Averaging Operators on Normed Köthe Spaces (*).

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Summary. – *Under study is the existence of averaging operators determined by measurable maps φ from a measure space (S, Σ, μ) into an arbitrary Hausdorff topological space T. The map φ induces a continuous map φ^e from the space $C_b(T)$ into the normed (Banach) function space $L_\varrho = L_\varrho(S, \Sigma, \mu)$ defined by $\varphi^e(f) = f \circ \varphi$ for all $f \in C_b(T)$. An integral representation for such operators is first studied. The existence is then determined by the existence of an averaging operator U_1 for the restriction of φ to a certain measurable subset B_1 of S. Utilizing a representation of $L_\varrho(S, \Sigma, \mu)$ as a Banach function space over a compact extremally disconnected Hausdorff space $\hat{S}$, we are able to give a definition for the concept of plural points and irreducible map. A significant upper bound is given for the operator U_1. Finally conditions are considered under which no bounded projection from L_ϱ onto the range of φ^e may exist. From a topological point of view the development is pursued in a general setting. Averaging operators have recently been used for the study of injective Banach spaces of the type $C_b(T)$ and in non-linear prediction and approximation theory relative to Tshebyshev subspaces of L_ϱ.*

1. – Introduction.

Let φ be a measurable map from the measure space (S, Σ, μ) into the arbitrary Hausdorff topological space T. Let us assume that φ induces a continuous map from the space $C_b(T)$ of bounded real valued continuous functions on T (with the sup norm topology), into the (as defined below) complete normed Köthe space $L_\varrho = L_\varrho(S, \Sigma, \mu)$ defined by $\varphi^e(f) = f \circ \varphi$ for all $f \in C_b(T)$. If U is a bounded linear operator from L_ϱ into $C_b(T)$ then U is called an *averaging operator* for the measurable map φ or it is said that φ *admits the averaging operator* U if

$$U \circ \varphi^e(f) = f$$

for all $f \in C_b(T)$ (1).

In Theorem 3 we show that the existence of an averaging operator for the measurable map φ from S into T, with φ^e injective, is determined by the existence of an averaging operator for φ_1 which is the restriction of φ to a certain measurable subset B_1 of S. This extends the literature (2) to a larger class of spaces.

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(1) This definition could readily be made by replacing $C_b(T)$ or L_ϱ by other spaces of functions defined on T or S respectively. For example as in [18], the case is studied for L_ϱ replaced by $C(S)$ where S and T are compact Hausdorff spaces.

(2) See [4], [5], [6], [12], and [18]. The analogy of course with our work, is that we have under investigation, the existence of projections from $L_\varrho(S, \Sigma, \mu)$ onto the range, $R(\varphi^e)$. Results for the historically interesting $L_\infty(S, \Sigma, \mu)$ case are herein obtained.

Before we can obtain this result (Theorem 3 in Section 3), we first show in Section 2 how any bounded linear operator from L_ϱ into $C_b(T)$ may have a so-called *integral representation.* This leads to an interesting characterization (Theorem 1) of averaging operators U as well as some computational assistance (Proposition 2) for such operators. In Section 4, significant upper bounds are obtained for the « representing » finitely additive set functions $\{\gamma_t\}_{t\in T}$ of U and a significant inequality is obtained between the norm of U and the norm of its restriction U_1 as given in Theorem 3. This depends much on Theorem 4, which tells us that these set functions may be replaced (via an isomorphism) by regular Borel measures $\{\beta_t\}_{t\in T}$ over a « relatively nice » extremally disconnected compact Hausdorff measure space $(\hat{S}, \hat{\Sigma}, \hat{\mu})$ with $\hat{\mu}$ a regular Borel measure on the field $\hat{\Sigma}$ of clopen subsets of $\hat{S}$. In fact $L_\varrho(S, \Sigma, \mu)$ and L-$(\hat{S}, \hat{\Sigma}, \hat{\mu})$ are isometric and lattice isomorphic. Finally in Section 5, we give conditions on φ under which no bounded pro^{e}ection from L_ϱ onto the range of φ^e may exist.

The spaces $L_\varrho = L_\varrho(S, \Sigma, \mu)$ on which we base our considerations have had considerable interest (see for example, the many papers in [14]). If *M is the collection of scalar valued μ-measurable functions on the measure space* (S, Σ, μ), then the *function norm* ϱ from M into the extended reals $\mathbb{R}^+$ is defined for all $f, g \in M$ as

(i) $0 \leqslant \varrho(f) \leqslant \infty$; $\varrho(f) = 0$ if and only if $f \equiv \mathfrak{L}$ (μ almost everywhere) where $\mathfrak{L}$ is the constant function on S taking all $s \in S$ to 0.

(ii) $\varrho(\alpha f) = |\alpha| \varrho(|f|)$ for all finite scalars α.

(iii) $\varrho(f + g) \leqslant \varrho(f) + \varrho(g)$.

(iv) $|f| \leqslant |g|$ (μ almost everywhere) on M implies that $\varrho(f) \leqslant \varrho(g)$.

We will denote by $L_\varrho = L_\varrho(S, \Sigma, \mu)$ the normed linear space of all functions $f \in M$ with $\varrho(f) < \infty$: The norm on L_ϱ is given by $\|f\|_\varrho = \varrho(|f|)$ and is called the *ϱ-norm* of L_ϱ. The spaces L_ϱ are called *normed Köthe spaces.*

In general the spaces L_ϱ are not complete. However under rather weak conditions, such as the weak Fatou property they may be made complete (see [14]). We will assume that L_ϱ has this property, that is, the spaces L_ϱ are complete in the ϱ-norm. Such complete normed Köthe spaces are called *Banach function spaces.* They include as examples the well-known Lebesgue spaces $\mathfrak{L}^p(S, \Sigma, \mu)$ $1 \leqslant p \leqslant \infty$ and the less well-known but equally important Orlicz spaces (see [15], [14], [10], [2]).

2. – Integral representations of operators.

Before proceeding to find conditions under which there exists (or fails to exist) a bounded projection from L_ϱ onto the range of φ^e, we need to first give some general results about bounded linear operators and their integral representations.

Let U be a bounded linear operator from L_ϱ into $C_b(T)$. Then for each $t\in T$, the operator U gives rise to the *point linear functionals* $U_t\in(L_\varrho)^*$ defined by $U_t(f)=\big(U(f)\big)(t)$ for all $f\in L_\varrho$. In [17] it is shown that there exists a unique finitely additive scalar valued set function γ_t defined on Σ, where γ is in the dual space $(L_\varrho)^*$, such that $U_t(f)=\int f d\gamma_t$ for all $f\in L_\varrho$: If δ_t represents the *point mass* at $t\in T$ then in [17], it is shown that for all $f\in L_\varrho$

$$U_t(f)=\big(U(f)\big)(t)=\langle U(f),\delta_t\rangle=\langle f, U^*(\delta_t)\rangle\,.$$

Thus

$$\langle f, U^*(\delta_t)\rangle=\langle f,\gamma_t\rangle$$

for all $f\in L_\varrho$, that is, $U^*(\delta_t)=\gamma_t$ (3). It is easy to see that the map now defined taking $t\in T$ to $\gamma_t\in(L_\varrho)^*$ is continuous when the weak* topology is placed on the dual space $(L_\varrho)^*$.

