The Internal Sphere Condition and the Capillary Problem (*) (**).

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Sunto. – *In un importante lavoro* [1], *M. Emmer ha dimostrato l'esistenza di superfici di equilibrio capillari, definite in domini con frontiera Lipschitziana, quando la costante di Lipschitz L e l'angolo di contatto γ soddisfano la relazione $L < |\operatorname{tg}\gamma|$. Questa condizione, che è noto essere necessaria in generale, può essere troppo restrittiva in alcuni casi particolari. In questo lavoro la condizione di Emmer è sostituita da una «condizione di sfera interna», che conduce ad una larga classe di domini che non rientrano nel risultato di Emmer. L'esistenza di una soluzione è dimostrata anche nel caso $\gamma = 0$, che non può essere trattato con il metodo di Emmer.*

1. – In a recent paper [1], EMMER proved that if a domain Ω has Lipschitz boundary Σ, with Lipschitz constant $L < |\tan\gamma|$, and if $\varkappa > 0$, then the functional

$$E(f) \equiv \int_\Omega (W + \tfrac{1}{2}\varkappa f^2)\,dx - \oint_\Sigma f\cos\gamma\; d\sigma\,, \tag{1}$$

$$W = \sqrt{1 + |\nabla f|^2}\,,$$

admits a unique minimizing function $u(x)$ in the class $BV(\Omega)$, the elements of which are functions in $L^1(\Omega)$ whose first distributional derivatives are Radon measures of bounded variation over Ω. The function $u(x)$ is real analytic interior to Ω (see also Pepe [2]) and has a trace on Σ in the class $L^1(\Sigma)$.

Although EMMER assumed $\gamma \equiv$ const on Σ, the situation in which γ is prescribed and continuous on Σ requires no significant change in the demonstration, and we shall discuss the problem in that generality. Emmer's condition then becomes $L < \min\limits_\Sigma |\tan\gamma|$.

In dimension $n = 2$ the variational problem for (1) arises from the physical problem of finding a capillary free surface over Ω, which meets the bounding cylinder Z over Σ in a contact angle γ. For background motivation and discussion see, e.g., [3] or [4]. In this paper we permit n to be arbitrary, $n \geqslant 1$.

The restriction $L < |\tan\gamma|$ is not accidental to the method. It can be shown that under conditions for which Σ is smooth except at a single point where the

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inequality is reversed, $E(f)$ may not be bounded below, and there will be no function for which $E(f)$ is stationary and finite (see the discussion in [4]).

Nevertheless, the restriction as it appears excludes cases for which a solution would be expected on physical grounds to exist. For example, if Σ is smooth one would expect to find a solution where $\gamma = 0$. This case is however excluded by Emmer's condition; the relation

$$\hat{E}(f) = \int_\Omega (W + \tfrac{1}{2}\varkappa f^2)\, dx < C_1 < \infty$$

for any sequence minimizing $E(f)$, which is basic to Emmer's method, is then not evidently satisfied.

Perhaps a more serious drawback is that the method excludes not only boundary corners with small angles (which is to be expected) but also corners with large opening angles, for which there is no a priori reason to expect difficulty.

Even in the case of a boundary corner with small angle, when no solution of Emmer's problem can exist, capillary surfaces are observed physically, and it is desirable to encompass them in the mathematical theory. We shall do this in § 7 ix).

Our principal achievement in the present work is to replace the condition $L < |\tan\gamma|$ by a different, but related requirement (*internal sphere condition*) and we show (roughly speaking) that the variational problem admits a solution in the sense of EMMER whenever this condition holds. We are led to a class of domains in some ways much broader than has previously been considered, and to less restrictive conditions on boundary data.

It has been our aim to obtain existence, uniqueness and a priori estimates for solutions without explicit hypotheses on boundary regularity. In this respect we have not entirely succeeded, and (for technical resons) we have had to impose a segment condition on the boundary. This is, however, the only explicit restriction we make, and the essential features of our results are obtained under implicit global geometric conditions. The requirements are verified readily in particular cases, some of which are discussed in § 7. They are in certain senses weakest possible for the results obtained.

