A Nonlinear Parabolic Free Boundary Problem (*) (**).

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Summary. — *When two immiscible fluids in a porous medium are in contact with one another, an interface is formed and the movement of the fluids results in a free boundary problem for determining the location of the interface along with the pressure distribution throughout the medium. The pressure satisfies a nonlinear parabolic partial differential equation on each side of the interface while the pressure and the volumetric velocity are continuous across the interface. The movement of the interface is related to the pressure through Darcy's law. Two kinds of boundary conditions are considered. In Part I the pressure is prescribed on the known boundary. A weak formulation of the classical problem is obtained and the existence of a weak solution is demonstrated as a limit of a sequence of classical solutions to certain parabolic boundary value problems. In Part II the same analysis is carried out when the flux is specified on the known boundary, employing special techniques to obtain the uniform parabolicity of the sequence of approximating problems.*

PART I

FIRST BOUNDARY VALUE PROBLEM

71. – Introduction.

FULKS and GUENTHER, in a recent paper [1], describe an extension of Muskat's model which concerns the motion of two incompressible and immiscible fluids in a porous medium. In the Russian literature [2] such a problem bears the name of Verigin. FULKS and GUENTHER considered the case in which the coefficients in the equation of motion were considered constants. The resulting problem was that of two heat equations with differing diffusivities in respective domains which are separated by a curve, the interface between the liquids, across which it was assumed that the pressures and the volumetric velocities were continuous. The motion of the interface was related to the volumetric velocities of the fluids. The resulting free boundary problem was solved using potential theoretic methods similar to those of FRIEDMAN [3].

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In this paper, we shall consider a generalization of the differential equations involved in the Fulks-Guenther extension of Muskat's model. Before stating the exact mathematical model, it is informative to reconsider some of the physics of the problem which are discussed at length in [4]. We shall assume the flow to be one-dimensional and horizontal. We shall assume that capillary effects and the effects of gravity are negligible (*). Since the fluids are assumed to be incompressible, and immiscible, there is a well defined interface, $x = s(t)$, which separates them, where x denotes the spatial variable and t denotes the time variable. To the left of the interface, we shall denote the volumetric velocity of the fluid by $u_1 = u_1(x, t)$ and its pressure by $p = p(x, t)$. To the right of the interface, we shall denote the volumetric velocity of the other fluid by $u_2 = u_2(x, t)$ and its pressure by $q = q(x, t)$. The pressures and volumetric velocities are related by Darcy's law:

$$(1.1)\qquad u_1 = -a\frac{\partial p}{\partial x}, \quad u_2 = -b\frac{\partial q}{\partial x},$$

where $a = a(x, p)$ and $b = b(x, q)$ are positive valued functions which are the ratios of the permeability of the medium to the viscosity of the respective fluid. Note that we are assuming the temperature as a fixed constant. The equation of conservation of mass and the constancy of the densities yields the equations of motion

$$(1.2)\qquad \begin{aligned} &\frac{\partial \varphi(x, p)}{\partial t} + \frac{\partial u_1}{\partial x} = 0\,, \quad 0 < x < s(t)\,, \quad 0 < t \leqslant T\,,\\ &\frac{\partial \varphi(x, q)}{\partial t} + \frac{\partial u_2}{\partial x} = 0\,, \quad s(t) < x < 1\,, \quad 0 < t \leqslant T\,, \end{aligned}$$

where u_i, $i = 1, 2$ are given by (1.1), $\varphi = \varphi(x, \cdot)$ is a positive valued function which represents the porosity of the medium which is assumed to be of finite length. Across the interface, we shall assume the continuity of the pressures and the volumetric velocities. Thus,

$$(1.3)\qquad \begin{aligned} p(s(t) - 0, t) &= q(s(t) + 0, t)\,,\\ u_1(s(t) - 0, t) &= u_2(s(t) + 0, t)\,. \end{aligned}$$

Clearly, the motion of the interface is related to the volumetric velocity at the interface through the equation

$$(1.4)\qquad \varphi\big(s(t), p(s(t), t)\big)\,\dot{s}(t) = u(s(t), t)\,,$$

where the dot above the s denotes differentiation of that function with respect to t.

(*) The neglect of capillary effects is quite significant [5, 6, 7] and limits the utility of the model here to fluids having similar capillary behavior.

The discussion above motivates the statement of our free boundary problem which is to find functions p, q and s which as a triple (p, q, s) satisfy

$$(1.5)\quad \begin{cases} (1) & \dfrac{\partial\varphi(x,p)}{\partial t} = \dfrac{\partial}{\partial x}\left[a(x,p)\dfrac{\partial p}{\partial x}\right], \quad 0<x<s(t),\ 0<t\leqslant T, \\ (2) & \dfrac{\partial\varphi(x,q)}{\partial t} = \dfrac{\partial}{\partial x}\left[b(x,q)\dfrac{\partial q}{\partial x}\right], \quad x(t)<x<1,\ 0<t\leqslant T, \\ & \qquad s(0)=s_0,\ 0<s_0<1, \\ (3) & p(x,0) = h_1(x), \quad 0\leqslant x\leqslant s_0, \\ (4) & q(x,0) = h_2(x), \quad s_0\leqslant x\leqslant 1, \\ (5) & p(0,t) = f_1(t), \quad 0<t\leqslant T, \\ (6) & q(1,t) = f_2(t), \quad 0<t\leqslant T, \\ (7) & p(s(t),t) = q(s(t),t), \quad 0<t\leqslant T, \\ (8) & a(s(t),p(s(t),t))\dfrac{\partial p}{\partial x}(s(t),t) = b(s(t),q(s(t),t))\dfrac{\partial q}{\partial x}(s(t),t), \quad 0<t\leqslant T, \end{cases}$$

and

$$(1.6)\qquad \varphi\big(s(t),p(s(t),t)\big)\dot{s}(t) = -a\big(s(t),p(s(t),t)\big)\frac{\partial p}{\partial x}(s(t),t), \qquad 0<t\leqslant T,$$

where the functions φ, a, b, h_1, h_2, f_1 and f_2 are given functions of their respective arguments and the s_0, $0<s_0<1$, is a specified constant. With respect to the functions φ, a and b, the physical situation suggests the following assumptions:

(A_1) there exist positive constants φ_0 and μ_1 such that for $0\leqslant x\leqslant 1$ and $\xi\geqslant 0$,

$$(1.7)\qquad 0<\varphi_0\leqslant\varphi(x,\xi)\leqslant 1, \quad 0\leqslant\frac{\partial\varphi}{\partial\xi}(x,\xi)\leqslant\mu_1,$$

and for each positive constant M, there exists a positive constant $\mu(M)$ such that

$$(1.8)\qquad 0<\mu(M)\leqslant\frac{\partial\varphi}{\partial\xi}(x,\xi) \quad \text{when } 0\leqslant x\leqslant 1 \text{ and } 0\leqslant\xi\leqslant M;$$

(B_1) there exist positive constants ν and ν_1 such that for $0\leqslant x\leqslant 1$ and $\xi\geqslant 0$,

$$(1.9)\qquad 0<\nu\leqslant a(x,\xi),\ b(x,\xi)\leqslant\nu_1 .$$

Under these assumptions equations (1.5)-(1) and (2) are nonlinear parabolic partial differential equations. Moreover, it should be noted that the physical assumption (1.7) implies that the problem must be nonlinear. We shall add additional assumptions on the data later.

In the next section we shall define what is meant by a classical solution of problem (1.5)-(1.6), and we shall derive a weak formulation of it. The remaining sections are devoted to the demonstration of the existence of a weak solution. This is accomplished by retarding the argument in the weak formulation of (1.6) and by applying the Ascoli-Arzela Theorem to sequences of solutions of a family of problems which formally approximate the weak formulation of (1.5)-(1.6).

REMARK 1. - For any given continuous function $s(t)$, $0 < s(t) < 1$, $0 \leqslant t \leqslant T$, the problem (1.5) is an example of a *diffraction problem* in the sense of [8], Chapter III, section 13.

REMARK 2. - The notation and definitions of LADYZENSKAJA, SOLONNIKOV, and URAL'CEVA's book on Linear and Quasilinear Equations of Parabolic Type [8] will be used extensively in this paper. As an aid to the reader, we shall list a few items here. Let $Q_T = \{(x, t): 0 \leqslant x \leqslant 1, 0 < t \leqslant T\}$. Then,

a) $W_2^{1,0}(Q_T)$ denotes the Hilbert space with scalar product

$$(u, v)_{W_2^{1,0}(Q_T)} = \int_{Q_T} \left(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \right) dx\, dt,$$

while $\overset{\circ}{W}{}_2^{1,0}(Q_T)$ denotes the subspace of $W_2^{1,0}(Q_T)$ whose elements vanish when $x = 0$ and $x = 1$;

b) $W_2^{1,1}(Q_T)$ denotes the Hilbert space with scalar product

$$(u, v)_{W_2^{1,1}(Q_T)} = \int_{Q_T} \left(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} \right) dx\, dt,$$

while $\overset{\circ}{W}{}_2^{1,1}(Q^T)$ denotes the subspace of $W_2^{1,1}(Q_T)$ whose elements vanish when $x = 0$ and $x = 1$;

c) $H^{2+\alpha, 1+\alpha/2}(\bar{Q}_T)$ denotes the Banach space of functions whose second x derivative is uniformly Holder continuous with exponent α and whose first t derivative is uniformly Holder continuous with exponent $\alpha/2$. The norm consists of the sum of the uniform norms of the function, its first two x derivatives, its t derivative, the α-Hölder semi-norm of the second x derivative, and the $\alpha/2$-Hölder semi-norm of the first t derivative.

2. - A weak formulation of problem (1.5) - (1.6).

We begin with the definition of a classical solution of problem (1.5)-(1.6).

DEFINITION. - A classical solution of problem (1.5)-(1.6) is a triple of functions (p, q, s) such that

i) $s = s(t)$ is continuous in $0 \leqslant t \leqslant T$ and continuously differentiable in $0 < t \leqslant T$;

ii) $p = p(x,t)$ is continuous in $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant T$, $\partial p/\partial x$ is continuous in $0 < x \leqslant s(t)$, $0 < t \leqslant T$, $\partial^2 p/\partial x^2$ and $\partial p/\partial t$ are continuous in $0 < x < s(t)$, $0 < t \leqslant T$;

iii) $q = q(x,t)$ is continuous in $s(t) \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$, $\partial q/\partial x$ is continuous in $s(t) \leqslant x < 1$, $0 < t \leqslant T$, $\partial^2 q/\partial x^2$ and $\partial q/\partial t$ are continuous in $s(t) < x < 1$, $0 < t \leqslant T$;

iv) the equations in (1.5)-(16.) are satisfied by p, q and s.

Obviously, the definition imposes some minimal assumptions upon the data.

REMARK 3. – For a given continuously differentiable s, a classical solution of (1.5) is a pair of functions (p, q) which satisfies ii), iii) above and the equations in (1.5).

