On Holomorphically Subprojective Kählerian Manifold, I (*).

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Summary. – *See the Introduction.*

0. – Introduction.

In an n-dimensional affinely coonnected manifold A_n is said to be k-fold projectiv if there exists a coordinate system with respect to which every geodesic can be given by means of k linear equations and $n-k-1$ equations that need not be linear. For $k=n-2$ it may happen that there exists a coordinate system such that every geodesic is given with respect to this system by $n-2$ homogeneous linear equations and one other equation that need not be linear. Then such A_n is called a subprojective manifold by B. KAGAN [3]. In a subprojective manifold A_n, a geodesic lies on a two-dimensional surface whose equations are given by then form

$$x^h = \alpha^h x^{n-1} + \beta^h x^n \qquad (h = 1, 2, \dots, n-2) \tag{0, 1}$$

for a suitable coordinate system (x^i) $(i = 1, 2, \dots, n)$, where α^h and β^h are constants. From (0.1) we find that the affine connection Γ^i_{jk} $(i, j, k, \dots, = 1, 2, \dots, n)$ takes the form

$$\Gamma^i_{jk} = \varphi_j \delta_k^{\,i} + \varphi_k \delta_j^{\,i} + \varphi_{jk} x^i \tag{0.2}$$

where φ_j and φ_{jk} are any covariant vector and symmetric tensor respectively. Conversely, if the affine connection is given by (0.2) for a suitable coordinate system, we can conclude that A_n is a subprojective manifold.

As a necessary and sufficient condition that a Riemannian manifold be subprojective P. RACHEVSKY introduced relations

$$R_{kjih} = T_{ih} g_{jk} + T_{jk} g_{ih} - T_{ik} g_{jh} - T_{jh} g_{ik}\,, \tag{A}$$

$$\nabla_j T_{ik} - \nabla_i T_{jk} = 0\,, \tag{A)'$$

$$T_{ij} = \varrho g_{ji} + \varrho_i \sigma_j\,, \tag{B}$$

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where ∇ means the operator of covariant derivation with respect to Riemannian connection defined by the Riemannian metric tensor g_{ij}, and we put

$$T_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)} g_{ij}\right),$$

$$\varrho_i = \partial_i \varrho , \qquad \sigma_i = \partial_i \sigma$$

and σ is a function of ϱ. However the conditions (A) and $(A)'$ express that the manifold is conformally flat, because (A) may be written as $C_{kji}{}^h = 0$, where $C_{kji}{}^h$ is the Weyl conformal curvature tensor.

Recently many authors have dealt with the Bockner curvature tensor of a Kählerian manifold as a curvature tensor which corresponds to the Weyl conformal curvature tensor of a Riemannian manifold and obtained the corresponding interesting theorems for a Kählerian manifold with vanishing Bochner curvature tensor to some results for a conformally flat manifold. As for such problems, Professor S. TACHIBANA has suggested one of present authors (S. YAMAGUCHI) the following question: « Can you complexify the definition of a subprojective Riemannian manifold so that it fits for Kählerian manifolds and obtain the corresponding theorems for such Kählerian manifolds to some results for subprojective Riemannian manifolds? ».

The purpose of this paper is to give an answer to this problem, taking account of the Bochner curvature tensor instead of the Weyl conformal curvature tensor. In § 1 we shall recall a Kählerian manifold M with complex coordinate systems and a holomorphically planar curve in M. The definition of a holomorphically subprojective Kählerian manifold will be given in § 2, and moreover we shall seek the necessary and sufficient condition for the Christoffel symbols of M in order that M is holomorphically subprojective. The notion of the Bochner curvature tensor introduced by S. TACHIBANA in real coordinate systems will be remembered in § 3. In § 4, we shall calculate the curvature tensor from the Christoffel symbols of a holomorphically subprojective Kählerian manifold with respect to a suitable real coordinate system and the identities which are necessary for what follows. The last section has proved the main theorem of this paper, that is, we shall show under certain condition that a holomorphically subprojective Kählerian manifold is a Kählerian manifold with vanishing Bochner curvature tensor.

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1. – Holomorphically planar curve.

In the first place, we agree to adopt the summation convention and the following ranges of indices throghout the paper: $h, i, j, \ldots, r, s, \ldots = 1, 2, \ldots, 2n$; $\lambda, \mu, \nu, \ldots = 1, 2, \ldots, n$; $\bar{\lambda} = n + \lambda$; $\alpha, \beta, \gamma = 1, 2, \ldots, n-2$. Consider an n complex dimensional Kählerian manifold M with metric

$$ds^2 = g_{ij} dz^i dz^j , \tag{1.1}$$

where (z^λ) is a local complex coordinate system and $z^{\bar\lambda} = \bar z^\lambda$ (= conjugate of z^λ). As the metric is Kählerian, g_{ij} satisfy the following conditions:

$$g_{\lambda\mu} = g_{\bar\lambda\bar\mu} = 0 , \quad g_{\lambda\bar\mu} = g_{\bar\mu\lambda} = \bar g_{\bar\lambda\mu} = \bar g_{\mu\bar\lambda} \tag{1.2}$$

and (1.1) becomes $ds^2 = 2g_{\lambda\bar\mu} dz^\lambda d\bar z^\mu$. g^{ji} satisfy the corresponding equations to (1.2). The Christoffel symbols Γ^i_{jk} vanish except $\Gamma^\lambda_{\mu\nu}$ and their conjugate.