Not every domain that verifies Emmer's conditions satisfies ours (cf. § 7 iv)) but our condition does include a number of cases of special interest that are excluded by Emmer's hypotheses. In particular we find the existence of a unique solution in domains with opening corners (§ 7 vii)), and also in the entire range $0 \leqslant \gamma \leqslant \pi$, whenever $\Sigma \in C^2$. If Σ is any Lipschitz boundary, a variant of our method extends Emmer's result to the limiting case $L \leqslant |\tan\gamma|$, which is important in some situations. Thus, our results can be said to extend all those of Emmer, although not by a unified approach.

We base our discussion not on the variational integral but on the variational condition satisfied by a stationary function. The condition is defined in a general situation under minimal regularity hypotheses near Σ, because of the particular nonlinearity in the problem. Our procedure then permits us to consider also cases, as indicated above, in which no solution with finite variational integral can exist (§ 7).

Our proofs use basically existence and regularity results of EMMER and of PEPE; we note, however, that we need these results in less generality than appears in their papers. For example, for our principal purposes Emmer's existence theorem is used only for a domain with smooth boundary, in which case his procedure can be simplified, see, e.g., [4, pp. 133-4]. We obtain our present results by an approximation procedure, using an a priori estimate for the solutions due to CONCUS and FINN [5].

2. – Let Ω be a bounded domain in $\boldsymbol{R}^n$, whose boundary $\Sigma \in C^1$. Let Z denote the vertical cylinder over Σ. Let $B_\delta \subset \boldsymbol{R}^n$ be a ball of radius δ, $\Gamma_\delta = \partial B_\delta$, and let Ξ_δ be the vertical cylinder over Γ_δ. Let S_δ be a lower hemisphere inscribed in Ξ_δ. S_δ will meet Z in an angle $\hat{\gamma}(x)$ over Σ, see Figure 1. We normalize $\hat{\gamma}$ so that $0 \leqslant \hat{\gamma} \leqslant \pi/2$