Suppose now that there exists a function $F = F(x,t)$ which is defined and continuous in $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$ and possesses continuous derivatives $\partial F/\partial x$, $\partial^2 F/\partial x^2$, and $\partial F/\partial t$ in $0 < x < 1$, $0 < t \leqslant T$. Furthermore, suppose that for $0 < t \leqslant T$, $F(0,t) = f_1(t)$ and $F(1,t) = f_2(t)$. For a given continuously differentiable s, $0 < s < 1$, $s(0) = s_0$, let (p, q) denote a classical solution of (1.5). Defining the functions $\bar{p}$ and $\bar{q}$ via

$$\bar{p}(x,t) = p(x,t) - F(x,t)\,, \quad 0 \leqslant x \leqslant s(t)\,, \quad 0 \leqslant t \leqslant T\,, \tag{2.1}$$

and

$$\bar{q}(x,t) = q(x,t) - F(x,t)\,, \quad s(t) \leqslant x \leqslant 1\,, \quad 0 \leqslant t \leqslant T\,, \tag{2.2}$$

we see that $\bar{p}(0,t) = \bar{q}(1,t) = 0$, $0 < t \leqslant T$, and we can now formulate the weak version of (1.5).

Selecting a function $\eta = (x,t)$ from $\mathring{W}_2^{1,1}(Q_T)$, where $Q_T = \{(x,t)\colon 0 \leqslant x \leqslant 1,\ 0 < t \leqslant T\}$ and $\eta(x,T) = 0$, we multiply both sides of (1.5)-(1) by η, substitute $p = \bar{p} + F$ and integrate over the domain $0 < x < s(t)$, $0 < t < T$. Integrating by parts we obtain

$$\int_0^T\!\!\int_0^{s(t)} \eta(x,t)\,\frac{\partial\varphi(x,\bar{p}+F)}{\partial t}\,dx\,dt = -\int_0^T\!\!\int_0^{s(t)} \frac{\partial\eta}{\partial t}\,\varphi(x,\bar{p}+F)\,dx\,dt - \int_0^{s_0} \eta(x,0)\,\varphi(x,h_1(x))\,dx - \int_0^T \dot{s}(t)\,\eta(s(t),t)\,\varphi\big(s(t),\bar{p}(s(t),t) + F(s(t),t)\big)\,dt\,, \tag{2.3}$$

and

$$\int_0^T\!\!\int_0^{s(t)} \eta(x,t)\,\frac{\partial}{\partial x}\left[a(x,\bar{p}+F)\,\frac{\partial(\bar{p}+F)}{\partial x}\right]dx\,dt = -\int_0^T\!\!\int_0^{s(t)} a(x,\bar{p}+F)\,\frac{\partial(\bar{p}+F)}{\partial x}\,\frac{\partial\eta}{\partial x}\,dx\,dt + \int_0^T \eta(s(t),t)\,a\big(s(t),\bar{p}(s(t),t) + F(s(t),t)\big)\,\frac{\partial(\bar{p}+F)}{\partial x}\bigg|_{(s(t),t)}dt\,. \tag{2.4}$$

In a similar manner with respect to (1.5)-(2), we obtain

$$\int_0^T \int_{s(t)}^1 \eta(x,t)\,\frac{\partial \varphi(x,\bar q + F)}{\partial t}\,dx\,dt = -\int_0^T \int_{s(t)}^1 \frac{\partial \eta}{\partial t}\,\varphi(x,\bar q + F)\,dx\,dt - \tag{2.5}$$
$$-\int_0^{s_0} \eta(x,0)\,\varphi(x,h_2(x))\,dx + \int_0^T \dot s(t)\,\eta(s(t),t)\,\varphi\big(s(t),\bar q(s(t),t) + F(s(t),t)\big)\,dt\,,$$

and

$$\int_0^T \int_{s(t)}^1 \eta(x,t)\,\frac{\partial}{\partial x}\left[b(x,\bar q + F)\,\frac{\partial(\bar q + F)}{\partial x}\right]dx\,dt = \tag{2.6}$$
$$-\int_0^T \int_{s(t)}^1 b(x,\bar q + F)\,\frac{\partial(\bar q + F)}{\partial x}\,\frac{\partial \eta}{\partial x}\,dx\,dt - \int_0^T \eta(s(t),t)\,b\big(s(t),\bar q(s(t),t)$$
$$+ F(s(t),t)\big)\,\frac{\partial(\bar q + F)}{\partial x}\bigg|_{(s(t),t)} dt\,.$$

Equating the right hand sides of (2.3) and (2.4) and the right hand sides of (2.5) and (2.6) and adding the resulting equations, it follows from (1.5)-(7) and (1.5)-(8) that the integrals along the curve $x = s(t)$ cancel each other. Hence, we obtain

$$-\int_0^T \left\{\int_0^{s(t)} \varphi(x,\bar p + F)\,\frac{\partial \eta}{\partial t}\,dx + \int_{s(t)}^1 \varphi(x,\bar q + F)\,\frac{\partial \eta}{\partial t}\,dx\right\}dt + \tag{2.7}$$
$$\int_0^T \left\{\int_0^{s(t)} a(x,\bar p + F)\,\frac{\partial(\bar p + F)}{\partial x}\,\frac{\partial \eta}{\partial x}\,dx + \int_{s(t)}^1 b(x,\bar q + F)\,\frac{\partial(\bar q + F)}{\partial x}\,\frac{\partial \eta}{\partial x}\,dx\right\}dt$$
$$= \int_0^{s_0} \eta(x,0)\,\varphi(x,h_1(x))\,dx + \int_{s_0}^1 \eta(x,0)\,\varphi(x,h_2(x))\,dx\,.$$

Defining

$$v(x,t) = \begin{cases} \bar p(x,t)\,, & 0 \leqslant x \leqslant s(t),\ 0 \leqslant t \leqslant T\,, \\ \bar q(x,t)\,, & s(t) \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T\,, \end{cases} \tag{2.8}$$

$$w(x,t) = v(x,t) + F(x,t)\,, \quad 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T\,, \tag{2.9}$$

$$\gamma(x,\xi;s) = \begin{cases} a(x,\xi)\,, & 0 \leqslant x \leqslant s(t),\ \xi \geqslant 0\,, \\ b(x,\xi)\,, & s(t) < x \leqslant 1\,,\ \xi \geqslant 0\,, \end{cases} \tag{2.10}$$

and

$$h(x) = \begin{cases} h_1(x)\,, & 0 \leqslant x \leqslant s_0\,, \\ h_2(x)\,, & s_0 < x \leqslant 1\,, \end{cases} \tag{2.11}$$

We can rewrite (2.7) as

$$(2.12)\qquad \int_0^T\int_0^1\left\{-\varphi(x, v+F)\frac{\partial\eta}{\partial t}+\gamma(x, v+F; s)\frac{\partial(v+F)}{\partial x}\frac{\partial\eta}{\partial x}\right\}dx\,dt=\int_0^1\eta(x, 0)\varphi(x, h(x))\,dx\,.$$

From the above calculations we have shown that any classical solution of (1.5), where $s=s(t)$ is continuously differentiable, satisfies (2.12) for every $\eta\in\mathring{W}_2^{1,1}(Q_T)$ such that $\eta(x, T)=0$. Note that if $v=v(x, t)$ is a solution of (2.12) in the sense that v satisfies (2.12) for every $\eta\in\mathring{W}_2^{1,1}(Q_T)$, $\eta(x, T)=0$, and if v possesses the smoothness of a classical solution, then v can be employed in an obvious way to define p and q which form a classical solution (p, q) of (1.5). The analysis is elementary and is omitted.

Returning now to the free boundary problem (1.5)-(1.6), suppose that (p, q, s) is a classical solution. Then, p and q in the form of v along with s must satisfy (2.12) for all $\eta\in\mathring{W}_2^{1,1}(Q_T)$, $\eta(x, T)=0$. Recall that the condition (1.6) was *not* employed in the formation of (2.12). We shall use it now to obtain a weak formulation of the free boundary condition. Consider (1.5)-(1) and multiply both sides by x. Integrating over the domain $0<x<s(\tau)$, $0<\tau<t$, where $t\in(0, \tau]$ it follows that

$$(2.13)\qquad \int_0^{s(t)}x\varphi(x, p(x, t))\,dx-\int_0^{s_0}x\varphi(x, h_1(x))\,dx-\int_0^t\dot{s}(\tau)s(\tau)\varphi\big(s(\tau), p(s(\tau), \tau)\big)\,d\tau=-\int_0^t\int_0^{s(\tau)}a(x, p)\frac{\partial p}{\partial x}\,dx\,d\tau+\int_0^t s(\tau, a\big(s(\tau), p(s(\tau), \tau)\big)\frac{\partial p}{\partial x}\bigg|_{(s(\tau),\tau)}d\tau\,.$$

Since (1.6) implies the cancellation of the integrals along s, it follows from (2.8)-(2.11) that (2.13) can be written as

$$(2.14)\qquad \int_0^{s(t)}x\varphi(x, v+F)\,dx=\int_0^{s_0}x\varphi(x, h(x))\,dx-\int_0^t\int_0^{s(\tau)}\gamma(x, v+F; s)\left(\frac{\partial v}{\partial x}+\frac{\partial F}{\partial x}\right)dx\,d\tau\,.$$

We conclude this section with our definition of a weak solution of the free boundary problem (1.5)-(1.6).

DEFINITION. – A pair of functions (v, s), $v=v(x,t)$ and $s=s(t)$, is a weak solution to the free boundary problem (1.5)-(1.6) provided that:

i) $s=s(t)$ is continuous in $[0,T]$, $0\leqslant s\leqslant 1$, and $s(0)=s_0$;

ii) $v\in \mathring{W}_2^{1,0}(Q_T)$ and v is bounded;

iii) the pair (v, s) satisfies (2.12) for every $\eta\in \mathring{W}_2^{1,1}(Q_T)$ that vanishes at $t=T$, and the pair (v,s) satisfies (2.14),

where F is a bounded function belonging to $W^{1,0}(Q_T)$ with traces $F(0,t)=f_1(t)$ and $F(1,t)=f_2(t)$.

3. – Statement of an existence theorem.