If U is now an averaging operator for the measurable function φ from S into T then for $f\in L_\varrho$,

$$\langle f,\delta_t\rangle=\langle U(\varphi^e(f)),\delta_t\rangle=\langle\varphi^e(f), U^*(\delta_t)\rangle=\langle f,[\varphi^e]^*(\gamma_t)\rangle\,.$$

Consequently $[\varphi^e]^*(\gamma_t)=\delta_t$ for all $t\in T$. Thus U is an averaging operator for φ if and only if $[\varphi^e]^*(\gamma_t)=\delta_t$ for all $t\in T$.

For such γ_t, we need to designate its decomposition as yielded in [17] by $\gamma_{t,1}+\gamma_{t,2}$: The (scalar) valued set function $\gamma_{t,1}$ is finitely additive, defined on $\Sigma_0=\{A\in\Sigma\colon \varrho(\chi_A)<\infty\}$ (χ_A represents *characteristic function* of A), vanishes on μ-null sets and $\varrho'(\gamma_{t,1})<\infty$ (4). The set function $\gamma_{t,2}$ is purely finitely additive, (scalar) valued, defined on Σ, vanishes on μ-null sets and its support is contained in the support of some $f\in L_\varrho\setminus M^\varrho$ designates the *closure of the span of all Σ_0-simple functions in L_ϱ*.

Thus we have almost shown completely the following theorem.

THEOREM 1. – *If U is a bounded linear operator from L_ϱ into $C_b(T)$, then for each $t\in T$ there is a unique $\gamma_t\in(L_\varrho)^*$ such that $\gamma_t=U^*(\delta_t)$. The operator U and the in-*

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(3) The operator U^* represents the adjoint of U which takes the dual space $M(T)$ of $C_b(T)$ into the dual space $(L_\varrho)^*$. Any appropriate $M(T)$ will do (dependent on the topological structure of T, naturally) just as long as the point mass $\delta_t\in M(T)$. Of course $\delta_t\in M(T)$ if and only if the *point evaluation map* ξ_t from $C_b(T)$ into the scalars, defined by $\xi_t(f)=f(t)$, is continuous.

(4) As in the case of the Lebesgue spaces $\mathfrak{L}^p$ it is natural to define for L_ϱ an *associate norm* ϱ' as either for $f\in L_\varrho$, $\varrho'(f)=\sup\{\int|fg|d\mu\colon g\in L_\varrho,\ \varrho(g)\le 1$ or$\}$ for γ a finitely additive set function on Σ_0 as $\varrho'(\gamma)=\sup\{|\int f d\gamma|\colon f$ in unit ball of $M^\varrho\}$. Of course if $g\in L_\varrho$ and if $d\gamma=g\,d\mu$ then $\varrho'(\gamma)=\varrho'(f)$ (see [17]). Analogously $L_{\varrho'}=\{f\in L_\varrho\colon\varrho'(f)<\infty\}$ is a Banach function space (see [14]).

tegral representation from t to γ_t are related by

$$(U(f))(t) = \gamma_t(f) \qquad \text{for } t \in L_\varrho \text{ and } t \in T$$

with

$$\|U\| = \sup\{\|U_t\| : t \in T\}\,.$$

This map from t to γ_t is weak continuous.*

If φ^e is continuous (5) *then U is an averaging operator for the measurable function φ from S into T if and only if $[\varphi^e]^*(\gamma_t) = \delta_t$. If the range $R(\varphi^e)$ of φ^e is contained in M^ϱ then U is an averaging operator for φ if and only if $\gamma_{t1}\circ\varphi^{-1}{}_0 = \delta_t$ (as elements f the dual space of $C_b(T)$).*

PROOF. – The last part is all that is left to check. If $R(\varphi^e) \subset M^\varrho$ then $\int F(\varphi^e(f))\, d\gamma_{t,2} = 0$ (see [17]). Thus for $f \in L_\varrho$

$$\langle f, \delta_t\rangle = \langle \varphi^e(f), \gamma_t\rangle = \int f(\varphi(s))\, d\gamma_{t,1} = \int f(t)\, d(\gamma_{t,1}\circ\varphi^{-1})\,.$$

Thus as elements of the dual of $C_b(T)$, $\delta_t = \gamma_{t,1}\circ\varphi^{-1}$. *This completes our proof.*

We should remark, that the above result is somewhat similar to that of PEŁCZYNSKI in [16]. Also the above proof will be established for even more general situations. In particular $C_b(T)$ may be replaced by even more general spaces, for example, by a Banach space $F(T)$ of functions on T where the map from f to $f(t)$ is continuous. Such is the case for the space of bounded functions on T under the supremum norm.

For the case where φ^e maps just $C(T)$ into $C(S)$ (as in [18]) where φ is a continuous map of S onto T, the points $t \in T$ for which the fiber $\varphi^{-1}(t)$ is a subset in S of more than one point play an important role in studying averaging operators. Such points t have been called *plural points in T* (see [18]). In our study where φ_e is defined on L_ϱ and φ is a measurable map, the concept as defined, is not satisfactory. Shortly we will redefine this taking into account the measure μ. For the time being let P_φ be that subset of T such that

$$P_\varphi = \{t \in T : \operatorname{card}(\varphi^{-1}(t)) > 1\}\,.$$

Let B be a Borel subset of T containing P: and let $B_1 = \varphi^{-1}(B)$. We assume that φ is measurable with respect to Σ_0, that is $\int f d\gamma_{t,1}$ exists (as defined in [9]). We now may give some formulas to compute $\langle\varphi^e(f), \gamma_1\rangle$ when φ^e has values in M^ϱ.

(5) Since we have assumed that φ induced a continuous map φ^e from $C_b(T)$ into $L_\varrho(S, \Sigma, \mu)$ by $\varphi^e(f) = f\circ\varphi$ for all $f \in C_b(T)$ we have $\varrho(f\circ\varphi) < \infty$. Hence φ^e is a bounded linear operator. Such a situation is rather easy to construct. For example if φ is a measurable map then the map φ^e defined above from $C_b(T)$ into the Lebesgue space $\mathfrak{L}^\infty(S, \Sigma, \mu)$ is a bounded linear operator. If $\mu\varphi^{-1}$ is of finite variation (with respect to Borel partitions T) then the map φ^e defined from $C_b(T)$ into the Lebesgue space $\mathfrak{L}^p(S, \Sigma, \mu)$, $1 \leqslant p < \infty$, is again a bounded linear operator.