We consider a curve in a Kählerian manifold M defined by parametric representation in a real parameter $z^\lambda = z^\lambda(t)$. If the curve satisfies

$$\frac{d^2 z^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \frac{dz^\mu}{dt} \frac{dz^\nu}{dt} = \varrho(t) \frac{dz^\lambda}{dt} , \tag{1.3}$$

in which $\varrho(t)$ is not real-valued, but complex-valued in general, we say the curve to be holomorphically planar [5[.

2. – Holomorphically subprojective Kählerian manifolds.

Let M be an n complex dimensional Kählerian manifold. In M if there exist a complex coordinate system such that every holomorphically planar curve is given with respect to this system by $n-2$ homogeneous linear equations and one other equation that need not be linear, then we shall call M a holomorphically subprojective Kählerian manifold. We assume that M is a holomorphically subprojective manifold in this section. In M, a holomorphically planar curve lies on a complex two-dimensional surface whose equations are given by the form

$$z^\alpha = a^\alpha z^{n-1} + b^\alpha z^n \tag{2.1}$$

for a suitable complex coordinate system (z^λ), where a^α and b^α are complex constants. We shall call this complex coordinate system a special complex coordinate system.

Now, we shall calculate the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ of M from (2.1). Operating d/dt and d^2/dt^2 to (2.1) respectively, we have

$$\begin{aligned} dz^\alpha/dt &= a^\alpha(dz^{n-1}/dt) + b^\alpha(dz^n/dt) , \\ d^2z^\alpha/dt^2 &= a^\alpha(d^2z^{n-1}/dt^2) + b^\alpha(d^2z^n/dt) , \end{aligned} \tag{2.2}$$

from which, by virtue of (2.1) and (2.2) we find

$$d^2z^\lambda/dt^2 = \alpha(t)\, dz^\lambda/dt + \beta(t)\, z^\lambda , \tag{2.3}$$

where $\alpha(t)$ and $\beta(t)$ are complex-valued. Consequently, taking account of (1.3) and (2.3), it holds that

$$\Gamma^{\gamma}_{\mu\nu}\xi^{\mu}\xi^{\nu} = \gamma(t)\,\xi^{\lambda} + 2(t)\,z^{\lambda}\,, \tag{2.4}$$

where we have put $\xi^{\mu} = dz^{\mu}/dt$ and $\gamma(t)$ and $\varepsilon(t)$ are complex-valued. If we assume that $\dim M = n \geqq 3$, then by the same method of B. KAHAN [3], we have for $\Gamma^{\lambda}_{\mu\nu}$ an expression of the form

$$\Gamma^{\lambda}_{\mu\nu} = \varrho_{\mu}\delta^{\lambda}_{\nu} + \varrho_{\nu}\,\delta^{\lambda}_{\mu} + f_{\mu\nu}z^{\lambda}\,, \qquad f_{\mu\nu} = f_{\nu\mu} \tag{2.5}$$

with respect to all special complex cordinate systems. If we apply the coordinate transformation

$$z^{\lambda'} = \varrho(z^{\lambda})\,P^{\lambda'}_{\mu}\,z^{\mu}\,, \qquad P^{\lambda'}_{\mu} = \text{complex constant,} \tag{2.6}$$

then we have

$$A^{\lambda'}_{\mu} = \partial z^{\lambda'}/\partial z^{\mu} = \varrho P^{\lambda'}_{\mu} + z^{\lambda'}(\partial \log \varrho/\partial z^{\mu})\,,$$

$$\begin{aligned}\partial A^{\lambda'}_{\mu}/\partial z^{\nu} = {} & (\partial\varrho/\partial z^{\nu})\,P^{\lambda'}_{\mu} + \varrho P^{\lambda'}_{\mu}(\partial \log \varrho/\partial z^{\mu}) \\ & + z^{\lambda'}(\partial \log \varrho/\partial z^{\mu})(\partial \log \varrho/\partial z^{\nu}) + z^{\lambda'}(\partial^{2} \log \varrho/\partial z^{\mu}\,\partial z^{\nu})\end{aligned}$$

and

$$\Gamma^{\lambda'}_{\mu'\nu'} = \varrho_{\mu'}\delta^{\lambda'}_{\mu'} + \varrho_{\nu'}\,\delta^{\lambda'}_{\mu'} + f_{\mu'\nu'}z^{\lambda'}$$

with

$$\varrho_{\lambda'} = A^{\mu}_{\lambda'}\varrho_{\mu} - \partial \log \varrho/\partial z^{\lambda'}\,,$$

$$f_{\mu'\nu'} = (1 + z^{\lambda}(\partial \log \varrho/\partial z^{\lambda}))\,(A^{\omega k}_{\mu'\nu'}f_{\omega k} - \varrho^{-1}(\partial^{2}\varrho/\partial z^{\mu'}\partial z^{\nu'}))\,.$$