We shall present here our assumptions on the data and the statement of our result. Recalling assumptions (A_1) and (B_1) in section 1, we add the following:

(A_2) there exist positive constants μ_2, μ_3 and μ_4 such that

$$\left|\frac{\partial\varphi}{\partial x}(x,\xi)\right|\leqslant\mu_2, \quad 0\leqslant x\leqslant 1,\ \xi\geqslant 0, \tag{3.1}$$

$$\left|\frac{\partial^2\varphi}{\partial x\partial_\xi}(x,\xi)\right|\leqslant\mu_3, \quad 0\leqslant x\leqslant 1,\ \xi\geqslant 0, \tag{3.2}$$

and

$$\left|\frac{\partial^2\varphi}{\partial \xi^2}(x,\xi)\right|\leqslant\mu_4, \quad 0\leqslant x\leqslant 1,\ \xi\geqslant 0; \tag{3.3}$$

(B_2) there exist positive constants ν_2 and ν_3 such that

$$\left|\frac{\partial a}{\partial x}(x,\xi)\right|\leqslant\nu_2, \left|\frac{\partial b}{\partial x}(x,\xi)\right|\leqslant\nu_2, \quad 0\leqslant x\leqslant 1,\ \xi\geqslant 0, \tag{3.4}$$

and

$$\left|\frac{\partial a}{\partial \xi}(x,\xi)\right|\leqslant\nu_3, \left|\frac{\partial b}{\partial \xi}(x,\xi)\right|\leqslant\nu_3, \quad 0\leqslant x\leqslant 1,\ \xi\geqslant 0; \tag{3.5}$$

(C_1) the data f_1, f_2, and h are bounded, positive, and Lipschitz continuous; in other words, there exist positive constants M_0, M_1, L_0 and L_1 such that

$$0< M_0\leqslant f_i(t)\leqslant M_1, \quad 0\leqslant t\leqslant T, \qquad i=1,2, \tag{3.6}$$

$$0< M_0\leqslant h(x)\leqslant M_1, \quad 0\leqslant x\leqslant 1, \tag{3.7}$$

$$|f_i(t_1)-f_i(t_2)|\leqslant L_0|t_1-t_2|, \qquad t_1,t_2\in[0,T],\ i=1,2, \tag{3.8}$$

and

$$|h(x_1) - h(x_2)| \leqslant L_1 |x_1 - x_2| \,, \qquad x_1\, x_2 \in [0, 1]; \tag{3.9}$$

and

(C_2) the data f_1, f_2, and h match in the corners; in other words

$$f_1(0) = h(0) \quad \text{and} \quad f_2(0) = h(1)\,. \tag{3.10}$$

THEOREM. – Under the assumptions (A_1), (A_2), (B_1), (B_2), (C_1) and (C_2) upon the data φ, a, b, f_1, f_2, h, and s_0, $0 < s_0 < 1$, it follows that for any δ such that $0 < \delta < s_0 < 1 - \delta$, there exists a $T_0 > 0$ such that in Q_{T_0} there exists at least one weak solution (v, s) to the free boundary value problem (1.5)-(1.6) such that $\delta \leqslant s(t) \leqslant 1 - \delta$, $0 \leqslant t \leqslant T_0$.

The proof of the existence theorem will be given in section 6. In section 4 we derive a family of approximations (v_m, s_m) which are solutions to problems which tend formally to the weak formulation of (1.5)-(1.6) as m tends to infinity. In section 5 we obtain uniform estimates upon the (v_m, s_m) which enable us to utilize arguments of the Ascoli-Arzela type in section 6 to demonstrate the existence of at least one weak solution to (1.5)-(1.6).

4. – A sequence of approximations.

We begin by extending the functions φ, a, and b to the domain $\{(x, \xi)\colon -\infty < x < \infty,\ -\infty < \xi < \infty\}$. The extension can be carried out so that the extended functions satisfy (A_1), (A_2), (B_1), and (B_2), where some of the constants appearing in these conditions may have been modified slightly. For any continuous function $s = s(t)$, $0 \leqslant t \leqslant T$, $s(0) = s_0$, we extend it by setting $s(t) \equiv s_0$ for $t < 0$ and $s(t) \equiv s(T)$ for $t > T$. Recalling now the definition of $\gamma = \gamma(x, \xi; s)$ given by (2.11), we utilize the extensions above to extend γ via (2.11) over the domain $\{(x, t, \xi)\colon -\infty < x < \infty,\ -\infty < t < \infty, \text{ and } -\infty < \xi < \infty\}$.

Next we mollify [3; p. 274] φ and γ to obtain sequences $\{\varphi_m\}$ and $\{\gamma_m\}$ of C^∞ functions which satisfy (A_1), (A_2), (B_1) and (B_2) and which are such that the φ_m converge uniformly to φ on compact subsets of $\{(x, \xi)\colon -\infty < x < \infty,\ -\infty < \xi < \infty\}$ while the γ_m converge uniformly to γ on compact subsets of $\{(x, \xi, t)\colon -\infty < x < \infty,\ -\infty < \xi < \infty,\ -\infty < t < \infty\}$ which do not intersect the surface $x = s(t)$. *Note that the mollification can be achieved by integrating the functions φ and γ against C^∞ kernels which have support in balls of radius* $1/2m$. This will be used in the discussion below.

In what follows, it is essential that we smooth the boundary and initial data in such a way as to be able to utilize the existence theorems of LADYZENSKAJA, SOLON-

NIKOV, and URAL'CEVA [8]. Consider f_i, $i=1,2$, and define

$$\bar{f}_{i,m}(t)=\begin{cases} f_i(0)\,, & -\infty<t\leqslant\dfrac{1}{m}\,,\\ f_i\left(t-\dfrac{1}{m}\right), & \dfrac{1}{m}\leqslant t\leqslant T\,,\\ f_i\left(T-\dfrac{1}{m}\right), & T\leqslant t\leqslant\infty\,,\end{cases}\tag{4.1}$$

$m=1,2,\ldots$. The functions $\bar{f}_{i,m}$ can be mollified to obtain C^∞ functions $f_{i,m}$ which satisfy the conditions stated in (C_1). Moreover, the conditions

$$\dot{f}_{i,m}(0)=0\,,\qquad m=1,2,\ldots\,,\tag{4.2}$$

can be achieved if the support of the mollifying kernel for each m is an interval of length $1/2m$. Clearly, the uniform convergence of the $f_{i,m}$ to the f_i follows from the Lipschitz continuity of the f_i. The transformation $x'=(1-2/m)x+1/m$ maps the interval $[0,1]$ 1-1 and onto the interval $[1/m,(1-1/m)]$. Considering the inverse transformation, we define

$$\bar{h}_m(x)=\begin{cases} h(0)\,, & -\infty<x\leqslant\dfrac{1}{m}\\ h\left(\left(1-\dfrac{2}{m}\right)^{-1}\left(x-\dfrac{1}{m}\right)\right), & \dfrac{1}{m}\leqslant x\leqslant\left(1-\dfrac{1}{m}\right),\\ h(1)\,, & \left(1-\dfrac{1}{m}\right)\leqslant x<\infty\,,\end{cases}\tag{4.3}$$

$m=3,4,5,\ldots$. The functions $\bar{h}_m(x)$ can be mollified to obtain C^∞ functions $h_m(x)$ which satisfy the conditions stated in (C_1), where the Lipschitz constant L_1 must be replaced by $2L_1$ for $m\geqslant4$. Obviously the h_m converge uniformly to h and, moreover,

$$h'_m(0)=h''_m(0)=h'_m(1)=h''_m(1)=0\tag{4.4}$$

can be obtained if the support of the mollifying kernel for each m is an interval of length $1/2m$.

We begin now the discussion of the procedure for generating the approximate solutions $\{(v_m,s_m)\}$. Set

$$s_m(t)\equiv s_0\,,\qquad 0\leqslant t\leqslant\frac{1}{m}\,.\tag{4.5}$$

For this $s=s_m(t)$, we can obtain a smooth γ_m in the strip $0\leqslant t\leqslant 1/2m$. Consider the problems

$$(4.6)\qquad \begin{cases} \dfrac{\partial\varphi_m(x, w_m)}{\partial t} = \dfrac{\partial}{\partial x}\left(\gamma_m(x, w_m, t)\dfrac{\partial w_m}{\partial x}\right), & 0 < x < 1,\ 0 < t \leqslant \dfrac{1}{2m}, \\ w_m(0, t) = f_{1,m}(t), & 0 < t \leqslant \dfrac{1}{2m}, \\ w_m(1, t) = f_{2,m}(t), & 0 < t \leqslant \dfrac{1}{2m}, \\ w_m(x, 0) = h_m(x), & 0 \leqslant x \leqslant 1, \end{cases}$$

$m = 4, 5, 6, \ldots$. By the maximum principle [8; p. 23], it follows that

$$(4.7)\qquad 0 < M_0 \leqslant w_m(x, t) \leqslant M_1 .$$

Hence, we can set

$$(4.8)\qquad \mu \equiv \mu(M_1) > 0 ,$$

and obtain

$$(4.9)\qquad \mu \leqslant \frac{\partial\varphi_m}{\partial\xi} \leqslant \mu_1 .$$

Also, since there exist functions

$$\chi_m(x, t) \in C^\infty \quad \text{in } 0 \leqslant x \leqslant 1 , \qquad 0 \leqslant t \leqslant T$$

such that $\chi_m(0, t) = f_{1,m}(t)$, $\chi_m(1, t) = f_{2,m}(t)$, and $\chi_m(x, 0) = h_m(x)$, and (4.2), (4.4) imply that the differential equation in (4.6) is satisfied for $w_m = \chi_m$ at $x = 0$, $t = 0$ and $x = 1$, $t = 0$, it follows [8; p. 452] that for each m there is a unique classical solution in a Hölder class in the closed rectangle $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant 1/2m$.

Next, we extend the boundary $x = s_m(t)$ to the interval $1/m \leqslant t \leqslant 1/m + 1/2m$ via a modification of the free boundary condition (2.15). First, we note that for $0 \leqslant t \leqslant 1/2m$ the equation

$$(4.10)\qquad \int_0^{\sigma(t)} x\varphi_m(x, w_m(x, t))\, dx = H(t)$$

defines a continuous function $\sigma(t)$ for any given continuous function $H(t)$ since φ_m is positive. Let δ denote a fixed positive constant chosen so that

$$(4.11)\qquad 0 < \delta < s_0 < (1 - \delta) .$$

Setting

$$(4.12)\qquad H_m(t) = \int_0^{s_0} x\varphi_m(x, h_m(x))\, dx - \int_0^t \int_0^{s_m(\tau)} \gamma_m(x, w_m(x, \tau), \tau)\frac{\partial w_m}{\partial x}(x, \tau)\, dx\, d\tau ,$$

$m=4, 5, 6, \ldots$, we solve (4.10) with $H(t)$ replaced by $H_m(t)$ for $\sigma(t)$ and define

$$s_m\left(t+\frac{1}{m}\right)=\sigma(t) \tag{4.13}$$

for $0\leqslant t\leqslant 1/2m$. If there is a t_m^* such that $s_m(t_m^*+1/m)=\delta$(or $s_m(t_m^*+1/m)=(1-\delta)$), then we set

$$s_m(t)\equiv\delta \qquad (\text{or } s_m(t)\equiv 1-\delta) \tag{4.14}$$

for $t\geqslant t_m^*+1/m$. In the cases of (4.14) the coefficients γ_m can be defined up to $t=T$ and problem (4.6) can be solved up to $t=T$. In the event that

$$0<\delta<s_m\left(t+\frac{1}{m}\right)<(1-\delta)$$

for $0\leqslant t\leqslant 1/2m$, then the boundary $x=s_m(t)$ can be defined in $0\leqslant t\leqslant 1/m+1/2m$ via

$$s_m(t)=\begin{cases} s_0\,, & 0\leqslant t\leqslant \dfrac{1}{m}\,,\\[2mm] \sigma\left(t-\dfrac{1}{m}\right), & \dfrac{1}{m}\leqslant t\leqslant \dfrac{1}{m}+\dfrac{1}{2m}\,. \end{cases} \tag{4.15}$$

Thus, the function γ can be defined up to $t=1/m+1/2m$ and the function γ_m obtained for $0\leqslant t\leqslant 1/m$. Note that γ_m here coincides with the γ_m in (4.6) for $0\leqslant t\leqslant 1/2m$. Consequently, we can replace $1/2m$ in (4.6) with $1/m$ and obtain a classical solution w_m in $0\leqslant x\leqslant 1$, $0\leqslant t\leqslant 1/m$. Considering (4.10) and (4.12) again along with (4.13) and (4.14), we see that the boundary can be extended to the interval $1/m+1/2m\leqslant t\leqslant 1/m+2(1/2m)$. By induction, we obtain a sequence $\{(w_m, s_m)\}$ such that s_m are continuous and satisfy

$$0<\delta\leqslant s_m(t)\leqslant 1-\delta\,, \qquad 0\leqslant t\leqslant T\,, \qquad m=4, 5, 6, \ldots \tag{4.16}$$

and that there exists an α_m, $0<\alpha_m<1$, such that

$$w_m\in H^{2+\alpha_m, 1+\alpha_m/2}(\bar{Q}_T)\,, \qquad m=4, 5, \ldots, \tag{4.17}$$

and w_m satisfies (4.6) with $1/2m$ replaced by T and γ_m defined from $\gamma(x, \xi; s_m)$.