PROPOSITION 2. – *Assume that $R(\varphi^e) \subset M^\varrho$ and that φ admits an averaging operator. Then the following hold.*

(1) *If $t \in B$ then* $\langle \varphi^e(f), \gamma_t \rangle = \int_{B_1} \varphi^e(f)\, d\gamma_{t,1}$.

(2) *If φ is surjective and if $t \notin B$ then* $\langle \varphi^e(f), \gamma_t \rangle = \int_{B_1} \varphi^e(f)\, d\gamma_{t\,1} + f(t)$.

PROOF. – If $t \in B$, then $\delta_t(\complement B) = 0$ (6). Thus $\gamma_{t,1}(\varphi^{-1}(\complement B)) = 0$. Moreover if H is measurable and $B \cap H = \emptyset$ then $\gamma_{t,1}(\varphi^{-1}(H)) = 0$. Consequently for $t \in B$,

$$\langle \varphi^e(f), \gamma_t \rangle = \int_{B_1} \varphi^e(f)\, dk_{t,1} + \int_{\complement B_1} \varphi^e(f)\, d\gamma_{t,1} = \int_{B_1} \varphi^e(f)\, d\gamma_{t,1}\,.$$

This shows statement (1). For statement (2), if $t \notin B$, then $\varphi^{-1}(t)$ is a singleton in S. Partition T into the Borel sets B, $\{t\}$, and the set A. Then S is partitioned into sets B_1, $B_2 = \{\varphi^{-1}(\{t\})\}$, and $\varphi^{-1}(A)$. On A, $\gamma_{t,1} \circ \varphi^{-1} \equiv \mathbf{0}$, so

$$\langle \varphi^e(f), \gamma_t \rangle = \int_{B_1} \varphi^e(f)\, d\gamma_{t,1} + \int_{B_2} \varphi^e(f)\, d\gamma_{t,1}\,.$$

The last integral is just $f(t)$ for $\gamma_{t,1} \circ \varphi^{-1} \equiv \mathbf{1}$ on $\{t\}$. This completes the proof.

The assumption in our proposition (and in other results) that $R(\varphi^e)$ be contained in M^ϱ is reasonable. For example in the class $L = L^\psi$ of Orlicz spaces where ψ satisfies the so called Δ_2 condition, one has that $M^\varrho = L_\varrho$ (see [15]).

The result in Proposition 2 for $\varphi^e(f)$ may be given more generally for any $h \in L_\varrho$. If $t \in B$, then

$$\langle h, \gamma_t \rangle = \int_{B_1} h\, d\gamma_{t,1} + h\varphi^{-1}(t)\,.$$

3. – Existence of averaging operators.

Let us now consider the question of the existence of an averaging operator for φ in terms of the existence of an averaging operator for the restriction φ_1 of φ to B_1.

In particular let B be a Borel subset of T (it need not contain P_φ at all!) and let $B_1 = \varphi^{-1}(B)$. Since φ is measurable, $B_1 \in \Sigma$. A new Banach function space $L_{\varrho_1}(B_1, \Sigma_1, \mu_1)$ may be defined as follows. Let $\Sigma_1 = \{A \cap B_1 \colon A \in \Sigma\}$ and let μ_1 be the restriction of μ to Σ_1. For f a function defined on B_1 and measurable with bepect to μ_1, we may define $\bar{f}$ on S by $\bar{f} \equiv f$ on B_1 and $\bar{f} \equiv \mathbf{0}$ on $\complement B_1$. Now ϱ_1 may be defined for such f by $\varrho_1(f) = \varrho(\bar{f})$.

Clearly L_{ϱ_1} is a Banach function space. Let φ_1 mapping B_1 into B be the restriction of φ to B_1.

(6) By $\complement B$ we mean $T \setminus B$.

For U a bounded linear operator from L_ϱ into $C_b(T)$, we will need the following two concepts for the next theorem. We will say that the operator U is *B extendable* if for every $g \in C_b(B)$, the map $\overline{U(\varphi_1^e(g))}$ in $C_b(T)$ is an extension of g. In particular if U is B-extendable then $\varphi_1^e(g) \in L_\varrho$. Motivated by this we will say that φ is *determined by* B_1 if for every $f \in L_\varrho$ such that $f|B_1 = \varphi_1^e(g)$ for some $g \in C_b(B)$ there is $g' \in C_b(T)$ such that $f = \varphi^e(g')$. Note that in this case g' need not be an extension of g.

What may be said if B does definitely contain the subset P_φ of T? In this case, as we will see in the next theorem, the fact that φ is determined by B_1, may be replaced by the following somewhat weaker statement. We will say that φ is *weakly determined by* B_1 if for all $f \in L_\varrho(S, \Sigma, \mu)$ there is $g \in C_b(B)$ such that $f|B_1 = \varphi_1^e(g)$ and if g' is defined on T to be $g'(t) = g(t)$ for $t \in B$ and $g'(t) = f(\varphi^{-1}(t))$ for $t \in \complement B$ then $g' \in C_b(T)$. Let us note that if S and T were both compact spaces and if B is a closed Borel subset of T containing P_φ then φ is always weakly determined by B_1.

THEOREM 3. – *Assume that $R(\varphi^e) \subset M^\varrho$. If φ admits an averaging operator U that is B-extendable, then φ_1 admits an averaging operator U_1 from $L_{\varrho_1}(B_1, \Sigma_1, \mu_1)$ into $C(B_1)$.*

Conversely if φ_1 admits an averaging operator U_1 and if φ^e is injective with φ determined by B_1 then φ admits an averaging operator. However if $P_\varphi \subset B$ then φ need not be determined by B_1 but need only be weakly determined by B_1.

PROOF. – Let U be an averaging operator for φ that is B-extendable and let $\{\gamma_t\}_{t\in T}$ be the family of associated set functions as described for Theorem 1. The operator U from $L_\varrho(S, \Sigma, \mu)$ into $C_b(T)$ induces an operator U_1 from $L_\varrho(B_1, \Sigma_1, \mu_1)$ into $C_b(B)$ defined by $(U_1(f))(t) = \int_{B_1} \bar{f} d\gamma_t$ for all $f \in L_{\varrho_1}$ and $t \in B$. Since $\varrho_1(f) \leqslant 1$ implies $\varrho(\bar{f}) \leqslant 1$, it follows that $\|U_1\| \leqslant \|U\|$, that is U_1 is a bounded linear operator. It is clear that $(U_1(f))(t) = (U(\bar{f}))(t)$ for all $t \in B$. Thus we have $U_1[\varphi_1^e(g)] = U[\varphi_1^e(g)]|B$ where the right side represents restriction to B. Since U is B-extendable, it follows that $U_1[\varphi_1^e(g)] = g$ for all $g \in C(B)$. Thus U_1 is an averaging operator for φ_1.