From these equations we see that a transformation of the form (2.6) transforms a special coordinate system to a special one with same origine. The ϱ_{λ} and $f_{\mu\nu}$ do not behave like a vector field or a tensor field resprectively and each of them forms a geometric object for transformations of the form (2.6) only. It is often convenient to consider a vector field and a symmetric tensor field defined by $u_{\lambda} = \varrho_{\lambda}$ and $u_{\lambda\mu} = f_{\lambda\mu}$ in a special complex coordinate system (z^{λ}) but these fields depend on the choice of (z^{λ}) and they change if another special complex coordinate system is introduced instead of (z^{λ}). We can easily prove from (2.5) that the vector ϱ_{λ} and the symmetric tensor $f_{\mu\nu}$ are self-conjugate.

Conversely, substituting (2.5) into (1.3), we get

$$\begin{aligned} & d^{2}z^{\alpha}/dt^{2} + p(t)(dz^{\alpha}/dt) + q(t)\,z^{\alpha} = 0\,, \\ & d^{2}z^{n-1}/dt^{2} + p(t)(dz^{n-1}/dt) + q(t)\,z^{n-1} = 0\,, \\ & d^{2}z^{n}/dt^{2} + p(t)(dz^{n}/dt) + q(t)\,z^{n} = 0\,, \end{aligned} \tag{2.7}$$

from which, eliminating $p(t)$ and $q(t)$ from (2.7), we get

$$\det\begin{pmatrix} d^2z^\alpha/dt^2 & dz^\alpha/dt & z^\alpha \\ d^2z^{n-1}/dt^2 & dz^{n-1}/dt & z^{n-1} \\ d^2z^n/dt^2 & dz^n/dt & z^n \end{pmatrix} = 0$$

and consequantly we may put

$$A^\alpha(t)z^\alpha + B^\alpha(t)z^{n-1} + C^\alpha(t)z^n = 0\,, \tag{2.8}$$

$$A^\alpha(t)\,dz^\alpha/dt + B^\alpha(t)\,dz^{n-1}/dt + C^\alpha(t)\,dz^n/dt = 0\,, \tag{2.9}$$

$$A^\alpha(t)\,d^2z^\alpha/dt^2 + B^\alpha(t)\,d^2z^{n-2}/dt^2 + C^\alpha(t)\,d^2z^n/dt^2 = 0\,, \tag{2.10}$$

from which, operating d/dt to (2.8) and compering with (2.9), we can find

$$dA^\alpha/dt = \psi(t)A^\alpha\,, \qquad dB^\alpha/dt = \psi(t)B^\alpha\,, \qquad dC^\alpha/dt = \psi(t)C^\alpha\,,$$

that is,

$$A^\alpha = k^\alpha \exp\left[\int\psi(t)\,dt\right], \qquad B^\alpha = l^\alpha \exp\left[\int\psi(t)\,dt\right], \qquad C^\alpha = m^\alpha \exp\left[\int\psi(t)\,dt\right],$$

where k^α, l^α and m^α are complex constants. Substituting these into (2.8), it follows that

$$z^\alpha = \beta^\alpha z^{n-1} + \gamma^\alpha z^n$$

and hence, this means that M is holomorphically subprojective.

Therefore we have

THEOREM 1. – *A Kählerian manifold of complex dimension n ($\geqq 3$) is holomorphically subprojective if and only if there exists a local coordinate system (z^λ) such that the Christoffel symbol $\Gamma^\lambda_{\mu\nu}$ of M takes the form*

$$\Gamma^\lambda_{\mu\nu} = \varrho_\mu\delta^\lambda_\nu + \varrho_\nu\delta^\lambda_\mu + f_{\mu\nu}z^\lambda\,,$$

where ϱ_λ and $f_{\mu\nu}$ are self-conjugate vector and symmetric tensor respectively.

3. – Bochner curvature tensor.

Let M be a real $2n$-dimensional Kahlerian manifold with complex structure J and Riemannian metric g which satisfy the followings:

$$J_j{}^rJ_r{}^i = -\delta_j{}^i\,, \qquad J_j{}^rJ_i{}^sg_{rs} = g_{ji}\,, \qquad \nabla_jJ_i{}^h = 0\,,$$

∇ being the operator of covariant derivation with respect to Riemannian connection defined by g.

We denote by $R_{kji}{}^h$ the Riemannian curvature tensor

$$R_{kji}{}^h = \partial_k \begin{Bmatrix} h \\ ij \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ki \end{Bmatrix} + \begin{Bmatrix} h \\ kr \end{Bmatrix}\begin{Bmatrix} r \\ ji \end{Bmatrix} - \begin{Bmatrix} h \\ jr \end{Bmatrix}\begin{Bmatrix} r \\ ki \end{Bmatrix}, \qquad \partial_k = \partial/\partial x^k ,$$

where (x^i) denotes real coordinate systems and by R_{ji}, R the Ricci tensor, the scalar curvature respectively.