5. – Some uniform properties of w_m and s_m.

We begin with the following lemma which will be used in obtaining a uniform estimate of w_m in $W_2^{1,0}(Q)_T$.

LEMMA 1. – There exists a positive constant C and an integer m_0 which depend on δ and the parameters in the set $\mathcal{T}=\{\varphi_0, \mu, \mu_1, \mu_2, \mu_3, \mu_4, \nu, \nu_1, \nu_2, \nu_3, M_0, M_1, L_0, L_1\}$ such that for any $m\geqslant m_0$,

$$\left|\frac{\partial w_m}{\partial x}(0,t)\right|\leqslant C\,,\left|\frac{\partial w_m}{\partial x}(1,t)\right|\leqslant C\,,\qquad 0\leqslant t\leqslant T\,. \tag{5.1}$$

PROOF. – Set

$$\bar{z}^{(n)}_{0,m}(x)=\begin{cases} f_1(0)+\dfrac{1}{n}, & 0\leqslant x\leqslant\dfrac{1}{2m},\\[2mm] L_2x+\left\{f_1(0)+\dfrac{1}{n}-\dfrac{1}{2m}L_2\right\}, & \dfrac{1}{2m}\leqslant x\leqslant x_{n,m}\,,\\[2mm] M_1+1\,, & x_{n,m}\leqslant x\leqslant\delta/2\,,\end{cases} \tag{5.2}$$

where

$$L_2=\max\{L_1, 8\delta^{+1}(M_1+1)\}\,, \tag{5.3}$$

$$x_{n,m}=L_2^{-1}\left\{M_1+1-\frac{1}{n}-f_1(0)+L_2\frac{1}{2m}\right\}, \tag{5.4}$$

$n=1,2,3,\ldots$, and $m=m_0+1,\ m_0+2,\ldots$, where m_0 is a positive integer which satisfies

$$m_0>4\delta^{-1}\,. \tag{5.5}$$

Let $z^{(n)}_{0,m}(x)$ denote the mollification of $\bar{z}^{(n)}_{0,m}(x)$ against a positive smooth kernel which has support in an interval of length less than $\frac{1}{2}\min(L_2^{-1}, 1/2m)$. Clearly, all of the derivatives of $z^{(n)}_{0,m}$ vanish at $x=0$ and $x=\delta/2$. Moreover, elementary considerations yield the fact that

$$z^{(n)}_{0,m}(x)>h_m(x) \tag{5.6}$$

For all $n=1,2,\ldots$ and $m>m_0$. Note also that the first derivative of $z^{(n)}_{0,m}$ is bounded by a constant that depends only on L_1 and δ.

We consider now

$$\begin{cases} \dfrac{\partial\varphi_m(x,z^{(n)}_m)}{\partial t}=\dfrac{\partial}{\partial x}\left(\gamma_m(x,z^{(n)}_m,t)\dfrac{\partial z^{(n)}_m}{\partial x}\right)+\dfrac{1}{n}, & 0<x<\dfrac{\delta}{2},\quad 0<t\leqslant T\,,\\[2mm] z^{(n)}_m(0,t)=f_{1,m}(t)+\dfrac{1}{n}+\dfrac{1}{n}\left[\dfrac{\partial\varphi}{\partial\xi}\left(0,f_1(0)+\dfrac{1}{n}\right)\right]^{-1}t, & 0\leqslant t\leqslant T\,,\\[2mm] z^{(n)}_m\left(\dfrac{\delta}{2},t\right)=M_1+1+\dfrac{1}{n}\left[\dfrac{\partial\varphi}{\partial\xi}\left(\dfrac{\delta}{2},M_1+1\right)\right]^{-1}t, & 0\leqslant t\leqslant T\,,\\[2mm] z^{(n)}_m(x,0)=z^{(n)}_{0,m}(x)\,, & 0\leqslant x\leqslant\delta/2\,.\end{cases} \tag{5.7}$$

Referring to the discussion following (4.6), we see that a unique classical solution $z_m^{(n)}$ exists. For $m > m_0$, it follows that the coefficient γ_m and its derivatives $\partial\gamma_m/\partial x$ and $\partial\gamma_m/\partial z_m^n$ can be estimated in terms of the bounds given in (B_1) and (B_2) since the boundary s_m is $\delta/2$ units away from the domain under consideration. Hence, we may apply the Lemma 3.1 [8; p. 535] and Theorem 4.2 [8; p. 444] to conclude that there exists a positive constant C_1 independent of m and n such that

$$\left|\frac{\partial z_m^{(n)}}{\partial x}\right| \leqslant C_1, \quad 0 \leqslant x \leqslant \delta/2, \quad 0 \leqslant t \leqslant T. \tag{5.8}$$

Moreover, Theorem 1.1 [5; p. 419] implies that the functions $z_m^{(n)}$ are equi-Hölder-continuous in $0 \leqslant x \leqslant \delta/2$, $0 \leqslant t \leqslant T$ with respect to n. Since the maximum principle can be applied to obtain a uniform bound for the $z_m^{(n)}$, it follows that for fixed $m > m_0$ there exists a subsequence $\{z_m^{(n')}\}$ which converges uniformly to a Hölder continuous function $z_m = z_m(x, t)$. From (5.8) it follows that z_m is Lipschitz continuous with respect to x with Lipschitz constant C_1. Applying the lemma of WESTPHAL-PRODI [3; p. 52] to $z_m^{(n)}$ and w_m, we obtain

$$z_m^{(n)}(x, t) > w_m(x, t), \quad 0 \leqslant x \leqslant \delta/2, \quad 0 \leqslant t \leqslant T, \tag{5.9}$$

which implies

$$z_m(x, t) \geqslant w_m(x, t), \quad 0 \leqslant x \leqslant \delta/2, \quad 0 \leqslant t \leqslant T, \tag{5.10}$$

while

$$z_m(0, t) = w_m(0, t) = f_{1,m}(t), \quad 0 \leqslant t \leqslant T. \tag{5.11}$$

Thus, z_m is a barrier for w_m at each point of the boundary $x = 0$. Consequently,

$$\frac{\partial w_m}{\partial x}(0, t) \leqslant C_1, \quad 0 \leqslant t \leqslant T, \ m > m_0. \tag{5.12}$$

By similar arguments, the remaining inequalities in (5.1) can be demonstrated.

LEMMA 2. – There exists a positive constant C which depends only upon δ and the parameters in the set $\mathcal{S}$ such that for $m > m_0$,

$$\|w_m\|_{W^{1,0}(Q_T)} \leqslant C. \tag{5.13}$$

PROOF. – We multiply both sides of the differential equation in (4.6) by $\varphi_m(x, w_m(x, t))$ and integrate over $0 < x < 1$ and $0 < \tau < t$. An integration by parts on the right hand side yields

$$(5.14)\quad \frac{1}{2}\int_0^1 \varphi_m^2(x, w_m(x,t))\,dx - \frac{1}{2}\int_0^1 \varphi_m^2(x, h_m(x))\,dx =$$
$$= \int_0^t \varphi_m(1, f_{2,m}(\tau))\frac{\partial w_m}{\partial x}(1,\tau)\,d\tau - \int_0^t \varphi_m(0, f_{1,m}(\tau))\frac{\partial w_m}{\partial x}(0,\tau)\,d\tau -$$
$$- \int_0^t\int_0^1 \gamma_m \frac{\partial w_m}{\partial x}\left\{\frac{\partial \varphi_m}{\partial x}(x, w_m) + \frac{\partial \varphi_m}{\partial \xi}\frac{\partial w_m}{\partial x}\right\} dx\,d\tau\,.$$

Using the assumptions (A_1), (A_2) and (B_1) it follows from (5.1), (5.14) and Schwartz's inequality that

$$(5.15)\quad \mu\nu\int_0^t\int_0^1 \left(\frac{\partial w_m}{\partial x}\right)^2 dx\,d\tau \leqslant \frac{1}{2} + 2TC + \mu_2\,\nu_1\sqrt{T}\left\{\int_0^t\int_0^1\left(\frac{\partial w_m}{\partial x}\right)^2 dx\,d\tau\right\}^{\frac{1}{2}};$$

whence, we conclude that

$$(5.17)\quad \left\{\int_0^t\int_0^1\left(\frac{\partial w_n}{\partial x}\right)^2 dx\,dt\right\}^{\frac{1}{2}} \leqslant \frac{\mu_2\,\nu_1\sqrt{T} + \sqrt{(\mu_2\,\nu_2)^2\,T + 2\mu\nu(1+4TC_1)}}{2\mu\nu}\,.$$

Combining (5.17) with (4.7). We obtain the result (5.13).

Writing the differential equation for w_m in the form

$$(5.18)\quad \frac{\partial w_m}{\partial t} = \frac{\partial}{\partial x}\left\{\gamma_m\left(\frac{\partial \varphi_m}{\partial \xi}\right)^{-1}\frac{\partial w_m}{\partial x}\right\} + \gamma_m\left(\frac{\partial \varphi_m}{\partial \xi}\right)^{-2}\left\{\frac{\partial w_m}{\partial x}\left[\frac{\partial^2 \varphi_m}{\partial x\,\partial \xi} + \frac{\partial^2 \varphi_m}{\partial \xi^2}\frac{\partial w_m}{\partial x}\right]\right\},$$

it follows from the assumptions (A_1), (A_2), (B_1) and (B_2) and from (4.7) that Theorem 1.1 of [8; p. 419] can be applied to obtain the following result.