If it is assumed now that φ^e is injective then to show that φ has an averaging operator U, it is sufficient to show the existence of a projection P from $L_\varrho(S, \Sigma, \mu)$ onto the range of φ^e (since there exists a one-to-one correspondence between projections from L_ϱ onto $R(\varphi^e)$ and averaging operators from L_ϱ into $C_b(T)$. If U_1 is an averaging operator for φ_1, define P_1 to be $\varphi_1^e \circ U_1$: Clearly P_1 is a bounded projection operator from $L_{\varrho_1}(B_1, \Sigma_1, \mu_1)$ onto $R(\varphi_1^e)$. Define a bounded linear operator T from $L_\varrho(S, \Sigma, \mu)$ into $L_{\varrho_1}(B_1, \Sigma_1, \mu_1)$ by $T(f) = f|B_1$ for all $f \in L_\varrho$. We now may define the required projection P. For $f \in L_\varrho(S, \Sigma, \mu)$, define

$$P(f) = f - \overline{P_1 T(f) - T(f)}\,.$$

Now $P^2(f) = P(f) - \overline{P_1 T(P(f)) - T(P(f))}$. For all $h \in L_{\varrho_1}(B_1, \Sigma_1, \mu_1)$, if $s \in \complement B_1$, then $\bar{h}(s) = 0$. Consequently $(P^2(f))(s) = (P(f))(s)$ for all $s \in \complement B_1$. Moreover for $s \in B_1$,

$\bar{h}(s) = h(s)$. Thus

$$(P_1 TP(f))(s) = (\varphi_1^e U_1 TP(f))(s) = (TP(f))(s)\,.$$

Consequently $(P^2(f))(s) = (P(f))(s)$ for all $s \in S$, that is, P is a projection.

The range of P and the range of φ^e coincide. For if $P(f) = f$ then $P_1 T(f) - T(f) = 0$. Consequently $T(f) \in R(\varphi_1^e)$. If now φ is determined by B_1 then $f \in R(\varphi^e)$. Conversely if $f = \varphi^e(h)$ for some $h \in C(T)$, then $P(f) = f - \overline{P_1 T(f) - T(f)}$. In this case $P_1 T(f) = T(f)$ and thus $P(f) = f$. Therefore $R(\varphi^e) = R(P)$.

If we now assume that the subset P_φ of T is contained in B, then the above arguments show there is a $g \in C(B)$ such that $T(f) = \varphi_1^e(g)$. If φ is now weakly determined by B_1, let g' be the function in $C_b(T)$ as defined in the definition. Hence $f = \varphi_1^e(g')$ and φ is now determined by B_1. *This completes our proof.*

Thus the existence of an averaging operator has somewhat been characterized in terms of a smaller, so to speak, averaging operator defined on an appropriate function space. Let us now consider more of a reduction type theorem where the set functions $\{\gamma_t\}_{t\in T}$ may be replaced, in some cases, by regular Borel measures $\{\beta_t\}_{t\in T}$ defined over a compact space.

We need to *assume that* $L_\varrho = M_\varrho$ (7) *and that* $\varrho(\chi_S) < \infty$: The last condition is needed to insure that there is an $f_0 \in L_\varrho(S, \Sigma, \mu)$ such that $f_0 > \mathbf{0}$ μ almost everywhere. This assumption permits us to make use of a spectral type theorem given in [17]. In particular, let $\hat{\Sigma}$ be the σ-field generated by the compact subsets of the locally compact Hausdorff space $\hat{S}$. Let $\hat{\mu}$ be a measure on $\hat{\Sigma}$ which is finite on compact sets. An adequate function norm $\hat{\varrho}$ is defined (see below) so that one may consider the appropriate function space $L_{\hat{\varrho}}(\hat{S}, \hat{\Sigma}, \hat{\mu})$. Every element in $L_{\hat{\varrho}}$ has σ-compact support. Further let B_ϱ be the *algebra of essentially bounded functions* in $L_\varrho(S, \Sigma, \mu)$ and let $\mathrm{cl}_\infty B_\varrho$ be its closure in $L_\infty(S, \Sigma, \mu)$ (where $L_\infty = L_\varrho$ for $\varrho = \varrho_\infty$ as discussed in [14]). In [17], it is shown that

(1) *there is a measure space* $(\hat{S}, \hat{\Sigma}, \hat{\mu})$ *(as defined above) such that* $L_\varrho(S, \Sigma, \mu)$ *is isometric and (lattice) isomorphic to* $L_\varrho(\hat{S}, \hat{\Sigma}, \hat{\mu})$.

Moreover if there is an $f_0 \in L_\varrho(S, \Sigma, \mu)$ *such that* $f_0 > \mathbf{0}$, μ *almost everywhere, then*

(2) *statement* (1) *holds where now* $\hat{S}$ *is a compact extremally disconnected Hausdorff space and where* μ *is a regular Borel measure* ($\mu(S) < \infty$) *on the* σ*-field generated by the clopen subsets of* $\hat{S}$.

(3) *there is an isomorphism* Φ *from* $\mathrm{cl}_\infty B_\varrho$ *onto* $C_b(\hat{S})$. *Also* $f = \mathbf{0}$, μ *almost everywhere if and only if* $\Phi(f) = \mathbf{0}$, $\hat{\mu}$ *almost everywhere and* $\|f\|_{\infty,\mu} = \|\Phi(f)\|_{\infty,\hat{\mu}}$. *In addition* Φ *takes characteristic functions in* $\mathrm{cl}_\infty B_\varrho$ *into characteristic functions in* $C_b(\hat{S})$.

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(7) As pointed out previously this is a reasonable assumption. The Orlicz space $L_\varrho = L^\Psi$ with Δ_2 condition have this property (see [15]).

Let us note that the condition on $f_0 \in L_\varrho$ given above to demonstrate statements (2) and (3), does hold rather generally, for example, in any σ-finite measure space.

Let $\hat{\Sigma}_c$ denote those clopen subsets in $\hat{\Sigma}$ contained in the compact extremally disconnected space $\hat{S}$, which are in one-to-one correspondence with the sets $A \in \Sigma$. For every $t \in T$ we may define the finitely additive scalar valued set functions β_t by $\beta_t(A) = \gamma_t(\hat{A})$ for $\hat{A} \in \hat{\Sigma}_e$, $A \in \Sigma_0 = \Sigma$. Since we are assuming that $M^\varrho = L_\varrho$, it is clear that $\gamma_t = \gamma_{t,1}$. Note also that $\varrho'(\beta_t) = \varrho'(\gamma_t)$ where

$$\varrho'(\beta_t) = \sup \{|\int \hat{f} d\beta_t| : \hat{f} \text{ is a } \hat{\Sigma}_e \text{ simple function; } \hat{\varrho}(\hat{f}) \leqslant 1\}.$$

Now for $\hat{A} \in \hat{\Sigma}_c$, $\hat{\mu}(\hat{A}) = 0$ if and only if $\mu(A) = 0$. Consequently $\hat{\mu}(\hat{A}) = 0$ implies $\beta_t(\hat{A}) = 0$.

