the following lemma adapted from STAIKOS and SFICAS [11].

LEMMA. – *Let $u(t)$ be the solution of the differential equation*

$$u' - \frac{\alpha}{t}u + \frac{h(t)}{t} = 0 \qquad \text{on } [T, \infty) \tag{1}$$

satisfying $u(T) = 0$, where α is a positive constant and $h(t)$ is continuous on $[T, \infty)$.

If $\lim_{t\to\infty} |h(t)| = h^*$ exists in the extended real line $R^\#$, then $\lim_{t\to\infty} |u(t)| = u^*$ exists in $R^\#$. In particular, if h^* is infinite, then so is u^*.

PROOF. – The solution $u(t)$ is given by

$$u(t) = -t^\alpha \int_T^t \frac{h(s)}{s^{\alpha+1}}\, ds\,.$$

The existence of h^* implies that the improper integral

$$\lim_{t\to\infty} \int_T^t \frac{h(s)}{s^{\alpha+1}}\, ds = H^*$$

exists in $R^\#$. If $H^* \neq 0$, then clearly $\lim_{t\to\infty} |u(t)| = \infty$. If $H^* = 0$, then by l'Hospital's rule

$$\lim_{t\to\infty} |u(t)| = \lim_{t\to\infty} \left| \left(-\int_T^t \frac{h(s)}{s^{\alpha+1}}\, ds\right)' \Big/ \left(\frac{1}{t^\alpha}\right)' \right| = \frac{h^*}{\alpha}.$$

In the sequel we use the notation: $a^+(t) = \max\{a(t), 0\}$, $a^-(t) = \max\{-a(t), 0\}$.

THEOREM 1. – *Assume that there exists an integer k, $0 \leqq k \leqq m-1$, such that*

$$\int^{\infty} t^k a^-(t)\, dt < \infty , \tag{2}$$

$$\int^{\infty} t^k |b(t)|\, dt < \infty , \tag{3}$$

$$\int^{\infty} \frac{t^{m-k-1}}{r(t)}\, dt = \infty , \tag{4}$$

$$\limsup_{t\to\infty} \frac{t^{m-k-1}}{r(t)} < \infty . \tag{5}$$

Then, for every bounded nonoscillatory solution $y(t)$ of (A), *the derivatives $y^{(i)}(t)$, $i = 1, \ldots, n-m$, tend to zero as $t \to \infty$.*

PROOF. – Let $y(t)$ be a bounded nonoscillatory solution of (A). Without loss of generality we may suppose that $y(t) > 0$ for $t \geqq t_0 > 0$. There exists $t_1 \geqq t_0$ such that $g(t) \geqq t_0$ for $t \geqq t_1$. Thus, $y(g(t)) > 0$ for $t \geqq t_1$. Multiplying (A) by t^k and integrating from t_1 to t, we obtain

$$\int_{t_1}^{t} s^k[r(s)y^{(n-m)}(s)]^{(m)}\, ds + \int_{t_1}^{t} s^k a^+(s) f\big(y(g(s))\big)\, ds = \int_{t_1}^{t} s^k b(s)\, ds + \int_{t_1}^{t} s^k a^-(s) f\big(y(g(s))\big)\, ds . \tag{6}$$

We examine the following two possible cases:

Case 1.
$$\int_{t_1}^{\infty} t^k a^+(t) f\big(y(g(t))\big)\, dt = \infty ,$$

Case 2.
$$\int_{t_1}^{\infty} t^k a^+(t) f\big(y(g(t))\big)\, dt < \infty .$$

Let Case 1 hold. Since the right-hand side of (6) is bounded because of (2), (3) and the boundedness of $y(t)$, we get

$$\lim_{t\to\infty} \int_{t_1}^{t} s^k[r(s)y^{(n-m)}(s)]^{(m)}\, ds = -\infty . \tag{7}$$

Now define

$$u_i(t) = \int_{t_1}^{t} s^{k-i}[r(s)y^{(n-m)}(s)]^{(m-i)}\, ds , \qquad i = 0, 1, \ldots, k .$$