LEMMA 3. – There exist positive constants H and α, $0 < \alpha < 1$, which depend only on the parameters in the set $\mathcal{T}$, such that

$$(5.19)\quad |w_m(x',t') - w_m(x'',t'')| \leqslant H\{|x'-x''|^{\alpha} + |t'-t''|^{\alpha/2}\}$$

holds for all $m = 1, 2, \ldots,$ and (x',t'), $(x'',t'') \in \overline{Q_T}$.

Based upon the results of Lemmas 1, 2 and 3, we can demonstrate the following lemma.

LEMMA 4. – There exists a positive constant C, which depends only on the parameters in the set $\mathcal{T}$ and on δ, such that for each $m > m_0$ and any t', $t'' \in (0, T]$,

$$(5.20)\quad |s_m(t') - s_m(t'')| \leqslant C|t'-t''|^{\alpha/2}\,,$$

where α is the exponent in (5.19).

PROOF. – Recalling the construction of s_m, it is clear that we can restrict our attention to the time interval $[1/m, t_m^*]$ in which s_m is defined via (4.10), (4.12) and (4.13). Selecting t' and t'' from $[1/m, t_m^*]$, it follows from (4.10), (4.12) and (4.13) that

$$\int_0^{s_m(t')} x\varphi_m\left(x, w_m\left(x, t'-\frac{1}{m}\right)\right)dx - \int_0^{s_m(t'')} x\varphi_m\left(x, w_m\left(x, t''-\frac{1}{m}\right)\right)dx = \int_{t'-1/m}^{t''-1/m}\int_0^{s_m(\tau)} \gamma_m(x, w_m(x,\tau),\tau)\frac{\partial w_m}{\partial x}(x,\tau)\,dx\,d\tau . \tag{5.21}$$

Hence,

$$\left|\int_{s_m(t'')}^{s_m(t')} x\varphi_m\left(x, w_m\left(x, t'-\frac{1}{m}\right)\right)dx\right| \leqslant \int_0^1\left|\varphi_m\left(x, w_m\left(x, t'-\frac{1}{m}\right)\right) - \varphi_m\left(x, w_m\left(x, t''-\frac{1}{m}\right)\right)\right|dx + \left|\int_{t'-1/m}^{t''-1/m}\int_0^{s_m(\tau)} \gamma_m(x, w_m(x,\tau),\tau)\frac{\partial w_m}{\partial x}(x,\tau)\,dx\,d\tau\right| = I_1 + I_2 . \tag{5.22}$$

Considering the left hand side of (5.22) and recalling assumption (A_1) and (4.11) we see that

$$\left|\int_{s_m(t'')}^{s_m(t')} x\varphi_m\left(x, w_m\left(x, t'-\frac{1}{m}\right)\right)dx\right| \geqslant \frac{\varphi_0}{2}\left|[s_m(t')]^2 - [s_m(t'')]^2\right| \geqslant \delta\varphi_0|s_m(t') - s_m(t'')| . \tag{5.23}$$

From lemma 3 and (1.7), I_1 the first term on the right hand side of (5.22), can be estimated to yield

$$I_1 \leqslant \mu_1 H|t'-t''|^{\alpha/2} . \tag{5.24}$$

Applying Schwartz's inequality to I_2 and using Lemma 2, we obtain

$$I_2 \leqslant \nu_1 C|t'-t''|^{\frac{1}{2}} . \tag{5.25}$$

Consequently, the result (5.19) follows from (5.22), (5.23), (5.24), and (5.25).

The following lemma is an immediate consequence of Lemma 4.

LEMMA 5. – There exists a time interval $[0, T_0]$, $T_0 > 0$ and independent of m, such that for $0 \leqslant t \leqslant T_0$ and $m > m_0$,

$$\delta < s_m(t) < 1-\delta , \tag{5.26}$$

where δ satisfies $0 < \delta < s_0 < 1-\delta$.

6. – Proof of the existence theorem.

Set

$$F(x, t) = (1-x) f_1(t) + x f_2(t)\,, \qquad 0 \leqslant x \leqslant 1\,, \qquad 0 < t \leqslant T_0\,, \tag{6.1}$$

and

$$F_m(x, t) = (1-x) f_{1,m}(t) + x f_{2,m}(t)\,, \qquad 0 \leqslant x \leqslant 1\,, \qquad 0 < t \leqslant T_0\,, \qquad m > m_0\,, \tag{6.2}$$

where T_0 is that of Lemma 5. Then, it is clear that $\{F_m\}$ is a sequence of C^∞ functions which converge uniformly to F. Moreover, $\{F_m\}$ converges strongly to F in the norm of $W_2^{1,0}(Q_{T_0})$. If we multiply the differential equation in (4.6) by a test function $\eta \in W_2^{1,1}(Q_{T_0})$ which vanishes when $x = 0$, $x = 1$, and $t = T_0$, perform the usual integration by parts over Q_{T_0}, and define

$$v_m = w_m - F_m\,, \tag{6.3}$$

then we obtain

$$\int_0^{T_0}\!\!\int_0^1 \left\{- \varphi_m(x, v_m + F_m) \frac{\partial \eta}{\partial t} + \gamma_m(x, v_m + F_m, t) \frac{\partial (v_m + F_m)}{\partial x} \frac{\partial \eta}{\partial x}\right\} dx\, dt = \int_0^1 \eta(x, 0) \varphi_m(x, h_m(x))\, dx\,. \tag{6.4}$$

In addition the equations (4.10), (4.12), and (4.13) can be rewritten in the form

$$\int_0^{s_m(t+1/m)} x \varphi_m(x, v_m(x, t) + F_m(x, t))\, dx = \int_0^{s_0} x \varphi_m(x, h_m(x))\, dx - \int_0^t \int_0^{s_m(\tau)} \gamma_m(x, v_m + F_m, \tau) \frac{\partial (v_m + F_m)}{\partial x}\, dx\, dT\,. \tag{6.5}$$

From (6.2) and the results and analysis of section 5, it follows that the v_m, $m > m_0$, form a sequence of functions which belong to $W_2^{1,0}(Q_{T_0})$ and which have their norms uniformly bounded. Moreover, the v_m, $m > m_0$, are uniformly bounded and are uniformly Hölder continuous in $\overline{Q_{T_0}}$. From Lemma 4 and Lemma 5, we see that the s_m are equibounded and equicontinuous in $0 \leqslant t \leqslant T_0$. By the usual diagonal process of the proof of the Ascoli-Arzela Theorem, there exist subsequences $\{v_{m'}\}$

and $\{s_{m'}\}$ such that

$$(6.6)\quad \begin{cases} \text{(1) the sequence } \{v_{m'}\} \text{ converges uniformly in } \overline{Q_{T_0}} \text{ to a Hölder continuous function } v, \\ \text{(2) the sequence } \{v_{m'}\} \text{ converges weakly to } v \text{ in } \mathring{W}_2^{1,0}(Q_{T_0}) \text{ which implies that } v \in \mathring{W}_2^{1,0}(Q_{T_0}), \text{ and} \\ \text{(3) the sequence } \{s_{m'}\} \text{ converges uniformly to a Hölder continuous function } s \text{ on } [0, T_0]. \end{cases}$$

In what follows, we shall use (6.4), (6.5), and (6.6) to show that (v, s) satisfies (2.13) and (2.15) and thus forms a weak solution of (1.5)-(1.6).

First, we relabel the subscripts of the sequences $\{v_{m'}\}$ and $\{s_{m'}\}$ to m, and note that from (6.6) it follows that

$$\lim_{m\to\infty} \varphi_m(x, v_m + F_m) = \varphi(x, v + F) \tag{6.7}$$

uniformly in $\overline{Q_{T_0}}$. Next, for each ε, $0 < \varepsilon < \delta/4$, select two piecewise linear functions $s_-^{(\varepsilon)} = s_-^{(\varepsilon)}(t)$ and $s_+^{(\varepsilon)} = s_+^{(\varepsilon)}(t)$ such that

$$\varepsilon < s(t) - s_-^{(\varepsilon)}(t) < 2\varepsilon\,, \qquad 0 \leqslant t \leqslant T_0\,, \tag{6.8}$$

and that

$$\varepsilon < s_+^{(\varepsilon)}(t) - s(t) < 2\varepsilon\,, \qquad 0 \leqslant t \leqslant T_0\,. \tag{6.9}$$

Set

$$Q^{(\varepsilon)} = \{(x, t) : s_-^{(\varepsilon)}(t) < x < s_+^{(\varepsilon)}(t)\,,\ 0 \leqslant t \leqslant T_0\} \tag{6.10}$$

and

$$Q_c^{(\varepsilon)} = \overline{Q_{T_0}} - Q^{(\varepsilon)}\,. \tag{6.11}$$

We remark that the measure of $Q^{(\varepsilon)}$ satisfies

$$\operatorname{mes} Q^{(\varepsilon)} = O(\varepsilon)\,. \tag{6.12}$$

Since the s_m converge uniformly to s, it follows that

$$\lim_{m\to\infty} \gamma_m(x, v_m + F_m, t) = \gamma(x, v + F; s) \tag{6.13}$$

uniformly in $Q_c^{(\varepsilon)}$. Consider the quantity

$$(6.14)\qquad J=\int_0^{T_0}\int_0^1\left\{-\varphi(x,v+F)\frac{\partial\eta}{\partial t}+\gamma(x,v+F;s)\frac{\partial(v+F)}{\partial x}\frac{\partial\eta}{\partial x}\right\}dx\,dt-\int_0^1\eta(x,0)\varphi(x,h(x))\,dx\,,$$

where the v and s are the limits of the sequences $\{v_m\}$ and $\{s_m\}$ whose subscripts have been relabeled. Using (6.4) we see that for each m,

$$(6.15)\qquad J=J_m^{(1)}+J_m^{(2)}+J_m^{(3)}\,,$$

where

$$(6.16)\qquad J_m^{(1)}=\int_0^{T_0}\int_0^1\{\varphi_m(x,v_m+F_m)-\varphi(x,v+F)\}\frac{\partial\eta}{\partial t}\,dx\,dt\,,$$

$$(6.17)\qquad J_m^{(2)}=\int_0^1\{\varphi_m(x,h_m(s))-\varphi(x,h(x))\}\,dx\,,$$

and

$$(6.18)\qquad J_m^{(3)}=\int_0^{T_0}\int_0^1\left\{\gamma(x,v+F;\varrho)\frac{\partial(v+F)}{\partial x}-\gamma_m(x,v_m+F_m,t)\frac{\partial(v_m+F_m)}{\partial x}\right\}\frac{\partial\eta}{\partial x}\,dx\,dt.$$

Now (6.7) implies that for m sufficiently large

$$(6.19)\qquad |J_m^{(1)}|,\ |J_m^{(2)}|<\varepsilon\,.$$

With respect to the contribution to $J_m^{(3)}$ from $Q^{(\varepsilon)}$, Lemma 2, and and application of Schwarz's lemma implies a contribution which tends uniformly to zero as ε tends to zero. Considering the contribution to $J_m^{(3)}$ from $Q_c^{(\varepsilon)}$, (6.13) and (6.6) imply that for m sufficiently large that contribution in absolute value can be made less than ε. Hence,

$$(6.20)\qquad J=0;$$

and we see that our limit pair (v,s) satisfies (2.13). With respect to equation (2.15) we can define

$$(6.21)\qquad \Gamma=\int_0^{s(t)}x\varphi(x,v+F)\,dx-\int_0^{s_0}x\varphi(x,h(x))\,dx+\int_0^t\int_0^{s(\tau)}\gamma(x,v+F;s)\frac{\partial(v+F)}{\partial x}\,dx\,d\tau.$$

Using (6.5) and an analysis similar to that above, it follows that

$$\Gamma = 0\,, \tag{6.23}$$

and we see that our limit pair (v, s) satisfies (2.15). Consequently, (v, s) is a weak solution to (1.5)-(1.6) in Q_{T_0}.