Let $|\beta_t|$ represent the *variation* of β_t, that is,

$$|\beta_t|(\hat{S}) = \sup \{|\sum \gamma_t(A_i)| : (A_i)_{i \in I} \text{ finite partition in } \Sigma\}.$$

Now this variation is finite. In fact if $(\alpha_i)_{i\in I}$ are a finite set of scalars such that $|\alpha_i| = 1$ and such that $\alpha_i \beta_t(A_i) = |\beta_t(A_i)|$ then

$$\sum |\gamma_t(A_i)| = \int \left(\sum \alpha_i \chi_{A_i}\right) d\gamma_t \leqslant \varrho(\chi:) \varrho'(\gamma_t) < \infty.$$

It is also clear that β_t is regular on $\hat{\Sigma}_c$. Let us see now how β_t may be extended to a regular Borel measure on $\hat{\Sigma}$. Since $\varrho(\chi_S) < \infty$ the ring $\hat{\Sigma}_e$ is *dense* in the power set of $\hat{S}$, that is, if K and G are respectively compact and open subsets of $\hat{S}$, then there is $\hat{A} \in \hat{\Sigma}$ such that $K \subset \hat{A} \subset G$. In [8], it is shown that such a situation yields β_t as countably additive on $\hat{\Sigma}_e$ and that a unique extension to $\hat{\Sigma}$ of β_t exists as a regular Borel measure. Furthermore the variation of the extension (considered as a Borel measure) is finite and coincides on $\hat{\Sigma}_c$ with the variation of β_t. For simplicity let us retain β_t as notation for this extension.

Let ψ be the correspondence that takes Σ simple functions into $\hat{\Sigma}_e$ simple functions as now

$$\int \left(\sum \alpha_i \chi_{A_i}\right) d\gamma_t = \int \left(\sum \alpha_i \chi_{\hat{A}_i}\right) d\beta_t.$$

Since $M^\varrho = L_\varrho$, ψ may be extended to all $f \in L_\varrho$ as f is then in the closure of Σ-simple functions. Since $\varrho'(\gamma_t) = \varrho'(\beta_t) < \infty$, a final limit argument will show that $\int f d\gamma_t = \int \hat{f} d\beta_t$ where $\hat{f} = \psi(f)$.

Note that what we have just proceeded to do, could be applied to more general situations. What is crucial here is that in addition to $M^\varrho = L_\varrho$, we need the variation $|\beta_t|$ finite, the field $\hat{\Sigma}_c$ dense in the power set of $\hat{S}$ and the existence of an $f_0 \in L_\varrho$ such that $f_0 > \mathbf{0}$, μ almost everywhere. If $M^\varrho \neq L_\varrho$, then the above arguments may be applied to $\beta_{t,1}$. More formally we have shown

THEOREM 4. – *If $L_\varrho = M^\varrho$ and if $\varrho(\chi_S) < \infty$ then there is an extremally disconnected compact Hausdorff space $\hat{S}$ with $\hat{\Sigma}$, its field of clopen subsets, and $\hat{\mu}$, a regular Borel measure on Σ, such that $L_\varrho(S, \Sigma, \mu)$ is isometric and lattice isomorphic with $L_\varrho(\hat{S}, \hat{\Sigma}, \hat{\mu})$. If ψ is this isomorphism and if U is an averaging operator for φ then for every $t \in T$ there exists a regular Borel measure β_t on $\hat{\Sigma}$ such that*

$$\int f\, d\gamma_t = \int \hat{f}\, d\beta_t$$

where $\hat{f} = \psi(f)$ for $f \in L_\varrho$ and γ_t is the additive set function associated with U as determined for Theorem 1.

4. – Upper bounds.

We are now in a position to give a reasonable definition of plurality as was indicated earlier. The above result also leads to a definition, for the present context, of the concept of an irreducible map (see [18] for the concepts in the more restricted cases).

Again we need to assume that $L_\varrho = M^\varrho$ and the existence of an $f_0 \in L_\varrho$ such that $f_0 > \mathbf{0}$, μ almost everywhere. Let U be an averaging operator for φ.

DEFINITION 5. – For $t \in T$, if $\varphi^{-1}(t) \in \Sigma_0$, let $\hat{\varphi}^{-1}(t)$ be the associated clopen set in the Stone space $\hat{S}$. The point $t \in T$ is called a *plural point* if

(*a*) whenever β_t is positive on subsets of $\hat{\varphi}^{-1}(t)$ then there is a set $\hat{A} \in (\hat{\Sigma}_c)_0$ such that $\hat{A} \subset \hat{\varphi}^{-1}(t)$ and $0 < \beta_t(\hat{A}) < 1$.

(*b*) whenever β is not positive on $\hat{\varphi}^{-1}(t)$ then for the part N of $\hat{\varphi}^{-1}(t)$ on which β_t is negative there is a subset $\hat{A} \in (\hat{\Sigma}_c)_0$ such that $\hat{A} \subset N$ and $0 < \beta_t(\hat{A}) < 1$.

Let Pl_φ be the *set of plural points of T*. The measurable map φ will be called *irreducible* ([8]) if for $A \subset S$, with $\mu(A) > 0$ there is a $t \in T$ such that $\emptyset \neq \varphi^{-1}(t) \subset A$.

Let us recall that since φ has an averaging operator and since $M^\varrho = L_\varrho$, Theorem 1 says that $\beta_t(\hat{\varphi}^{-1}(t)) = 1$. Also let us note that as we have defined it, saying that t is a plural point amounts to saying that $\hat{\varphi}^{-1}(t)$ is not an atom for β_t. An interesting relation between non atomicity and the Darboux property may be found in [8].

We now make use of our ideas to establish an upper bound for the variation of the set functions γ_t in terms of the norm for U.

([8]) Our measure theoretical concept has its topological analogue as the following: the continuous map φ from the topological space S onto the topological space T is *irreducible* if for every non-empty open set G is S there is a point $t \in T$ such that $\emptyset \neq \varphi^{-1}(t) \subset G$.

THEOREM 6. – *Let* φ *have an averaging operator* U *and assume that*

(1) $M^\varrho = L_\varrho$;

(2) $\varrho(\chi_S) < \infty$;

(3) t *is plural*;

(4) φ *is irreducible*;

then $|\gamma_t|(S) < \varrho(\chi_S)\|U\| - 1$.