Now we shall consider a tensor $K_{kji}{}^h$ defined by

$$(3.1) \qquad K_{kji}{}^h = R_{kji}{}^h + \frac{1}{2(n+2)}(L_{ki}\delta_j^h - L_{ji}\delta_k^h + g_{ki}L_j{}^h - g_{ji}L_k{}^h + M_{ki}J_j{}^h - M_{ji}J_k{}^h + J_{ki}M_j{}^h - J_{ji}M_k{}^h + 2M_{kj}J_i{}^h + 2J_{kj}M_i{}^h),$$

where we have put

$$(3.2) \qquad L_{ji} = R_{ji} - \frac{R}{4(n+1)}g_{ji}, \qquad M_{jr} = J_j{}^r L_{ri} .$$

Then we can prove that the tensor $K_{kji}{}^h$ has components of the tensor given by S. BOCHNER with respect to complex coordinate systems. This tensor is introduced in real coordinate systems by S. TACHIBANA [9] and called the Bochner curvature tensor.

4. – Identities of a holomorphically subprojective Kählerian manifold.

In a real $2n$ $(n \geqq 3)$ dimensional Kählerian manifold M, by making use of Theorem 1, we can easily prove the following:

THEOREM 1'. – *In order that M is holomorphically subprojective, it is necessary and sufficient that there exists a local coordinate system (x^h) such that the Christoffel symbol $\begin{Bmatrix} h \\ j\ i \end{Bmatrix}$ of M takes the form*

$$(4.1) \qquad \begin{Bmatrix} h \\ j\ i \end{Bmatrix} = \varrho_{(j}\,\varrho_{i)}{}^h + \tilde{\varrho}_{(j}J_{i)}{}^h + f_{ji}x^h - f_{ji}J_i{}^r\tilde{x}^h ,$$

$$(4.2) \qquad f_{[ji]} = 0 , \quad f_{r[j}J_{ri]}{}^r = 0 ,$$

where ϱ_i and f_{ji} are covariant vector and tensor respectively and $\tilde{x}^h = J_r{}^h x^r$ and $\tilde{\varrho}_i = -J_i{}^r\varrho_r$ and (ij) resp. $[ij]$) means the symmetric part (resp. skew-symmetric part) with respect to i and j, for example

$$u_{(ji)} = u_{ji} + u_{ij} \quad (\text{resp. } u_{[ji]} = u_{ji} - u_{ij}) .$$

Now we calculate the curvature tensor with respect to (4.1). Then, by a straightforward and rather complicated computations, we obtain

$$R_{kji}{}^{h}=-P_{[kj]}\delta_i^h+H_{i[k}\delta_{j]}^h+Q_{kji}x^h+J_i^hP_{r[j}J_{k]}{}^r-J_i{}^rH_{r[k}J_{j]}{}^h-J_i{}^rQ_{kjr}\tilde{x}^h \tag{4.3}$$

or

$$R_{kjih}=-P_{[kj]}g_{ih}+H_{i[k}g_{j]h}+Q_{kji}x_h+J_{ih}P_{r[j}J_{k]}{}^r-J_i{}^rH_{r[k}J_{j]h}-J_i{}^rQ_{kjr}\tilde{x}_h\,, \tag{4.3)'}$$

where we have put

$$P_{ik}=\partial_k\varrho_i-\varrho_i\varrho_k+\tilde{\varrho}_i\tilde{\varrho}_k\,, \tag{4.4}$$

$$F_{ik}=-(1+\varrho')f_{ik}+\varrho''J_i{}^rf_{rk}\,,\qquad \varrho'=\varrho_rx^r,\ \varrho''=\varrho_r\tilde{x}^r\,, \tag{4.5}$$

$$H_{ik}=P_{ik}+F_{ik}\,, \tag{4.6}$$

$$Q_{kji}=\partial_{[k}f_{j]i}+f'_{[k}f_{j]r}-f''_{[k}f_{j]r}J_i{}^r\,,\qquad f'_k=f_{kr}x^r,\ f''_k=f_{kr}\tilde{x}^r\,. \tag{4.7}$$

By virtue of $(4.2)\sim(4.7)$, we can easily prove the followings which are necessary for what follows.

$$P_{r[j}J_{k]}{}^r=H_{r[j}J_{k]}{}^r\,,\qquad -(n+1)P_{[kj]}+Q_{kjr}x^r=0\,, \tag{4.8}$$

$$Q_{(kj)i}=0\,,\qquad Q_{[kji]}=0\,,\qquad J_{[i}{}^rQ_{kj]r}=0\,. \tag{4.9}$$

In the first place, let us prove that $P_{[kj]}=0$, that is, ϱ_i is closed in M. Making use of $R_{kj(ih)}=0$, it holds from (4.3), that

$$2H_{[jk]}g_{hi}+H_{(h|[k}g_{j]|i)}+Q_{kj(i}x_{h)}-H_{r[k}J_{j](h}J_{i)}{}^r-Q_{kjr}J_{(i}{}^r\tilde{x}_{h)}=0 \tag{4.10}$$

or

$$2H_{[jk]}g^{hi}+H^{(h}{}_{[k}\delta_{j]}^{i)}+Q_{kj}{}^{(i}x^{h)}-H_{r[k}J_{j]}^{(h}J^{i)r}+Q_{kjr}J^{r(i}\tilde{x}^{h)}=0\,. \tag{4.10)'}$$