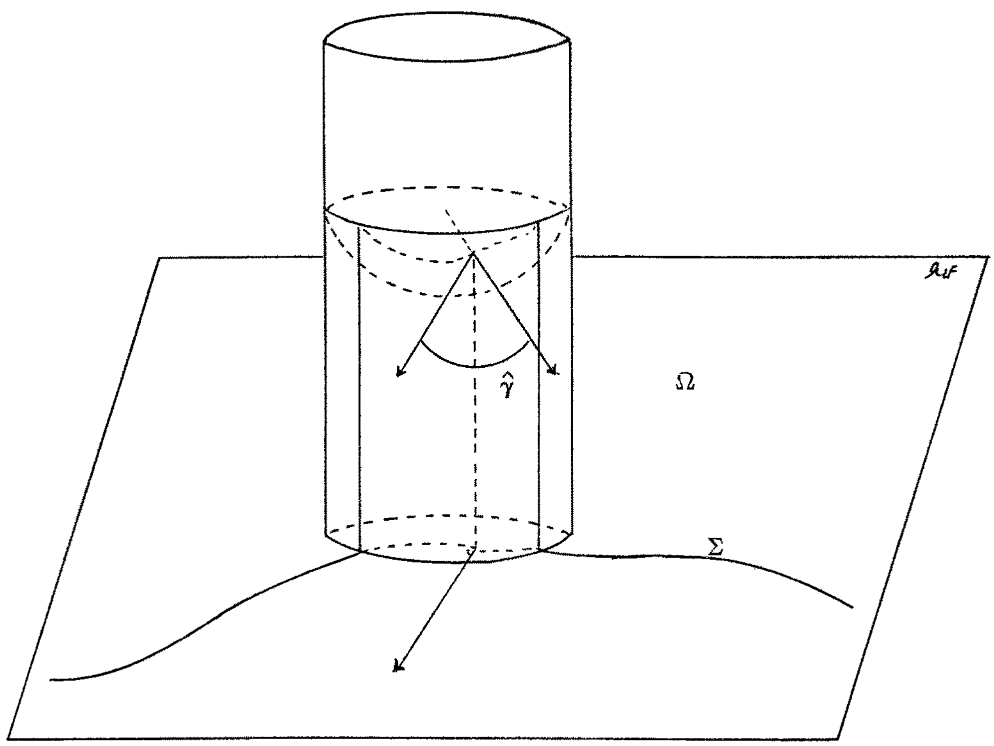

Figure 1

DEFINITION. – Ω will be said to satisfy an *internal sphere condition with radius* δ *and angle* γ_0, $0 \leqslant \gamma_0 \leqslant \pi/2$, if every $x \in \Omega$ lies in a ball B_δ, such that $\hat{\gamma}(\xi) \leqslant \gamma_0(\xi)$ at every $\xi \in \Sigma \cap B_\delta$.

We write $\Omega\subset ISC_{\delta,\gamma_0}$. In case $\gamma_0=0$ this means every $x\in\Omega$ lies in a ball $B_\delta\subset\bar\Omega$. We then say Ω satisfies an internal sphere condition with radius δ, $\Omega\subset ISC_\delta$.

The following result appears in [5].

LEMMA 1. – *Let $\Sigma\in C^1$ and suppose $\Omega\subset ISC_{\delta,\gamma_0}$. Let $u(x)$ be a solution in Ω of the Euler equation*

$$\operatorname{div} Tu=\varkappa u\,,\qquad Tu=\frac{1}{W}\nabla u\,,\qquad \varkappa>0\,, \tag{2}$$

associated with (1), *and suppose $u(x)$ is of class C^1 up to Σ, defining an angle $\gamma(x)\geqslant\gamma_0(x)$ with Z on Σ. Then*

$$u(x)<\frac{n}{\varkappa\delta}+\delta\qquad in\ \Omega\,. \tag{3}$$

COROLLARY 1. – *Let $\Omega\subset ISC_\delta$. Then any solution $u(x)$ of* (2) *in Ω satisfies* (3).

An alternative proof of boundedness under some conditions appears in [6].

3. – Let Ω have Lipschitz boundary Σ, with constant $L<\min_{\sigma\in\Sigma}|\tan\gamma(\sigma)|$. We are then assured by [1, 6] of the existence of a unique minimizing function $u(x)$ for (1), which is real analytic in Ω, and has a trace $u^t(\sigma)\in L^1(\Sigma)$. Further, there holds $\hat E(u)<\infty$, so that, in particular, $u\in H^{1,1}(\Omega)$. We note [7, 16] that every $f\in H^{1,1}(\Omega)$ has a trace $f^t(\sigma)\in L^1(\Sigma)$.

LEMMA 2. – *Under the above conditions, $u(x)$ satisfies the variational condition*

$$\int_\Omega (W_{p_i}\zeta_i+\varkappa u\eta)\,dx-\oint_\Sigma \eta\cos\gamma\,d\sigma=0\,,\qquad p_i=u_{x_i},\ \zeta_i=\eta_{x_i}\,, \tag{4}$$

for any $\eta(x)\in H^{1,1}(\Omega)$ (¹).

PROOF. – Since $u(x)$ minimizes $E(f)$, there holds

$$\delta E\equiv E(u+\varepsilon\eta)-E(u)\geqslant 0\,,$$

for any $\eta\in H^{1,1}(\Omega)$ and $\varepsilon\in\boldsymbol{R}$. We have

$$\delta E=\int_\Omega\Big(A_i(\varepsilon;\eta)\zeta_i+\varepsilon\varkappa\eta u+\varepsilon^2\frac{\varkappa}{2}\eta^2\Big)\,dx-\varepsilon\oint_\Sigma\eta\cos\gamma\,d\sigma \tag{5}$$