PROOF. – To simplify the notation in this proof we will replace the operator $\hat{\varphi}^{-1}$ by ξ. As noted above $\beta_t(\xi(t)) = 1$. Plurality of t finds a clopen set $\hat{A} \subset \hat{S}$ such that $1 = \beta_t(A) + \beta_t(\xi(t) - A)$ where $0 \neq |\beta_t(\hat{A})| \neq 1$, $0 \neq \beta_t[\xi(t) - \hat{A}] \neq 1$. Thus, in short, there is a clopen set $\tilde{A}$ (which may be either $\hat{A}$ or $\xi(t) - \hat{A}$) such that for $\varepsilon > 0$, $\beta_t(\tilde{A}) < \varepsilon + \frac{1}{2}$. Actually there is a compact set $K \subset \tilde{A}$ such that $0 < \beta_t(K) < \frac{1}{2} + \varepsilon$ and $|\beta_t(K)| = |\beta_t|(K)$. If β_t is positive on $\xi(t)$, the regularity of β_t assures the existence of a compact set $K \subset \tilde{A}$ such that

$$|\beta_t(K) - \beta_t(\tilde{A})| < \varepsilon + \tfrac{1}{2} - \beta_t(\tilde{A}).$$

Since $0 < \beta_t(\tilde{A}) < 1$, K may be chosen so that $\beta_t(K) \neq 0$. In addition $\beta_t(K) = \beta_t(\tilde{A}) + \beta_t(K) - \beta_t(\tilde{A}) < \varepsilon + \frac{1}{2}$. Clearly $\beta_t(K) = |\beta_t(K)| = |\beta_t|(K)$ as β_t is countably additive. If β_t is not positive on $\xi(t)$, let N be the negative part as in the definition. Again by the regularity of β_t, a compact set $K \subset N$ may be obtained so that $|\beta_t(K) - \beta_t(N)| < \frac{1}{2}$. Again it may be assumed that $\beta_t(K) \neq 0$. Now $\beta_t(K) \leqslant \beta_t(N) + \frac{1}{2} < \frac{1}{2}$. Since $-\beta_t$ is positive on N, $-\beta_t(K) = |\beta_t(K)| = |\beta_t|(K)$.

Now the regularity of $|\beta_t|$ permits us to pick a clopen set $C \subset \hat{S}$ such that $K \subset C$ and $|\beta_t|(C \setminus K) < \varepsilon$. Incidentally $\chi_C \in C(\hat{S})$ and $\hat{\varrho}(\chi_C) < \infty$. A finite pairwise disjoint family of clopen sets $C_i \subset \hat{S} \setminus C$, $i \in I$, may be chosen such that

$$|\beta_t|(\hat{S} \setminus K) - \varepsilon < \sum_{i=1}^{n} |\beta_t(C_i)|.$$

Of course $\hat{\varrho}(\chi_{C_i}) < \infty$.

Let α_i be scalars such that $|\alpha_i| = 1$ and $\alpha_i \beta_t(C_i) = |\beta_t(C_i)|$. Now

$$(1) \qquad |\beta_t|(\hat{S} \setminus C) - \varepsilon < \int \Big(\sum \alpha_i \chi_{C_i}\Big) d\beta_t;$$

$$(2) \qquad \int \chi_C \, d\beta_t \leqslant \beta_t(K) + |\beta_t(C \setminus K) < \beta_t(K) + \varepsilon \ (^7).$$

Since the map from t to γ_t is weak* continuous (Theorem 1), it follows that there is some neighborhood V of t such that for all $y \in V$

$$(3) \qquad |\beta_t|(\hat{S} \setminus G) - \varepsilon < \int \Big(\sum \alpha^i \chi_{C_i}\Big) d\beta_y;$$

$$(4) \qquad \int \chi_C \, d\beta_y < \beta_t(K) + \varepsilon.$$

If $D \in \Sigma$ is the correspondent of C, then $\mu[\varphi^{-1}(V) \cap D] > 0$. In fact $\varphi^{-1}(V) \cap D \supset \supset \varphi^{-1}(t) \cap D$. The last set corresponds to $\xi(t) \cap G$ which contains K. Now $\hat{\mu}(K) > 0$ or else $\beta_t(K) = 0$ which is a contradiction. Thus

$$\hat{\mu}[\xi(t) \cap G] > 0 \quad \text{and} \quad \mu[\varphi^{-1}(V) \cap V] > 0 .$$

The irreducibility of φ assures $q \in T$ such that $\varphi^{-1}(q) \neq 0$ and $\varphi z^1(q) \subset \varphi^{-1}(V) \cap D$. Hence $q \in V$ and

$$|\beta_t|(S \setminus G) - \varepsilon < \int \sum \alpha_i \chi_{C_i} d\beta_q ; \qquad \int \chi_C \, d\beta_q < \beta_t(K) + \varepsilon .$$

Since $\varphi^{-1}(q) \subset D$, $\xi(q) \subset G$ and

$$\int \chi_C \, d\beta_y = \int_{C-\xi(q)} \chi_C \, d\beta_q + \beta_q[\xi(q)] .$$

Consequently $\beta_q(C) = \beta_q(C - \xi(q)) + 1$, and by (4)

$$\int_{C-\xi(q)} \chi_C \, d\beta_q = \beta_q(C) - 1 < \beta_t(K) + \varepsilon - 1 .$$

Recall that $0 < |\beta_t(K)| = |\beta_t|(K) < \frac{1}{2} + \varepsilon$. If $\beta_t(K) < 0$ for sufficiently small $\varepsilon < 0$ then

$$|\beta_q|[C \setminus \xi(q)] > |\beta_t(K) + \varepsilon - 1| > |\beta_t(K)| - 3\varepsilon.$$

If $0 < \beta_t(K) < \varepsilon + \frac{1}{2}$ for sufficiently small $\varepsilon > 0$, then

$$|\beta_q|(C \setminus \xi(q)) \leqslant |\beta_q[C \setminus \xi(q)]| > \tfrac{1}{2} - 3\varepsilon > |\beta_t(K)| - 3\varepsilon$$

since

$$|\beta_q[C \setminus \xi(q)] < \beta_t(K) + \varepsilon - 1 < \tfrac{1}{2} + 2\varepsilon$$

Thus in all cases for sufficiently small $\varepsilon > 0$

$$|\beta_q|[C \setminus \xi(q)] > |\beta_t|(K) - 3\varepsilon .$$

Now

$$\|U\| > \sup \{ |\int f \, d\gamma_q| : f \in M_1^\varrho \} =$$
$$= \sup \{ |\sum \beta_q(\hat{A}_i)\alpha_i| : \hat{f} = \sum \alpha_i \chi_{\hat{A}_i} \text{ a } \hat{\Sigma}_c\text{-step function, } \varrho(\hat{f}) < \infty \} .$$

Picking scalars β_i, $|\beta_i| = 1$ and $\beta_i \gamma_q(A_i) = |\gamma_q(A_i)|$ we have

$$\varrho\left(\sum \chi_{\hat{A}_i} \circ \beta_i\right) \leqslant \hat{\varrho}(\chi_{\hat{S}}) = \varrho(\chi_S) .$$

Since $\hat{\Sigma}_c$ is dense in the power set of $\hat{S}$, we have

$$\|U\| > \frac{1}{\varrho(\chi_S)}\left[|\beta_q|(\hat{S}\setminus C) + |\beta_q|(C\setminus\xi(q)) + 1\right].$$

Thus

$$\|U\| > \frac{1}{\varrho(\chi_S)}\left[\int\left(\sum\chi_{C_i}\alpha_i\right)d\beta_q + 1 - 4\varepsilon\right] > \frac{1}{\varrho(\chi_S)}\left[|\beta_t|(\hat{S}) + 1 - 5\varepsilon\right].$$

It finally follows that $|\gamma_t|(S) \leq \varrho(\chi_S)\|U\| - 1$ and our proof is now complete.