PART II

SECOND BOUNDARY VALUE PROBLEM

1. – Introduction.

Referring to Part I for a discussion of the physics and symbols, we consider in this paper the problem of finding functions p, q and s which as a triple (p, q, s) satisfy

$$(1.1)\quad \begin{cases} (1) & \dfrac{\partial\varphi(x,p)}{\partial t} = \dfrac{\partial}{\partial x}\left[a(x,p)\dfrac{\partial p}{\partial x}\right], \quad 0<x<s(t),\ 0<t\leqslant T,\ 0<s(t)<1\,,\\[2mm] (2) & \dfrac{\partial\varphi(x,q)}{\partial t} = \dfrac{\partial}{\partial x}\left[b(x,q)\dfrac{\partial q}{\partial x}\right], \quad s(t)<x<1,\ 0<t\leqslant T\,,\\[2mm] (3) & p(x,0) = h_1(x)\,, \quad 0\leqslant x\leqslant s_0,\ s(0)=s_0,\ 0<s_0<1\,,\\[2mm] (4) & q(x,0) = h_2(x)\,, \quad s_0\leqslant x\leqslant 1\,,\\[2mm] (5) & a(0,p(0,t))\dfrac{\partial p}{\partial x}(0,t) = g_1(t)\,, \quad 0<t\leqslant T\,,\\[2mm] (6) & b(1,q(1,t))\dfrac{\partial q}{\partial x}(1,t) = g_2(t)\,, \quad 0<t\leqslant T\,,\\[2mm] (7) & p(s(t),t) = q(s(t),t)\,, \quad 0<t\leqslant T\,,\\[2mm] (8) & a\big(s(t),p(s(t),t)\big)\dfrac{\partial p}{\partial x}(s(t),t) = b(s(t),q(s(t),t)\dfrac{\partial q}{\partial x}(s(t),t)\,, \quad 0<t\leqslant T\,, \end{cases}$$

and

$$\varphi\big(s(t),p(s(t),t)\big)\dot{s}(t) = -\,a\big(s(t),p(s(t),t)\big)\frac{\partial p}{\partial x}(s(t),t)\,, \quad 0<t\leqslant T\,, \tag{1.2}$$

where the functions φ, a, b, h_1, h_2, g_1 and g_2 are given functions of their respective arguments and the s_0, $0<s_0<1$, is a specified constant. As in Part I, the physical situation suggests the following assumptions on the φ, a, and b:

(A_1) there exist positive constants φ_0 and μ_1 such that for $0\leqslant x\leqslant 1$ and $-\infty<\xi<\infty$,

$$0<\varphi_0\leqslant\varphi(x,\xi)\leqslant 1\,, \qquad 0<\frac{\partial\varphi}{\partial\xi}(x,\xi)\leqslant\mu_1, \tag{1.3}$$

and for each positive constant M, there exists a positive constant $\mu(M)$ such that

$$0<\mu(M)\leqslant\frac{\partial\varphi}{\partial\xi}(x,\xi) \tag{1.4}$$

when $0\leqslant x\leqslant 1$ and $|\xi|\leqslant M$;

(B_1) there exist positive constants ν and ν_1 such that for $0\leqslant x\leqslant 1$ and $-\infty<\xi<\infty$,

$$0<\nu\leqslant a(x,\xi)\,, \qquad b(x,\xi)\leqslant\nu_1\,. \tag{1.5}$$

Under these assumptions equations (1.1)-(1) and (2) are nonlinear parabolic partial differential equations. Again, it should amphasized that the physical condition (1.3) implies that the problem must be nonlinear. We shall add additional assumptions on the data later.

In the next section we shall define what is meant by a classical solution of problem (1.1)-(1.2), and we shall derive a weak formulation of it. As in Part I the remaining sections are devoted to the demonstration of the existence of a weak solution. *Although much of the analysis used in Part I will apply to the problem* (1.1)-(1.2), *there is a delicate difference in that the maximum principle cannot be applied directly to obtain an a priori estimate on the solution so that* (1.4) *can be applied to obtain a uniform parabolicity. Different techniques are needed.*

The structure of this paper follows that of Part I. In section 2 the definition of a classical and weak solution of (1.1)-(1.2) is given. Section 3 is devoted to the statement of the existence theorem along with the hypotheses on the data. Section 4 is devoted to the presentation of the derivation of a sequence of approximation via a retarded argument method presented in Part I. In this paper the construction of the approximating sequence is considerably more delicate than in Part I. In Section 5 some estimates on the approximations are obtained which enable us to conclude the proof of existence of a weak solution in Section 6.

As in Part I, we shall use the notation and definitions of LADYZENSKAJA, SOLONIKOV, and URAL'CEVA's book on Linear and Quasilinear Equations of Parabolic Type [8].

2. – A weak formulation of problem (1.1) - (1.2).

We begin with the definition of a classical solution of problem (1.1)-(1.2).

DEFINITION. – A classical solution of problem (1.1)-(1.2) is a triple of functions (p,q,s) such that

i) $s=s(t)$ is continuous in $0\leqslant t\leqslant T$ and continuously differentiable in $0<t\leqslant T$;

ii) $p = p(x, t)$ is continuous in $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant T$, $\partial p/\partial x$ is continuous in $0 \leqslant x \leqslant s(t)$, $0 < t \leqslant T$, $\partial^2 p/\partial x^2$ and $\partial p/\partial t$ are continuous in $0 < x < s(t)$, $0 < t \leqslant T$;

iii) $q = q(x, t)$ is continuous in $s(t) \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$, $\partial q/\partial x$ is continuous in $s(t) \leqslant x \leqslant 1$, $0 < t \leqslant T$, $\partial^2 q/\partial x^2$ and $\partial q/\partial t$ are continuous in $s(t) < x < 1$, $0 < t \leqslant T$;

iv) the equations in (1.1)-(1.2) are satisfied by p, q, and s.

Obviously, the definition imposes some minimal assumptions upon the data.

REMARK 1. – For a given continuously differentiable s, a classical solution of (1.1) is a pair of functions (p, q) which satisfies ii), iii) above and the equations in (1.1).

If we consider a classical solution of (1.1) for a given continuously differentiable function $s = s(t)$, and if we define

$$w(x, t) = \begin{cases} p(x, t)\,, & 0 \leqslant x \leqslant s(t),\ 0 \leqslant t \leqslant T\,, \\ q(x, t)\,, & s(t) \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T\,, \end{cases} \tag{2.1}$$

$$\gamma(x, \xi; s) = \begin{cases} a(x, \xi)\,, & 0 \leqslant x \leqslant s(t),\ -\infty < \xi < \infty\,. \\ b(x, \xi)\,, & s(t) < x \leqslant 1,\ -\infty < \xi < \infty\,. \end{cases} \tag{2.2}$$

and

$$h(x) = \begin{cases} h_1(x)\,, & 0 \leqslant x \leqslant s_0\,, \\ h_2(x)\,, & s_0 < x \leqslant 1\,, \end{cases} \tag{2.3}$$

then it can be shown as in section 2 of Part I that

$$\int_0^T\!\!\int_0^1 \left\{ -\varphi(x, w(x, t))\frac{\partial \eta}{\partial t} + \gamma(x, w(x, t); s)\frac{\partial w}{\partial x}\frac{\partial \eta}{\partial x} \right\} dx\, dt = \int_0^1 \varphi(x, h(x))\eta(x, 0)\, dx + \int_0^T \{g_2(t)\eta(1, t) - g_1(t)\eta(0, t)\}\, dt \tag{2.4}$$

must be satisfied for every $\eta \in W_2^{1,1}(Q_T)$ which vanishes when $t = T$.

Integrating equation (1.1)-(1) over the domain $0 \leqslant x \leqslant s(\tau)$, $0 \leqslant \tau \leqslant t$ and using equation (1.2) and (2.1), follows that

$$\int_0^{s(t)} \varphi(x, w(x, t))\, dx = \int_0^{s_0} \varphi(x, h(x))\, dx - \int_0^t g_1(\tau)\, d\tau\,. \tag{2.5}$$

Consequently, a classical solution of (1.1)-(1.2) must satisfy via (2.1), (2.2) and (2.3) the equations (2.4) and (2.5). Thus, we are motivated to make the following definition.

DEFINITION. – A weak solution of problem (1.1)-(1.2) is a pair of functions (w, s) such that

i) $s = s(t)$ is continuous in $[0, T]$, $0 \leqslant s \leqslant 1$, and $s(0) = s_0$;

ii) $w \in W_2^{1,0}(Q_T)$ and is bounded;

iii) the pair (w, s) satisfies (2.4) for every $\eta \in W_2^{1,1}(Q_T)$ that vanishes at $t = T$, and the pair (w, s) satisfies (2.5).

REMARK 2. – Clearly any weak solution of (1.1)-(1.2) which has the necessary smoothness of a classical solution can be shown to generate a classical solution.

3. – Statement of an existence theorem.

It is convenient here to present the remainder of our assumptions on the data prior to the statement of our result. Recalling assumptions (A_1) and (B_1) in section 1, we add the following:

(A_2) there exist positive constants μ_2, μ_3, and μ_4 such that

$$\left|\frac{\partial\varphi}{\partial x}(x,\xi)\right| \leqslant \mu_2, \qquad 0 \leqslant x \leqslant 1, \ -\infty < \xi < \infty, \tag{3.1}$$

$$\left|\frac{\partial^2\varphi}{\partial x\,\partial\xi}(x,\xi)\right| \leqslant \mu_3, \qquad 0 \leqslant x \leqslant 1, \ -\infty < \xi < \infty, \tag{3.2}$$

and

$$\left|\frac{\partial^2\varphi}{\partial\xi^2}(x,\xi)\right| \leqslant \mu_4, \qquad 0 \leqslant x \leqslant 1, \ -\infty < \xi < \infty; \tag{3.3}$$

(B_2) there exist positive constants ν_2 and ν_3 such that

$$\left|\frac{\partial a}{\partial x}(x,\xi)\right| \leqslant \nu_2, \ \left|\frac{\partial b}{\partial x}(x,\xi)\right| \leqslant \nu_2, \qquad 0 \leqslant x \leqslant 1, \ -\infty < \xi < \infty, \tag{3.4}$$

and

$$\left|\frac{\partial a}{\partial\xi}(x,\xi)\right| \leqslant \nu_3, \ \left|\frac{\partial b}{\partial\xi}(x,\xi)\right| \leqslant \nu_3, \qquad 0 \leqslant x \leqslant 1, \ -\infty < \xi < \infty; \tag{3.5}$$

(C_1) the data h is bounded, positive and Lipschitz continuous; in other words, there exist positives constants M_0, M_1, L_0 such that

$$0 < M_0 \leqslant h(x) \leqslant M_1, \qquad 0 \leqslant x \leqslant 1, \tag{3.6}$$

and

$$|h(x_1) - h(x_2)| \leqslant L_0|x_1 - x_2|, \qquad x_1, x_2 \in [0,1]; \tag{3.7}$$

and

(C_2) there exists a positive constant G_0 such that

$$|g_1(t)|\,,\quad |g_2(t)|< G_0\,,\qquad 0<t\leqslant T\,, \tag{3.8}$$

and the g_i are measurable in the sense of Lebesgue.