An integration by parts yields

$$u_i(t) = t^{k-i}[r(t)\,y^{(n-m)}(t)]^{(m-i-1)} - t_1^{k-i}\,[r(s)\,y^{(n-m)}(s)]_{s=t_1}^{(m-i-1)}$$

$$- (k-i)\int_{t_1}^{t} s^{k-i-1}\,[r(s)\,y^{(n-m)}(s)]^{(m-i-1)}\,ds$$

$$= t u'_{i+1}(t) - (k-i)\,u_{i+1}(t) - t_1^{k-i}\,[r(s)\,y^{(n-m)}(s)]_{s=t_1}^{(m-i-1)},$$

which shows that $u_{i+1}(t)$ is a solution of the differential equation (1) with $\alpha = k-i$ and

$$h(t) = -\,u_i(t) - t_1^{k-i}\,[r(s)\,y^{(n-m)}(s)]_{s=t_1}^{(m-i-1)}\,.$$

Obviously $u_{i+1}(t)$ satisfies the inital condition $u_{i+1}(t_1) = 0$,so that, by the lemma, it follows that $\lim_{t\to\infty} |u_{i+1}(t)| = \infty$ whenever $\lim_{t\to\infty} |u_i(t)| = \infty$. Since $\lim_{t\to\infty} |u_0(t)| = \infty$ by (7) we obtain $\lim_{t\to\infty} |u_1(t)| = \infty$, and continuing in this way, we arrive at

$$\lim_{t\to\infty} |u_k(t)| = \lim_{t\to\infty} \left| \int_{t_1}^{t} [r(s)\,y^{(n-m)}(s)]^{(m-k)}\,ds \right| = \infty,$$

which implies that

$$\lim_{t\to\infty} |[r(t)\,y^{(n-m)}(t)]^{(m-k-1)}| = \infty\,.$$

Consequently, there are positive numbers μ and $t_2 \geqq t_1$ such that

$$(8) \qquad |r(t)\,y^{(n-m)}(t)| \geqq \mu t^{m-k-1} \qquad \text{for } t \geqq t_2\,.$$

Dividing (8) by $r(t)$, integrating from t_2 to t and letting $t \to \infty$, we have by (4)

$$(9) \qquad \lim_{t\to\infty} |y^{(n-m-1)}(t)| = \infty\,, \qquad \text{and hence } \lim_{t\to\infty} |y(t)| = \infty\,.$$

This contradicts the boundedness of $y(t)$ and it follows that Case 1 is impossible.

In Case 2, letting $t \to \infty$ in (6), we see that

$$\lim_{t\to\infty} |u_0(t)| = \lim_{t\to\infty} \left| \int_{t_1}^{t} s^k [r(s)\,y^{(n-m)}(s)]^{(m)}\,ds \right|$$

exists as a finite number. Therefore, by the lemma,

$$\lim_{t\to\infty} |u_1(t)| = \lim_{t\to\infty} \left| \int_{t_1}^{t} s^{k-1} [r(s)\,y^{(n-m)}(s)]^{(m-1)}\,ds \right|$$

exists in the extended real line. This limit must be finite, since otherwise we would be led to a contradiction as in Case 1. Applying this argument repeatedly, we obtain the finite limit:

$$\lim_{t\to\infty} |u_k(t)| = \lim_{t\to\infty} \left| \int_{t_1}^{t} [r(s) y^{(n-m)}(s)]^{(m-k)} ds \right| .$$

From this we readily conclude that

$$\lim_{t\to\infty} |[r(t) y^{(n-m)}(t)]^{(m-k-1)}| = 0 . \tag{10}$$

For, if the limit (10) is not zero, then there exist positive numbers μ and t_2 such that (8) holds, and we are led to (9), an impossibility. It is a matter of elementary calculus to deduce from (10) that

$$\lim_{t\to\infty} \left| \frac{r(t)}{t^{m-k-1}} y^{(n-m)}(t) \right| = 0 ,$$

which, on account of (5), gives

$$\lim_{t\to\infty} |y^{(n-m)}(t)| = 0 . \tag{11}$$

If $n-m=1$, the proof is complete. If $n-m>1$, then since $y(t)$ is bounded and (11) holds, we are able to apply a classical inequality of Kolmogorov (cf. SCHOENBERG [8]) to conclude that $\lim_{t\to\infty} |y^{(i)}(t)| = 0$ for $i = 1, 2, \ldots, n-m-1$. This completes the proof.