We now have the answer to a rather natural question. What is the relation between the norm of the averaging operator U and the norm of U_1 as defined above. We quickly obtain it below.

COROLLARY 7. – *Assuming the hypotheses of Theorem* 8 *and assuming that* φ *and* φ_1 *admit the averaging operators* U *and* U_1 *respectively, then*

$$\|U_1(f)\| \leq \|\bar{f}\|_\infty[\varrho(\chi_S)\|U\| - 1]$$

where $B = \mathrm{cl}\,Pl_\varphi$ *as needed for* U_1.

PROOF. Following through the proof of Theorem 3 we see that $(U_1(f))(t) = \int_{B_1}\bar{f}\,d\gamma_t$. Then pick $\hat{f} \in C(\hat{S})$, $\mathbf{0} \leq \hat{f} < \mathbf{1}$ with $\int\hat{f}\,d\beta_t > \|\beta_t\| - \varepsilon$ where $\varepsilon > 0$ is given, t is a fixed point in B_1, and $\|\beta_t\| = |\beta_t|(\hat{S})$. Now if G is an open set containing t and if $r \in G \cap Pl_\varphi$, then the theorem yields

$$\varrho(\chi_S)\|U\| - 1 > |\beta_r|(\hat{S}) \geq \int\hat{f}\,d\beta_r.$$

The weak* continuity of the map that takes r to γ_r (restricting to G if necessary) yields

$$\varrho(\chi_S)\|U\| - 1 > \int\hat{f}\,d\beta_r > \|\beta_t\| - \varepsilon$$

for all $r \in G$. Now for every $t \in \mathrm{cl}\,Pl_\varphi$ it follows that $\|\beta_t\| \leq \varrho(\chi_S)\|U\| - 1$. Since

$$(U_1(f))(t) = \int_{B_1}\bar{f}\,d\gamma_t = \int_{B_1}\hat{f}\,d\beta_t$$

it follows that

$$\|U_1(f)(t)\| \leq \|\bar{f}\|_\infty \sup\{\|\beta_t\| : t \in \mathrm{cl}\,P1_\varphi\{ \leq \|\bar{f}\|_\infty[\varrho(\chi_S)\|U\| - 1].$$

This completes the proof.

5. – Projection problem.

We are now led to the consideration of obtaining conditions on φ with which we will know that no bounded projection will exist onto the range of φ^e. For this remaining section we will need to assume that L_ϱ is reflexive.

A function $f \in L_\varrho$ is said to be of *absolutely continuous norm* if the sequence $\{\varrho(f_n)\}_{n\in N}$ is monotonically decreasing and convergent to zero whenever the sequence $\{f_n\}_{n\in N} \in L_\varrho$ is monotonically decreasing and pointwise convergent μ almost everywhere to zero with $f_1 \leqslant |f|$.

Let L^a_ϱ represent all $f \in L_\varrho$ which are of absolutely continuous norm. It can be shown that L^a_ϱ is a norm closed order ideal in L_ϱ (see [19], Chapter 15 for its significance). For our purposes, its significance will be in its determination of the reflexivity of L_ϱ.

The function norm ϱ is said to be *absolutely continuous* if $L_\varrho = L^a_\varrho$. The space L_ϱ is reflexive if and only if both ϱ and ϱ' are absolutely continuous and ϱ has the weak Fatou property.

Now we will assume that L^a_ϱ is identified with $L^a_{\varrho'}$ and $R(\varphi^e)$ is considered as a subset of $L[L^a_\varrho, \mathbf{C}]$ the set of bounded linear operators from L^a_ϱ into the complex scalars $\mathbf{C}$, or as a subset of L^a_ϱ. The following operators will be needed.

Let U be an arbitrary element of $L[L^\alpha_\varrho, \mathbf{C}]$. For E in a partition $\mathcal{E}$ of Σ_0 and for $f \in L^\alpha_{\varrho'}$, define the operator $U_E \in L[L^a_\varrho, \mathbf{C}]$ by

$$U^E(f) = U\left[\left(\int\limits_E f\, d\mu\right)\chi_E\right]$$

and define the linear operator $A_\mathcal{E} \in L[^a_{\varrho'}, L^a_{\varrho'}]$ by

$$A_\mathcal{E}(f) = f_\mathcal{E} = \sum_{E\in\mathcal{E}}\left(\int\limits_E \frac{|f|}{\mu(E)}\, d\mu\right)\chi_E\,.$$

We may also define

$$A_E(f) = \frac{1}{\mu(E)}\left(\int\limits_E f\, d\mu\right)\chi_E\,.$$

The function norm ϱ is said to be *weakly leveling* if for each partition $\mathcal{E}$ in Σ_0, $\varrho(f_\mathcal{E}) \leqslant \varrho(f)$. All well known Banach function spaces such as the Orlicz spaces (and in particular the Lebesgue spaces) have weakly leveling function norms. In [11] this concept was referred to as ϱ *having property* (J). The present terminology appropriate in comparison to the concept of *leveling* as discussed in [10].

If $R(\varphi^e)$ is closed, we will let P be a bounded projection of $L[L^a_{\varrho'}, \mathbf{C}]$ onto $R(\varphi^e)$. Thus if P^* is the adjoint of P, then P^* is a bounded linear map from L^a_ϱ into L^a_ϱ.

Asking that $R(\varphi^e)$ be closed is not much of an assumption. Specifically this occurs when φ admits an averaging operator.

Any linear operator K from L^a_ϱ into L^a_ϱ may be written (see [19]) as $K(f)=\int gf\,d\mu$ for some $g\in L^a_{\varrho''}=L^a_\varrho$. The important assumption here is that for a certain class of operators K, g may be chosen in $R(\varphi^e)$.

THEOREM 10. – *Assume the following conditions*

(1) L_ϱ *is reflexive with* $(L^a_\varrho)^*\cong L^a_{\varrho'}$.

(2) $R(\varphi^e)$ *is closed*;

(3) ϱ' *has the weak leveling property*;

(4) *For every* $E\in\mathcal{E}$ *there is* $f_E\in C_b(T)$ *such that*

$$U_E(f)=\langle\varphi^e(f_E),f\rangle\Big(=\int\varphi^e(f_E)\,f\,d\mu\Big)\,.$$

Then either φ^e *is surjective or no bounded projection* P *exists from* $L_\varrho(S,\Sigma,\mu)$ *onto* $R(\varphi^e)$ *such that*

$$P[UA_{\mathcal{E}}]=P(U)A_{\mathcal{E}}\,.$$

In particular either φ^e *is surjective or no bounded projection* P *from* L^a_ϱ *onto* $R(\varphi^e)$ *exists such that* P^* *commutes with* A_E.