Hence we can take four unit vectors y^h, $\tilde{y}^h$, z^h and $\tilde{z}^h$ which are orthogonal each other and also to x^h and $\tilde{x}^h$, being $2n\geqq 6$. Mulitiplying y^hz^i to (4.10) and summing for h and i, we find

$$z^rH_{r[k}y_{j]}+y^rH_{r[k}x_{j]}+\tilde{z}^rH_{r[k}\tilde{y}_{j]}+\tilde{y}^rH_{r[k}\tilde{z}_{j]}=0 \tag{4.11}$$

and furthermore, contracting this with y^j, z^j, $\tilde{y}^j$ and $\tilde{z}^j$ respectively

$$z^rH_{rk}=y_kH(z,y)+z_kH(y,y)+\tilde{y}_kH(\tilde{z},y)+\tilde{z}_kH(\tilde{y},y)\,, \tag{4.12}$$

$$y^rH_{rk}=y_kH(z,z)+z_kH(y,z)+\tilde{y}_kH(\tilde{z},z)+\tilde{z}_kH(\tilde{y},z)\,, \tag{4.13}$$

$$\tilde{z}^rH_{rk}=y_kH(z,\tilde{y})+z_kH(y,\tilde{y})+\tilde{y}_kH(\tilde{z},\tilde{y})+\tilde{z}_kH(\tilde{y},\tilde{y})\,, \tag{4.14}$$

$$\tilde{y}^rH_{rk}=y_kH(z,\tilde{z})+z_kH(y,\tilde{z})+\tilde{y}_kH(\tilde{z},\tilde{z})+\tilde{z}_kH(\tilde{y},\tilde{z})\,, \tag{4.15}$$

where we have put $H(y,z)=y^r z^s H_{rs}$ and etc. On the other hand, if we contract (4.10) with $y^i y^h$ and $z^i z^h$ respectively, it follows that

$$-H_{[kj]}+y^r H_{r[k}y_{j]}+\tilde{y}^r H_{r[k}\tilde{y}_{j]}=0\,, \tag{4.16}$$

$$-H_{[kj]}+z^r H_{r[k}z_{j]}+\tilde{z}^r H_{r[k}\tilde{z}_{j]}=0 \tag{4.17}$$

and, making use of (4.13), (4.15) and (4.16), we get

$$\begin{aligned}-H_{[kj]}+y_{[j}\tilde{y}_{k]}(H(\tilde{z},z)-H(z,\tilde{z}))+y_{[j}z_{k]}H(y,z)\\ +y_{[j}\tilde{z}_{k]}H(\tilde{y},z)+\tilde{y}_{[j}\tilde{z}_{k]}H(\tilde{y},\tilde{z})+\tilde{y}_{[j}z_{k]}H(y,\tilde{z})=0\,.\end{aligned} \tag{4.18}$$

Similarly we can readily obtain from (4.12), (4.14) and (4.17)

$$\begin{aligned}-H_{[kj]}+z_{[j}\tilde{z}_{k]}(H(\tilde{y},y)-H(y,\tilde{y}))+z_{[j}y_{k]}H(z,y)\\ +z_{[j}\tilde{y}_{k]}H(\tilde{z},y)+\tilde{z}_{[j}y_{k]}H(z,\tilde{y})+\tilde{z}_{[j}\tilde{y}_{k]}H(\tilde{z},\tilde{y})=0\end{aligned} \tag{4.19}$$

and hence, the equations (4.18) and (4.19) mean that

$$\begin{aligned}&H(\tilde{y},y)=H(y,\tilde{y})\,, &&H(\tilde{z},z)=H(z,\tilde{z})\,,\\ &H(y,z)+H(z,y)=0\,, &&H(\tilde{y},z)+H(z,\tilde{y})=0\,,\\ &H(\tilde{y},\tilde{z})+H(\tilde{z},\tilde{y})=0\,, &&H(y,\tilde{z})+H(\tilde{z},y)=0\,.\end{aligned}$$

By transvecting $y^k z^j$, (4.18) and (4.19) imply by virtue of the above equations

$$H(y,z)=H(z,y)=H(y,\tilde{z})=H(\tilde{y},\tilde{z})=0\,,$$

which show that $H_{[jk]}=0$ or $P_{[jk]}=0$. Therefore the equation (4.8) may be rewritten as follows:

$$Q_{kjr}x^r=0\,,\qquad P_{[jk]}=0\,. \tag{4.20}$$

5. Main Theorem.

The purpose of this section is to prove the our main Theorem after some complicated computations. In the first step, making use of (4.3) and (4.3′), we shall derive three kinds of the form of the Ricci tensor R_{jk} in a holomorphically subprojective Kählerian manifold M. By contraction over h and k in (4.3), it holds that

$$R_{ij}=-2nP_{ij}-2(n-1)F_{ij}-2(\phi P)_{ij}+Q_{rji}x^r-J_i{}^r Q_{sjr}\tilde{x}^s\,, \tag{5.1}$$