From the proof of Theorem 1 it is not difficult to see that the same conclusion holds if the hypothesis (2) is replaced by

$$\int^{\infty} t^k a^+(t)\, dt < \infty . \tag{12}$$

THEOREM 2. – *Assume that there exists an integer k, $0 \leqq k \leqq m-1$, such that* (3)-(5) *and* (12) *hold.*

Then, for every bounded nonoscillatory solution $y(t)$ of (A), *the derivatives $y^{(i)}(t)$, $i = 1, \ldots, n-m$, tend to zero as $t \to \infty$.*

The following theorem is one of the main results of this paper.

THEOREM 3. – *In addition to* (2)-(5) *assume that $g'(t)$ is nonnegative and bounded above and that there exists a positive number $\delta > 0$ such that*

$$\liminf_{t\to\infty} \int_t^{t+\delta} s^k a^+(s)\, ds > 0 . \tag{13}$$

Then every bounded nonoscillatory solution of (A) *tends to zero as $t \to \infty$.*

PROOF. – Our proof is based on an adaptation of the techniques used by HAMMETT [4] and SINGH [9]. Let $y(t)$ be a bounded nonoscillatory solution of (A) such that $y(g(t)) > 0$ for $t \geqq T$. A parallel argument holds if $y(t)$ is assumed to be eventually negative. From the proof of Theorem 1 we must have

$$\int_T^\infty t^k a^+(t) f\big(y(g(t))\big)\, dt < \infty . \tag{14}$$

Note that (13) implies

$$\int_T^\infty t^k a^+(t)\, dt = \infty . \tag{15}$$

From (14) and (15) it follows that

$$\liminf_{t\to\infty} y\big(g(t)\big) = \liminf_{t\to\infty} y(t) = 0 , \tag{16}$$

and so it remains to show that

$$\limsup_{t\to\infty} y\big(g(t)\big) = \limsup_{t\to\infty} y(t) = 0 . \tag{17}$$

If this is not the case, then there exists $\eta > 0$ such that

$$\limsup_{t\to\infty} y\big(g(t)\big) > 2\eta > 0 . \tag{18}$$

In view of (16) and (18) there is an increasing sequence of numbers $\{t_\nu\}_{\nu=1}^\infty$ with the following properties:

(i) $\lim\limits_{\nu\to\infty} t_\nu = \infty$;

(ii) For each ν, $y(g(t_\nu)) > 2\eta$;

(iii) For each ν, there exists a number t'_ν such that $t_{\nu-1} < t'_\nu < t_\nu$ and $y(g(t'_\nu)) < \eta$.

Let σ_ν be the largest number less than t_ν such that $y(g(\sigma_\nu)) = \eta$, and let τ_ν be the smallest number larger than t_ν such that $y(g(\tau_\nu)) = \eta$. By the mean value theorem there exists, for each ν, a $\sigma'_\nu \in (\sigma_\nu, t_\nu)$ such that

$$y'\big(g(\sigma'_\nu)\big) g'(\sigma'_\nu) = \frac{y\big(g(t_\nu)\big) - y\big(g(\sigma_\nu)\big)}{t_\nu - \sigma_\nu} > \frac{\eta}{\tau_\nu - \sigma_\nu} . \tag{19}$$

Since $g'(t)$ is bounded and $\lim\limits_{t\to\infty} y'(g(t)) = 0$ by Theorem 1, it follows from (19) that

$$\lim_{\nu\to\infty} (\tau_\nu - \sigma_\nu) = \infty . \tag{20}$$

By the definition of σ_ν and τ_ν, $y(g(t)) \geqq \eta$ on $[\sigma_\nu, \tau_\nu]$ and hence there is a constant $c(\eta) > 0$ such that

$$f\big(y(g(t))\big) \geqq c(\eta) \quad \text{on } [\sigma_\nu, \tau_\nu]. \tag{21}$$