(¹) The existence of (4) in this generality is assured by the uniform boundedness of $u(x)$ in Ω.

with

$$A_i(\varepsilon;\eta)=\int_0^\varepsilon W_{p_i}(p+\lambda\zeta)\,d\lambda\,.$$

There holds

$$\lim_{\lambda\to 0} W_{p_i}(p+\lambda\zeta)=W_{p_i}(p)\,,$$

for almost all $x\in\Omega$, while $|W_{p_i}(p+\lambda\zeta)|<1$ for all λ. Thus, the bounded convergence theorem implies the existence of

$$\text{(6)}\qquad \frac{\partial E}{\partial\varepsilon}\bigg|_{\varepsilon=0}=\int_\Omega (W_{p_i}\zeta_i+\varkappa u\eta)\,dx-\oint_\Sigma \eta\cos\gamma\,d\sigma=\lim_{\varepsilon\to 0}\frac{E(u+\varepsilon\eta)-E(u)}{\varepsilon}=0$$

since the numerator is nonnegative for both positive and negative ε.

4. – DEFINITION. – A domain Ω will be called *admissible* if

a) it satisfies a segment condition (cf. [8]), and

b) it can be exhausted by an expanding sequence $\Omega^j\subset\Omega$, such that

(i) $\Sigma^j\in C^2$, all j.

(ii) $\Omega^j\subset ISC_{\delta,\gamma_0}$ with δ, γ_0 independent of j, $0\leqslant\gamma_0\leqslant\pi/2$, $\delta>0$.

(iii) there exists a fixed integer N such that each Σ^j can be covered by N local parametrizations $x^{jk}(\alpha)$, $k=1,\dots,N$, in the ball $|\alpha|<1$, and such that the x^{jk} converge pointwise and in measure in each ball.

Some examples of admissible (and of inadmissible) Ω are considered in § 7. We note for later reference that if $\Omega\subset ISC_{\delta,\gamma_0}$ then $\Omega\subset ISC_{\delta,\gamma}$ for any γ satisfying $\gamma_0\leqslant\gamma\leqslant\pi/2$.

DEFINITION. – A boundary condition $\gamma(\sigma)$, $\sigma\in\Sigma$, will be called *admissible* if $0\leqslant\gamma\leqslant\pi$, and if γ is the trace (from Ω) of a uniformly continuous function $\gamma(x)$ satisfying $0<\gamma<\pi$ for $x\in\Omega$. We write also $\beta=\cos\gamma$ and refer to admissible β.

REMARK. – Constant boundary data are always admissible.

DEFINITION. – We introduce the class $Q(\Omega)\equiv L^\infty(\Omega)\cap H^{1,1}(\Omega)$.

The following maximum principle may be of general interest, and applies to a much broader class of functions than those for which existence will be proved (²).

(²) The former author wishes to thank Mario MIRANDA for a number of discussions relating to the appropriate formulation of this principle.

LEMMA 3. – *Let Ω^j exhaust Ω, let $\varkappa > 0$, let $u(x), v(x)$ be functions in $H^{1,1}_{\mathrm{loc}}(\Omega)$, for which*

$$\liminf_{j\to\infty} \int_{\Omega^j} \{[W_{p_i}(p) - W_{p_i}(q)]\zeta_i + \varkappa(u - v)\eta\}\, dx \leqslant 0\,, \tag{7}$$

$$p = \nabla u,\ q = \nabla v\,,$$

for any $\eta \in L^\infty(\Omega) \cap H^{1,1}_{\mathrm{loc}}(\Omega)$ with $\eta \geqslant 0$. Then

$$u(x) \leqslant v(x)\,, \qquad \text{a.e. } in\ \Omega\,. \tag{8}$$

If $\varkappa = 0$, then either $u(x) \leqslant v(x)$ a.e. in Ω, or $u(x) \equiv v(x) +$ const. a.e. If $u, v \in H^{1,1}(\Omega)$, it suffices to consider $\eta \in Q(\Omega)$, $\eta \geqslant 0$.