where we have put $(\phi P)_{ij} = J_i{}^r J_j{}^s P_{rs}$ and used (4.5) and (4.6). Transvecting g^{ij} to (4.3)′, we find

$$R_{kh} = -H_r{}^r g_{kh} - P_{kh} - (\phi P)_{kh} + Q_{kr}{}^r x_h - J^{rs} Q_{krs} \tilde{x}_h \,, \tag{5.2}$$

and, interchanging k and h in (5.2) and subtracting the equation thus obtained from (5.2), it follows that

$$Q_{kr}{}^r x_h - Q_{hr}{}^r x_k = J^{rs}(Q_{krs} \tilde{x}_h - Q_{hrs} \delta_k) \,,$$

from which we get

$$Q_{kr}{}^r = \lambda x_k \,, \qquad Q_{krs} J^{rs} = \mu \tilde{x}_k \,, \tag{5.3}$$

where we have used (4.9) and put $|x|^2 \lambda = Q_{kr}{}^r x^k$ and $|x|^2 \mu = Q_{krs} J^{rs} \tilde{x}^k$. Finally if we contract (4.3) with $J_h{}^i$ and consider the well known relation $R_{kj} = -\frac{1}{2} J_j{}^t R_{ktrs} J^{rs}$, then we can obtain

$$R_{kj} = -(n+1)\big(P_{kj} + (\phi P)_{kj}\big) + J_j{}^t Q_{ktr} \tilde{x}^r \,. \tag{5.4}$$

In the second step, let us determine the form of H_{jk}. Subtracting (5.1) from (5.4) and regarding to (4.9) and (4.20), we can readily find

$$(n-1)\overline{H}_{jk} - Q_{rkj} x^r - J_j{}^s Q_{srk} \tilde{x}^r = 0 \,,$$

which shows that

$$\overline{H}_{jr} x^r = 0 \tag{5.5}$$

with $\overline{H}_{ij} = H_{ij} - (\phi H)_{ij}$. Now if the identity of Bianchi is applied to (4.3), with respect to indices k, j and h, then we have

$$x_{[h} Q_{kj]i} - P_{r[j} J_{k}{}^r J_{h]i} - 2 J_i{}^r H_{r[k} J_{jh]} - \tilde{x}_{[h} Q_{kj]r} J_i{}^r = 0$$

from which, contracting this with J^{ih} and $J^{ih} J^{kj}$ respectively, it holds that

$$2 Q_{kjr} \tilde{x}^r = (\lambda - \mu) \tilde{x}_{[j} x_{k]} + 2n P_{r[j} J_{k]}{}^r - 2 H_r{}^r J_{kj} \,, \tag{5.6}$$

$$(\lambda + \mu)|x|^2 = 2n F_r{}^r \,, \tag{5.7}$$

because of (5.3). By the way, transvecting (4.10), by x^h and $x^h x^i$ respectively, it follows that

$$H_{i[k} x_{j]} + x^r H_{r[k} g_{j]i} + |x|^2 Q_{kji} + J_i{}^r H_{r[k} \tilde{x}_{j]} - \tilde{x}^r H_{r[k} J_{j]i} - Q_{kjr} \tilde{x}^r \tilde{x}_i = 0 \,, \tag{5.8}$$

$$x^r H_{r[k} x_{j]} + \tilde{x}^r H_{r[k} \tilde{x}_{j]} = 0 \,, \tag{5.9}$$

because of (4.9) and (4.20), and furthermore, contracting (5.9) with x^j and $\tilde{x}^j$ respectively, we have

$$x^r H_{rk} = a x_k + b \tilde{x}_k , \qquad \tilde{x}^r H_{rk} = b x_k + c \tilde{x}_k$$

with $|x|^2 a = H(x, x)$, $|x|^2 b = H(\tilde{x}, x)$ and $|x|^2 c = H(\tilde{x}, \tilde{x})$. Regarding to (5.5), we have $a = c$ and $b = 0$. Consequently, the above equations can be reduced to

$$x^r H_{rk} = a x_k , \qquad \tilde{x}^r H_{rk} = a \tilde{x}_k . \tag{5.10}$$

From (4.10), we can find

$$\delta_{[m}^{(l} Q_{kj]}{}^{(i} x^{h))} + \delta_{[m}^{(l} H_{[k}^{|r|} J_{j]}^{(h} J_r{}^{i))} + \delta_{[m}^{(l} Q_{kj]}^{|r|} J^{(i} x^{h))} = 0$$

being $\delta_{[m}^{(l} H_{[k}^{(i} \delta_{j]}^{h))} = 0$. Putting $l = m$ in this, then we have

$$\begin{aligned} 2n[Q_{kj}{}^{(i} x^{h)} + Q_{kj}{}^r J_r^{(i} \tilde{x}^{h)} + H_{[k}^r J_{j]}^{(h} J_r{}^{i)}] + \lambda \delta_{[k}^{(i} x_{j]} x^{h)} \\ + \delta_{[k}^{(i} [Q_{j]r} x^{h)} - \bar{H}_{j]}^{h)} + Q_{j]m}{}^{|r|} J_r{}^{h)} \tilde{x}^m + Q_{j]m}{}^{|r|} J_r{}^{|m|} \tilde{x}^{h)}] = 0 , \end{aligned} \tag{5.11}$$