Using (13), (14), (20) and (21), we finally obtain

$$\infty > \int_T^\infty t^k a^+(t) f\big(y(g(t))\big)\, dt \geqq \sum_{\nu=1}^\infty \int_{\sigma_\nu}^{\tau_\nu} t^k a^+(t) f\big(y(g(t))\big)\, dt > c(\eta) \sum_{\nu=1}^\infty \int_{\sigma_\nu}^{\tau_\nu} t^k a^+(t)\, dt = \infty .$$

This, however, is a contradiction and (17) follows. Thus the proof is complete.

On the basis of Theorem 2 we can prove in a similar way the following theorem.

THEOREM 4. – *In addition to* (3)-(5) *and* (12) *assume that* $g'(t)$ *is nonnegative and bounded above and that there exists a positive number* $\delta > 0$ *such that*

$$\liminf_{t\to\infty} \int_t^{t+\delta} s^k a^-(s)\, ds > 0 . \tag{22}$$

Then every bounded nonoscillatory solution of (A) *tends to zero as* $t \to \infty$.

When specialized to the case $r(t) \equiv 1$, Theorems 3 and 4 give the following nonoscillation result for the equation

$$y^{(n)}(t) + a(t) f\big(y(g(t))\big) = b(t) . \tag{B}$$

COROLLARY. – *Suppose* $g'(t)$ *is nonnegative and bounded above. Every bounded nonoscillatory solution of* (B) *tends to zero as* $t \to \infty$ *if there exist an integer* k, $0 \leqq k \leqq n-2$, *and a positive number* δ *such that either*

$$\int^\infty t^k a^-(t)\, dt < \infty , \qquad \int^\infty t^k |b(t)|\, dt < \infty , \qquad \liminf_{t\to\infty} \int_t^{t+\delta} s^k a^+(s)\, ds > 0 ,$$

or

$$\int^\infty t^k a^+(t)\, dt < \infty , \qquad \int^\infty t^k |b(t)|\, dt < \infty , \qquad \liminf_{t\to\infty} \int_t^{t+\delta} s^k a^-(s)\, ds > 0 .$$

REMARK. – This corollary generalizes a recent result of SINGH [9] for the fourth order case.

We conclude with examples which illustrate our principal results.

EXAMPLES. – 1) Consider the equation

$$(t^2y''(t))^{(4)} + t^{-2}y^3(\gamma t) = (48 + \gamma^{-3})t^{-5}\,, \tag{23}$$

where γ is a positive constant (possibly greater than 1). Here $n = 6$, $m = 4$, $f(y) = y^3$, $g(t) = \gamma t$,

$$r(t) = t^2\,, \quad a^+(t) = t^{-2}\,, \quad a^-(t) = 0 \quad \text{and} \quad b(t) = (48 + \gamma^{-3})t^{-5}\,.$$

All the conditions of Theorem 3 are satisfied with $k = 2$ Hence all bounded nonoscillatory solutions of (23) tend to zero as $t \to \infty$. In fact, $y(t) = t^{-1}$ is a bounded nonoscillatory solution of (23).

2) Consider the equation

$$(ty'''(t))''' - t^{-1}y(\log t) = 32(2t - 3)\exp[-2t] - t^{-3}\,. \tag{24}$$

Here $n = 6$, $m = 3$, $f(y) = y$, $g(t) = \log t$,

$$r(t) = t\,, \quad a^+(t) = 0\,, \quad a^-(t) = t^{-1} \quad \text{and} \quad b(t) = 32(2t - 3)\exp[-2t] - t^{-3}\,,$$

and the conditions of Theorem 4 are satisfied with $k = 1$. It follows that all bounded nonoscillatory solutions of (24) tend to zero as $t \to \infty$. This equation has a bounded nonoscillatory solution $y(t) = \exp[-2t]$.

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