THEOREM. – Under the assumptions (A_1), (A_2), (B_1), (B_2), (C_1) and (C_2) upon the data φ, a, b, h, g_1, g_2 and s_0, $0<s_0<1$, it follows that for any δ such that $0<\delta<s_0<1-\delta$ there exists $T>0$ such that in Q_T there exists at least one weak solution (w, s) to the free boundary problem (1.1)-(1.2) such that

$$\delta\leqslant s(t)\leqslant 1-\delta\,,\qquad 0\leqslant t\leqslant T\,.$$

4. – A sequence of approximations.

We begin as in Part I by extending the functions φ, a, and b to the domain $\{(x,\xi)\colon -\infty<x<\infty,\ -\infty<\xi<\infty\}$. The extension can be carried out so that the extended functions satisfy (A_1), (A_2), (B_1) and (B_2), where some of the constants appearing in these conditions may have been modified slightly. For any continuous function $s=s(t)$, $0\leqslant t\leqslant T$, $s(0)=s_0$, we extend it by setting $s(t)\equiv s_0$ for $t<0$ and $s(t)\equiv s(T)$ for $t>T$. Recalling now the definition of $\gamma=\gamma(x,\xi;s)$ given by (2.2), we utilize the extensions above to extend γ via (2.2) over the domain $\{(x,t,\xi)\colon -\infty<x<\infty,\ -\infty<t<\infty, \text{ and } -\infty<\xi<\infty\}$.

Next, we must define a modified porosity $\varphi^*(x,\xi)$. Pick a constant

$$P>M_1\,, \tag{4.1}$$

where M_1 is defined in (C_1) and define

$$\varphi^*(x,\xi)=\begin{cases}\varphi(x,-P-1)+(\xi+P+1)\dfrac{\partial\varphi}{\partial\xi}(x,-P-1)\,, & -\infty<x<\infty,\\ & -\infty<\xi<-P-1\,,\\ \varphi(x,\xi)\,, \quad -\infty<x<\infty,\ |\xi|\leqslant P+1\,, & \\ \varphi(x,P+1)+(\xi-P-1)\dfrac{\partial\varphi}{\partial\xi}(x,P+1)\,, & -\infty<x-\infty\,,\\ & P+1<\xi<\infty\,.\end{cases} \tag{4.2}$$

From (1.3) and (1.4), it follows that

$$\mu=\min\{1,\mu(P+1)\}\leqslant\frac{\partial\varphi^*}{\partial\xi}\leqslant\mu_1\,. \tag{4.3}$$

We mollify [3; p. 274] φ^* and γ to obtain sequences $\{\varphi_m^*\}$ and $\{\gamma_m\}$ of C^∞ functions which satisfy (A_1), (A_2), (B_1) and (B_2) with the exception that φ_m^*, $m=1,2,\ldots$, are not bounded and that the φ_m^*, $m=1,2,\ldots$, satisfy (4.3). Moreover, the φ_m^*, $m=1,2,\ldots$, converge uniformly to φ^* on compact subsets of $\{(x,\xi): -\infty<x<\infty, -\infty<\xi<\infty\}$ while the k_m, $m=1,2,\ldots$, converge uniformly to γ on compact subsets of $\{(x,\xi,t): -\infty<x<\infty, -\infty<\xi<\infty, -\infty<t<\infty\}$ which do not intersect the surface $x=s(t)$. *Note that the mollification can be achieved by integrating the functions φ^* and γ against C^∞ kernels which have support in balls of radius* $1/2m$.

In a manner similar to that in Part I, we can obtain C^∞ approximations $g_{i,m}$ of g_i and h_m of h such that

$$\lim_{m\to\infty}\int_0^T |g_{i,m}(\tau)-g_i(\tau)|\,d\tau = 0\,, \qquad i=1,2\,, \tag{4.4}$$

$$|h_m'|\leqslant L_0\,, \qquad 0\leqslant x\leqslant 1\,, \tag{4.5}$$

$$g_{1m}(0)=\gamma_m(0,h_m(0),0)\,h_m'(0)\,, \tag{4.6}$$

$$g_{2m}(0)=\gamma_m(1,h_m(1),0)\,h_m'(1)\,. \tag{4.7}$$

Now, set

$$s_m(t)\equiv s_0\,, \qquad 0\leqslant t\leqslant\frac{1}{m}\,, \tag{4.8}$$

and use this definition of s in γ to calculate the γ_m for $0\leqslant t\leqslant 1/2m$. Then, consider the problems

$$\begin{cases} \dfrac{\partial\varphi_m^*(x,w_m)}{\partial t}=\dfrac{\partial}{\partial x}\left[\gamma_m(x,w_m,t)\dfrac{\partial w_m}{\partial x}\right], & 0<x<1,\ 0<t\leqslant\dfrac{1}{2m},\\[2mm] \gamma_m(0,w_m(0,t),t)\dfrac{\partial w_m}{\partial x}(0,t)=g_{1,m}(t)\,, & 0<t\leqslant\dfrac{1}{2m},\\[2mm] \gamma_m(1,w_m(1,t),t)\dfrac{\partial w_m}{\partial x}(1,t)=g_{2,m}(t)\,, & 0<t\leqslant\dfrac{1}{2m},\\[2mm] w_m(x,0)=h_m(x)\,, & 0\leqslant x\leqslant 1,\ m=1,2,\ldots. \end{cases} \tag{4.9}$$

In order to obtain the existence of a classical solution of (4.9), we write our differential equation in the form considered in Chapter V, Section 7, of [8; p. 475]:

$$\frac{\partial w_m}{\partial t}=\left(\frac{\partial\varphi_m^*}{\partial\xi}\right)^{-1}\gamma_m\frac{\partial^2 w_m}{\partial x^2}+\left(\frac{\partial\varphi_m^*}{\partial\xi}\right)^{-1}\left\{\frac{\partial\gamma_m}{\partial x}\frac{\partial w_m}{\partial x}+\frac{\partial\gamma_m}{\partial\xi}\left(\frac{\partial w_m}{\partial x}\right)^2\right\}.$$

In order to apply theorem 7.4 [1; p. 491], we note that a modification of the term $(\partial\gamma_m/\partial\xi)(\partial w_m/\partial x)^2$ must be made. This is accomplished by replacing the coefficient

$\partial\gamma_m/\partial\xi$ by $f(\xi)(\partial\gamma_m/\partial\xi)$, where f is identically equal to one for $|\xi| < \bar{P}$, identically equal to zero for $|\xi| > 2\bar{P}$, and varies smoothly between one and zero elsewhere in such a way as to form a smooth function. Hence, we may apply theorem 7.4 [8; p. 491] to obtain a classical solution w_m of the modified equation. We note that when $|w_m| \leqslant \bar{P}$, w_m satisfies (4.9). Also, we note that $\bar{P}$ is a parameter which is ours to choose. In the remainder of the paper it is convenient to select $\bar{P} > 2P$, where P is the parameter which is used to modify the function φ. With this choice of $\bar{P}$, the function f will never again appear below since we shall restrict our attention to values of w_m which are less than $2P$ in absolute value.

Following the procedure of section 4, Part I, we select a positive constant δ such that

$$0 < \delta < s_0 < 1 - \delta \tag{4.10}$$

and define s_m via

$$\int\limits_0^{s_m(t+1/m)} \varphi_m(x, w_m(x,t))\,dx = \int\limits_0^{s_0} \varphi_m(x, h_m(x))\,dx - \int\limits_0^{t} g_{1,m}(\tau)\,d\tau \tag{4.11}$$

for as long as s_m satisfies

$$\delta < s_m < 1 - \delta \tag{4.12}$$

and if ever $s_m = \delta$ or $s_m = 1-\delta$, then we define $s_m \equiv \delta$ or $s_m \equiv 1-\delta$ for all t following that of attaining the value of δ or $1-\delta$. Having obtained the definition of s_m over $1/m \leqslant t \leqslant 1/m + 1/2m$, we can obtain the φ_m^* and γ_m for $0 \leqslant t \leqslant 1/m$ and solve the equations in (4.9) up to $t = 1/m$. Returning to the discussion following (4.10), we can extend s_m up to $t = 1/m + 2(1/2m)$ and return to (4.9). In this manner we can assume that s_m has been defined for $0 \leqslant t \leqslant T$ and that there exists a smooth classical solution of (4.9) in $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$.

5. – Uniform estimates on w_m and s_m.

We begin by demonstrating the following result.

LEMMA 1. – There exists a positive constant T_0, which is independent of m, and there exists a positive integer m_0 such that for $m \geqslant m_0$,

$$|w_m(x,t)| \leqslant P\,, \qquad 0 \leqslant x \leqslant 1\,, \qquad 0 \leqslant t \leqslant T_0\,, \tag{5.1}$$

where P is a constant independent of m which is chosen sufficiently larger then M_1.

PROOF. – Since the $w_m(x,t)$ are continuous, set

$$T_m(P) = \max\{t\colon 0 \leqslant t \leqslant T;\ w_m(x,\tau) \leqslant 2P,\ 0 \leqslant x \leqslant 1,\ 0 \leqslant \tau \leqslant t\}\ , \tag{5.2}$$

$m=1,2,3,\ldots$. Since P is taken larger than M_1, it follows that $T_m(P)>0$. Next we consider the function

$$z(x,t)=At+B_1x^2+B_2x+B_3\,,\qquad A>0\,, \tag{5.3}$$

and we shall determine the coefficients A, B_i, $i=1,2,3$ in such a way that

$$\left.\frac{\partial z}{\partial x}\right|_{x=0}<-\frac{G_0}{\nu} \tag{5.4}$$

$$\min_{0\leqslant x\leqslant 1} z(x,0)>M_1\,, \tag{5.5}$$

$$z\left(\frac{\delta}{2},t\right)>2P\,, \tag{5.6}$$

and

$$\frac{\partial z}{\partial t}>\left\{\frac{\partial\varphi_m^*}{\partial\xi}(x,z)\right\}^{-1}\left\{\gamma_m(x,z,t)\frac{\partial^2 z}{\partial x^2}+\left[\frac{\partial\gamma_m}{\partial x}(x,z,t)+\frac{\partial\gamma_m}{\partial\xi}(x,z,t)\frac{\partial z}{\partial x}\right]\frac{\partial z}{\partial x}\right\}. \tag{5.7}$$

Then, it will follow from the lemma of Westphal-Prodi [2; p. 52] that

$$w_m(x,t)<z(x,t) \tag{5.8}$$

for $0\leqslant x\leqslant\delta/2$ and $0\leqslant t\leqslant T_m$.