PROOF. – If φ^e is not surjective, let P be a bounded projection of $L[L^a_{\varrho'},\mathbf{C}]$ onto $R(\varphi^e)$ with $P[UA_{\mathcal{E}}]=P(U)A_{\mathcal{E}}$. Now

$$UA_{\mathcal{E}}(f)=U\Big[\sum_{\mathcal{E}}\Big(\frac{1}{\mu(E_i)}\int_{E_i}f\,d\mu\Big)\chi_{E_i}\Big]$$

$$=\sum_{\mathcal{E}}\frac{1}{\mu(E_i)}\,U_{E_i}(f)=\sum_{\mathcal{E}}\frac{1}{\mu(E_i)}\,\langle\varphi^e(f_{E_i}),f\rangle\,.$$

Thus if $h=\sum_{\mathcal{E}}(1/\mu(E_i))\,f_{E_i}$ then $UA_{\mathcal{E}}(f)=\langle\varphi^e(h),f\rangle$. Consequently $UA_{\mathcal{E}}\in R(\varphi^e)$ and $P[UA_{\mathcal{E}}]=UA_{\mathcal{E}}$. Since ϱ' has the weak leveling property, we obtain from [11], that $A_{\mathcal{E}}(f)$ converges to f in the ϱ norm as $\mathcal{E}$ gets finer. Thus

$$\lim_{\mathcal{E}}P[UA_{\mathcal{E}}](f)=U(f)$$

and so

$$\lim_{\mathcal{E}}P[UA_{\mathcal{E}}](f)=\lim_{\mathcal{E}}P(U)A_{\mathcal{E}}(f)=P(U)(f)\,.$$

Hence $P(U)=U$ which contradicts the assumption of φ^e being not surjective.

To complete the proof we need show that if P^* commutes with A_E then $P[UA_{\mathcal{E}}] = P(U)A_{\mathcal{E}}$. Now

$$\langle P[UA_{\mathcal{E}}], f\rangle = \langle UA_{\mathcal{E}}, P^*(f^*)\rangle = \langle U, \sum_{\mathcal{E}} \frac{1}{\mu(E_i)} \int_{E_i} P^*(f)\, d\mu \chi_{E_i}\rangle$$

$$= \langle U, \sum_{\mathcal{E}} \frac{1}{\mu(E_i)} A_{E_i} P^*(f)\rangle = \langle U, P^* \left[\sum_{\mathcal{E}} \frac{1}{\mu(E_i)} A_{E_i}(f)\right]\rangle$$

$$= \langle P(U), \sum_{\mathcal{E}} \frac{1}{\mu(E_i)} \int_{E_i} f\, d\mu\, \chi_{E_i}\rangle = \langle P(U)A_{\mathcal{E}}, f\rangle .$$

Thus $P[UA_{\mathcal{E}}] = P(U)A_{\mathcal{E}}$.

BIBLIOGRAPHY

[1] R. A. ALÒ - A. DE KORVIN - L. KUNES, *Topological aspects of q-regular measures*, Studia Math., **48** (1973), pp. 49-60.

[2] R. A. ALÒ - A. DE KORVIN, *Topologies associated with operators on normed function spaces*, Proc. Conference on General Topology, Rome, 1973, Symposia Mathematica, **16** (1975), pp. 209-243.

[3] R. A. ALÒ - L. I. SENNOTT, *Extending linear space-valued functions*, Math. Ann., **191** (1971), pp. 79-86.

[4] D. AMIR, *Projections onto continuous function spaces*, Proc. Amer. Math. Soc., **15** (1964), pp. 396-402.

[5] D. AMIR, *On projections and simultaneous extensions*, Israel J. Math., **2** (1964), pp. 245-248.

[6] R. ARENS, *Projections on continuous function spaces*, Duke Math. J., **32** (1965), pp. 469-478.

[7] D. W. DEAN, *Subspaces of C(H) which are direct factors of C(H)*, Proc. Amer. Math. Soc., **16** (1965), pp. 237-242.

[8] N. DINCULEANU, *Vector Measures*, Pergamon Press, Berlin, 1967.

[9] N. DUNFORD - J. T. SCHWARTZ, *Linear Operators*, part I, New York, 1958.

[10] H. W. ELLIS - I. HALPERIN, *Function spaces determined by a leveling length function*, Canad. J. Math., **5** (1953), pp. 576-592.

[11] N. GRETSKY, *Representation theorems on Banach function spaces*, Memoirs of the Amer. Math. Soc., **84** (1968), pp. 56.

[12] J. R. ISBELL - Z. SEMADENI, *Projection constants and spaces of continuous functions*, Trans. Amer. Math. Soc., **107** (1963), pp. 38-48.

[13] S. KAKUTANI, *Concrete representation of abstract (M) spaces*, Ann. Math., (**2**) **42** (1941), pp. 994-1024.

[14] W. A. J. LUXEMBURG - A. C. ZAANEN, *Notes on Banach function spaces*, Proc. Acad. Sci. Amsterdam (Indag. Math.), Note I, A **66** (1963), pp. 135-147; Note II, A **66** (1963), pp. 148-153; Note III, A **66** (1963), pp. 239-250; Note IV, A **66** (1963), pp. 251-263; Note V, A **66** (1963), pp. 496-504; Note VI, A **66** (1963), pp. 655-668; Note VII, A **66** (1963), pp. 669-681; Note VIII, A **67** (1964), pp. 104-119; Note IX, A **67** (1964), pp. 360-376; Note X, A **67** (1964), pp. 493-506; Note XI, A **67** (1964), pp. 507-518; Note XII, A **67** (1964), pp. 519-529; Note XIII, A **67** (1964), pp. 530-543; Note XIV,

A **68** (1965), pp. 229-248; Note XV, A **68** (1965), pp. 415-446; Note XVI, A **68** (1965), pp. 646-667.

[15] W. ORLICZ, *Absolute continuity of set functions with respect to a finitely subadditive measure*, Amer. Soc. Math. Polonae, Series I, **14** (1970), pp. 101-118.

[16] A. PEŁCZYNSKI, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertations Math., **58** (1968).

[17] M. M. RAO, *Linear operations, tensor products and contractive projections in function spaces*, Stud. Math., **38** (1970), pp. 131-186.

[18] J. WOLFE, *Injective Banach spaces of type $C(T)$*, Ph. D. Thesis, University of Califonia, Berkeley, 1971.

[19] A. C. ZAANEN, *Integration* (Second Edition), North Holland, Amsterdam, 1967.

[20] T. ANDO, *Contractive projections in L^p-spaces*, Pac. J. Math., **17** (1966), pp. 391-405.