which yields the followings:

$$\begin{aligned} 2n[|x|^2 Q_{kji} - Q_{kjr} \tilde{x}^r \tilde{x}_i - J_{ri} H_{[k}^r \tilde{x}_{j]} - a \tilde{x}_{[k} J_{j]i}] + g_{i[k} (\lambda |x|^2 x_{j]} - Q_{j]mr} \tilde{x}^m \tilde{x}^r) \\ + x_{[k} (Q_{j]ri} x^r - \bar{H}_{j]i} + Q_{j]m}{}^r J_{ri} \tilde{x}^m - \mu \tilde{x}_{j]} \tilde{x}_i) = 0 , \end{aligned} \tag{5.12}$$

$$\begin{aligned} (2n-1)|x|^2 Q_{kri} x^r - 2n Q_{krs} x^r \tilde{x}^s \tilde{x}_i \\ + (\lambda |x|^4 + Q_{mjr} \tilde{x}^m x^j \tilde{x}^r) g_{ki} - x_i (\lambda |x|^2 x_k - Q_{kmr} \tilde{x}^r \tilde{x}^m) \\ + x_k Q_{jm}{}^r J_{ri} x^j \tilde{x}^m + |x|^2 (\bar{H}_{ki} - Q_{km}{}^r J_{ri} \tilde{x}^m + \mu \tilde{x}_k \tilde{x}_i) = 0 , \end{aligned} \tag{5.13}$$

$$(2n-1)|x|^2 Q_{kri} \tilde{x}^k x^r = [(2n-1) Q_{krs} \tilde{x}^k x^r \tilde{x}^s - (\lambda + \mu)|x|^4] \tilde{x}_i . \tag{5.14}$$

It follows from (5.14) that

$$\lambda + \mu = 0 \quad \text{and} \quad Q_{kri} \tilde{x}^k x^r = \varepsilon |x|^2 \tilde{x}_i \tag{5.15}$$

with $|x|^4 \varepsilon = Q_{krs} \tilde{x}^k x^r \tilde{x}^s$, from which, by virtue of (5.7) we can see that $F_r{}^r = 0$. By the way, we obtain from (5.4)

$$J_j{}^s Q_{ks} r \tilde{x}^r = J_k{}^s Q_{jsr} \tilde{x}^r ,$$

from which, considering (4.9) and (5.15),

$$Q_{ksr} \tilde{x}^s \tilde{x}^r = -\varepsilon |x|^2 x_k \tag{5.16}$$

and therefore the equation (5.13) can be written as follows:

$$(5.17)\qquad (2n-1)Q_{kri}x^r-(2n\varepsilon+\lambda)x_kx_i+(\lambda+\varepsilon)|x|^2g_{ki}-\lambda x_ix_k-Q_{km}{}^rJ_r{}^i\tilde{x}^m+\bar{H}_{ki}=0$$

by virtue of (5.15) and (5.16). Transvecting $\tilde{x}^j$ to (5.12) and taking account of (5.16), we obtain

$$(5.18)\qquad Q_{kji}\tilde{x}^j+\varepsilon x_k\tilde{x}_i+J_i{}^rH_{rk}+aJ_{ki}=0$$

or

$$(5.19)\qquad Q_{krs}J_i{}^s\tilde{x}^r+\varepsilon x_kx_i-H_{ki}+ag_{ki}=0\ ,$$

from which we have

$$(5.20)\qquad Q_{kir}\tilde{x}^r+\varepsilon(x_k\tilde{x}_i-x_i\tilde{x}_k)+J_i{}^rH_{rk}-J_k{}^rH_{ri}+2aJ_{ki}=0$$

by (4.9) and, substituting (5.19) into (5.17), it follows that

$$(5.21)\qquad (2n-1)Q_{kri}x^r-(2n\varepsilon+\lambda)\tilde{x}_k\tilde{x}_i+\big((\lambda+\varepsilon)|x|^2-a\big)g_{ik}$$
$$-(\lambda+\varepsilon)x_ix_k+\bar{H}_{ki}+H_{ki}=0\ .$$

Considering (5.14) $\sim$ (5.16) and (5.19) $\sim$ (5.21), we can get

$$(5.22)\qquad (2n-1)\big(|x|^2Q_{kji}+J_i{}^rH_{r[k}\tilde{x}_{j]}-a\tilde{x}_{[k}J_{j]i}\big)$$
$$+(2n\varepsilon+\lambda)\tilde{x}_ix_{[k}\tilde{x}_{j]}+(2n-1)\tilde{x}_i(H_{r[k}J_{j]}{}^r+2aJ_{kj})$$
$$+\big((\lambda+\varepsilon)|x|^2-a\big)g_{i[k}x_{j]}+(\bar{H}_{i[k}+H_{i[k}x_{j]}=0\ .$$