Condition (5.4) can be satisfied by selecting

$$B_2=-2\frac{G_0}{\nu}. \tag{5.9}$$

Next, with the aim of obtaining $z(0,t)<P$, we select

$$B_3=\lambda_1 P\,, \tag{5.10}$$

where $\lambda_1<1$. Now, we consider (5.6) and determine B_1 from

$$B_1\left(\frac{\delta}{2}\right)^2-\frac{G_0}{\nu}+\lambda_1 P=\lambda_2 P\,, \tag{5.11}$$

where $\lambda_2>2$. Solving (5.11) for B_1, we obtain

$$B_1=\frac{4}{\delta^2}\{(\lambda_2-\lambda_1)P+G_0\nu^{-1}\}\,. \tag{5.12}$$

Since the minimum of $z(x, 0)$ occurs at $x = G_0 \nu^{-1} B_1^{-1}$, we obtain

$$z(x, 0) \geqslant z(G_0 \nu^{-1} B_1^{-1}, 0) = -G_0^2 \nu^{-1} B_1^{-1} + \lambda_1 P\,. \tag{5.13}$$

From (5.12) we see that B_1 is an increasing function of P. Consequently, we can select P sufficiently large so that

$$-G_0^2 \nu^{-1} B_1^{-1} + \lambda_1 P > M_1\,. \tag{5.14}$$

Hence, (5.5) is satisfied. In order to satisfy (5.7) we first observe that for all $m > m_0 = [1/\delta] + 1$, the derivatives of γ_m are uniformly bounded by the constants in (B_1) and (B_2) since the domains of integration for the mollification do not intersect s_m. Thus, for $0 \leqslant x \leqslant \delta/2$ and $0 \leqslant t \leqslant T_m(P)$,

$$\begin{aligned}&\left\{\frac{\partial \varphi^* m}{\partial \xi}(x, z)\right\}^{-1}\left\{\gamma_m(x, z, t)\frac{\partial^2 z}{\partial x^2} + \left[\frac{\partial \gamma_m}{\partial x}(x, z, t) + \frac{\partial \gamma_m}{\partial \xi}(x, z, t)\frac{\partial z}{\partial x}\right]\frac{\partial z}{\partial x}\right\}\\ &\quad < \{\mu(P+1)\}^{-1}\{2\nu_1 B_1 + [\nu_2 + \nu_3(B_1\delta + 2\nu^{-1}G_0)]\cdot(B_1\delta + 2\nu^{-1}G_0)\} < A = \frac{\partial z}{\partial t}\end{aligned}$$

can be achieved by selecting a constant A sufficiently large. Note that the choice can be made independently of m. From such a choice of A it follows that (5.8) is valid for $0 \leqslant x \leqslant \delta/2$ and $0 \leqslant t \leqslant T_m(P)$. In particular

$$w_m(0, t) < At + B_3 \tag{5.15}$$

and since $B_3 < P$, it follows that

$$w_m(0, t) \leqslant P\,, \qquad 0 \leqslant t \leqslant T^*\,, \tag{5.16}$$

where

$$T^* = \min\,(T_m(P), T_1) \tag{5.17}$$

and

$$T_1 = A^{-1}(P - B_3)\,. \tag{5.18}$$

By a similar analysis, it follows that

$$w_m(1, t) \leqslant P\,, \qquad 0 \leqslant t \leqslant T^*\,. \tag{5.19}$$

A straigh forward application of the maximum principle implies that

$$w_m(x, t) \leqslant P\,, \qquad 0 \leqslant x \leqslant 1\,, \qquad 0 \leqslant t \leqslant T^*\,, \tag{5.20}$$

and that

$$T_m(P) > T_1 . \tag{5.21}$$

By an argument similar to the one above we can show that for $m > m_0$

$$-P \leqslant w_m(x, t) , \qquad 0 \leqslant x \leqslant 1 , \qquad 0 \leqslant t \leqslant T_2 , \tag{5.22}$$

where T_2 is independent of m. Setting.

$$T_0 = \min(T_1, T_2) , \tag{5.23}$$

the result (5.1) is obtained.

REMARK. – From the definition of φ^* we see that mollification of φ and φ^* with respect to kernels having support contained in balls of radius $1/2m$ will yield the same functions φ_m whenever $|\xi| \leqslant P$ since φ agrees with φ^* for $|\xi| \leqslant P + 1$. Consequently, it follows from (4.1), (4.2), and (4.3) that our selection of P and the result of the Lemma imply that φ_m^* in (4.9) can be replaced by φ_m.

Using the analysis of Lemma 2 of Part I we obtain a similar result.

LEMMA 2. – There exists a positive constant C which depends only upon δ and the parameters in the set $\mathfrak{T} = \{\varphi_0, \mu(P), \mu_1, \mu_2, \mu_3, \mu_4, \nu, \nu_1; \nu_2, \nu_3, M_1, L_0, G_0, L_1\}$ such that for $m > m_0$,

$$\|w_m\|_{W^{1,0}(Q_T)} \leqslant C . \tag{5.24}$$

The next result is an immediate consequence of Lemma 1 and Theorem 1.1 in [8; p. 419].

LEMMA 3. – There exist positive constants H and α, $0 < \alpha < 1$, which depend only upon δ, the parameters in $\mathfrak{T}$, and the positive distance d, from the boundaries $x = 0$, $x = 1$ such that for $m > m_0$ and (x', t'), $(x'', t'') \in \{(x, t): 0 < d \leqslant x \leqslant 1 - d, 0 \leqslant t \leqslant T_0\}$

$$|w_m(x', t') - w_m(x'', t'')| \leqslant H\{|x' - x''|^\alpha + |t' - t''|^{\alpha/2}\} . \tag{5.25}$$

We note that Lemma 3 here differs from Lemma 3 of Part I since we do not obtain a uniform Hölder continuity in Q_{T_0}. Consequently, we must use a different argument here to obtain the equi-Hölder-continuity of the s_m.

LEMMA 4. – There exists a positive constant C, which depends only upon δ and the parameters in $\mathfrak{T}$, such that for $m > m_0$,

$$|s_m(t') - s_m(t'')| \leqslant C|t' - t''|^{\alpha/2} , \qquad t', t'' \in [0, T_0] , \tag{5.26}$$

where α is defined in Lemma 3 for $d = \delta/4$.

PROOF. – From the construction of the s_m, it is clear that we may restrict our attention to the time interval in which (4.11) holds. We see that

$$\int_0^{s_m(t'+1/m)} \varphi_m(x, w_m(x, t'))\,dx - \int_0^{s_m(t''+1/m)} \varphi_m(x, w_m(x, t''))\,dx = -\int_{t''}^{t'} g_{1,m}(\tau)\,d\tau\,. \tag{5.27}$$

This can be rewritten as

$$\int_{s_m(t''+1/m)}^{s_m(t'+1/m)} \varphi_m(x, w_m(x, t'))\,dx = -\int_0^{s_m(t''+1/m)} \{\varphi_m(x, w_m(x, t')) - \varphi_m(x, w_m(x, t''))\}\,dx - \int_{t''}^{t'} g_{1m}(\tau)\,d\tau\,. \tag{5.28}$$

Using (A_1) and (C_2), we obtain

$$\begin{aligned}\varphi_0\left|s_m\left(t'+\frac{1}{m}\right) - s_m\left(t''+\frac{1}{m}\right)\right| \leqslant G_0|t'-t''| &+ \\ &+ \int_{\delta/4}^{s_m(t''+1/m)} |\varphi_m(x, w_m(x, t')) - \varphi_m(x, w_m(x, t''))|\,dx \\ &+ \left|\int_0^{\delta/4} \{\varphi_m(x, w_m(x, t')) - \varphi_m(x, w_m(x, t''))\}\right| dx\,.\end{aligned} \tag{5.29}$$

From (A_1) and Lemma 3, we obtain

$$\int_{\delta/4}^{s_m(t''+1/m)} |\varphi_m(x, w_m(x, t')) - \varphi_m(x, w_m(x, t''))|\,dx \leqslant \mu_1 H |t'-t''|^{\alpha/2}\,, \tag{5.30}$$

where d in Lemma 3 has been taken as $\delta/4$. Next, by integrating the differential equation in (4.9) over the region $0 \leqslant x \leqslant \delta/4$, $t_1 = \min(t', t'') \leqslant t \leqslant \max(t', t'') = t_2$, we obtain

$$\int_0^{\delta/4} \{\varphi_m(x, w_m(x, t_2)) - \varphi_m(x, w_m(x, t_1))\}\,dx = \int_{t_1}^{t_2} \gamma_m \frac{\partial w_m}{\partial x}\left(\frac{\delta}{4}, \tau\right) d\tau - \int_{t_1}^{t_2} g_{1,m}(\tau)\,d\tau\,. \tag{5.31}$$

From Theorem 3.1 [8; p. 437] and (A_1), (A_2), (B_1), and (B_2) it follows that there exists a positive constant C_1 which depends only upon the parameters in the set $\mathcal{T}$ and δ such that

$$\left|\frac{\partial w_m}{\partial x}\left(\frac{\delta}{4}, \tau\right)\right| \leqslant C_1\,, \qquad 0 \leqslant \tau \leqslant T_0\,. \tag{5.32}$$

Using (5.32), the result (5.26) follows from (5.31), (5.30), and (5.29).

As a consequence of Lemma 4, we obtain the following result.

LEMMA 5. – There exists a positive constant $T \leqslant T_0$ such that

$$\delta \leqslant s_m(t) < 1 - \delta \;. \tag{5.33}$$

holds for all $m \geqslant m_0$ and $0 \leqslant t \leqslant T$. Moreover, for $1/m \leqslant t \leqslant T$, s_m is determined by (4.11).

6. – Existence of a weak solution.

The results of the previous section imply that the w_m and s_m satisfy the compactness criterion of Ascoli-Arzela. Consequently, there exist subsequences $\{w_{m'}\}$ and $\{s_{m'}\}$ such that the $s_{m'}$, $m' \geqslant m_0$, converge uniformly to a Hölder continuous function $s = s(t)$ defined on $0 \leqslant t \leqslant T$ and that the $w_{m'}$, $m' \geqslant m_0$, converge subuniformly to a Hölder continuous function w and weakly in the norm of $W_2^{1,0}(Q_T)$ to w. By an analysis similar to that of Section 6, Part I, it follows that (w, s) is a weak solution of problem (1.1)-(1.2).

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