On the other hand, from (5.8) we have by transvection

$$(5.23)\qquad (2n-3)H_{ij}+(\phi H)_{ij}+(\lambda+\varepsilon)(x_ix_j+\tilde{x}_i\tilde{x}_j)-\big(2(n-1)a+(\lambda+\varepsilon)|x|^2\big)g_{ij}=0$$

or

$$(5.23)'\qquad (2n-3)(\phi H)_{ij}+H_{ij}+(\lambda+\varepsilon)(x_ix_j+\tilde{x}_i\tilde{x}_j)-\big(2(n-1)a+(\lambda+\varepsilon)|x|^2\big)g_{ij}=0$$

because of (5.10) and (5.20), from which we obtain

$$(5.24)\qquad \bar{H}_{ij}=0\quad\text{ or }\quad H_{ij}=(\phi H)_{ij}$$

and at last, the equation (5.23) may be reduced to

$$(5.25)\qquad 2(n-1)H_{ij}=[2(n-1)a+(\lambda+\varepsilon)|x|^2]g_{ij}-(\lambda+\varepsilon)(x_ix_j+\tilde{x}_i\tilde{x}_j)\ .$$

Thus we have determined the form of H_{ij}.

In the last step, we shall prove the following main theorem of the paper.

THEOREM 2. – *A holomorphically subprojective Kählerian manifold M $(n \geqq 3)$ with $2\lambda + (n+1)\varepsilon = 0$ is a Kählerian manifold with vanishing Bochner curvature tensor, where we have put $|x|^2\lambda = Q_{kr}{}^r x^k$ and $|x|^4\varepsilon = Q_{kji}\tilde{x}^k x^j \tilde{x}^i$.*

PROOF. – Regarding to (5.24) and (5.25), the equation (5.22) can be rewritten as follows:

$$2(n-1)|x|^2 Q_{kji} + (\lambda+\varepsilon)|x|^2(J_{i[k}\tilde{x}_{j]} + 2\tilde{x}_i J_{jk} + g_{i[k}x_{j]}) + 2(2\lambda+(n+1)\varepsilon)\,\tilde{x}_i\tilde{x}_{[j}x_{k]} = 0\,,$$

which means that

$$(5.26)\qquad 2(n-1)|x|^2(Q_{kji}x_k - Q_{kjr}J_i{}^r\tilde{x}_h) + (\lambda+\varepsilon)|x|^2[(J_{i[k}\tilde{x}_{j]} + 2J_{jk}\tilde{x}_i + g_{i[k}x_{j]})\,x_h - (J_{i[k}x_{j]} + 2J_{jk}x_i + g_{i[j}\tilde{x}_{k]})\,\tilde{x}_h] + 2(2\lambda+(n+1)\varepsilon)\,x_{[k}\tilde{x}_{j]}\tilde{x}_{[i}x_{h]} = 0$$

and therefore by virtue of (5.25) and (5.26) we get from (4.3)′

$$(5.27)\qquad \begin{aligned} R_{kjih} = {} & \left(a + \frac{(\lambda+\varepsilon)|x|^2}{2(n-1)}\right)[g_{i[k}g_{j]h} - J_{i[k}J_{j]h} + 2J_{ih}J_{kj}] \\ & - \frac{\lambda+\varepsilon}{2(n-1)}[(J_{i[k}\tilde{x}_{j]} + 2J_{jk}\tilde{x}_i - g_{i[j}x_{k]})x_h \\ & - (J_{i[k}x_{j]} + 2J_{jk}x_i + g_{i[j}\tilde{x}_{k]})\tilde{x}_h \\ & + x_i x_{[k}g_{j]h} + \tilde{x}_i\tilde{x}_{[k}g_{j]h} - 2J_{ih}\tilde{x}_{[k}x_{j]} + \tilde{x}_i x_{[k}J_{j]h} - x_i\tilde{x}_{[k}J_{j]h}] \\ & - \frac{2\lambda+(n+1)\varepsilon}{(n-1)|x|^2}\,x_{[k}\tilde{x}_{j]}\tilde{x}_{[i}x_{h]}\,. \end{aligned}$$

Taking account of our assumption $2\lambda + (n+1)\varepsilon = 0$, we have

$$R_{kh} = -\left(2(n+1)a - \frac{n}{2}\varepsilon|x|^2\right)g_{kh} - \frac{n+2}{2}\,\varepsilon(x_k x_h + \tilde{x}_k\tilde{x}_h)\,,$$
$$R = (n+1)(-4na + (n-2)\,\varepsilon|x|^2)\,,$$

and these mean that the tensor L_{jk} defined by (3.2) reduces to

$$L_{ij} = -(n+2)\left[(a - \tfrac{1}{4}\varepsilon|x|^2)\,g_{ij} + \tfrac{1}{2}\varepsilon(x_i x_j + \tilde{x}_i\tilde{x}_j)\right]$$

and consequently, substituting this into (5.27) and considering $M_{ij} = J_i{}^r L_{rj}$ and $2\lambda + (n+1)\varepsilon = 0$, we have $K_{kjih} = 0$. This completes the proof of Theorem 2.

As a corollary of Theorem 2, we have

COROLLARY. – *A holomorphically subprojective Kählerian manifold with $\lambda = \varepsilon = 0$ is a manifold of constant holomorphic sectional curvature.*

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