PROOF. – Suppose there exists $m > 0$ such that the set

$$\Omega_m = \{x \in \Omega\colon 0 < u - v < m\}$$

has positive measure. Choose

$$\eta(x) = \begin{cases} 0\ , & u - v \leqslant 0\,, \\ u - v\,, & 0 < u - v < m\,, \\ m\ , & u - v \geqslant m\,. \end{cases}$$

Then, for j sufficiently large, the set $\Omega_j \cap \Omega_m$ has positive measure, and

$$\int_{\Omega^{\alpha_j} \cap \Omega_m} \{[W_{p_i}(p) - W_{p_i}(q)]\, \zeta_i + \varkappa\eta^2\}\, dx \leqslant \varepsilon_{\alpha_j} \tag{9}$$

with $\lim_{j\to\infty} \varepsilon_{\alpha_j} = 0$, for a suitable subsequence $\{\alpha_j\} \subset \{j\}$. But in Ω_m,

$$W_{p_j}(p) - W_{p_i}(q) = \zeta_j \int_0^1 W_{p_i p_j}(q + \lambda(p - q))\, d\lambda$$

wherever p, q are defined. One verifies that $W_{p_i p_j}\xi_j\xi_j \geqslant |\xi|^2/W^3$ for any choice of argument in $W_{p_i p_j}$; thus the integrand in (9) is positive in Ω_m, the integral is increasing in j, and from this the result follows.

LEMMA 4. – *If Ω, β are admissible, $\varkappa > 0$, there is at most one function $u(x) \in \in H^{1,1}(\Omega)$ satisfying the variational condition*

$$\liminf_{j\to\infty} \left| \int_{\Omega^j} (W_{p_i}\zeta_i + \varkappa u\eta)\, dx - \oint_{\Sigma^j} \beta\eta\, d\sigma \right| = 0 \tag{10}$$

for any $\eta \in Q(\Omega)$. *If in addition* $u(x) \in Q(\Omega)$, *then* $u(x)$ *minimizes the functional* $E(f)$ *in the sense that for any* $v(x) \in Q(\Omega)$,

$$\limsup_{j\to\infty} \{E_j(v) - E_j(u)\} \geqslant 0 \tag{11}$$

where $E_j(f)$ *is the functional* (1) *defined over* Ω_j; *equality holds in* (11) *only if* $v \equiv u$ a.e. *in* Ω. *If* $\varkappa = 0$, *then* $u(x)$ *is determined by* (10) *up to an additive constant, and equality holds in* (11) *only if* $v \equiv u +$ const. a.e. *in* Ω.

PROOF. – For given $\eta \in Q(\Omega)$, denote by α_j and β_j minimizing sequences for (10), corresponding to solutions $u(x)$, $v(x)$. We may assume $\Omega^{\alpha_j} \subset \Omega^{\beta_j}$. Since Ω satisfies a segment condition, any $\eta \in H^{1,1}(\Omega)$ can be approximated in $H^{1,1}$ by a function $\hat\eta$ uniformly continuous in Ω [8, 9]. For given η and prescribed $\varepsilon > 0$, we may (since $|W_{p_i}| < 1$) choose $\hat\eta$ so that

$$\int_\Omega \{|W_{p_i}(p) - W_{p_i}(q)|\,|\zeta_i - \hat\zeta_i| + \varkappa |u - v|\,|\eta - \hat\eta|\}\,dx < \varepsilon \tag{12}$$

$$\int_\Omega \{|W_{p_i}(p)|\,|\zeta_i - \hat\zeta_i| + \varkappa |u|\,|\eta - \hat\eta|\}\,dx < \varepsilon$$

with a corresponding inequality in $v(x)$.

For fixed $\hat\eta$, we have

$$\lim_{j\to\infty}\left|\oint_{\Sigma\alpha_j} \beta\hat\eta\,d\sigma - \oint_{\Sigma\beta_j} \beta\hat\eta\,d\sigma\right| = 0$$

because of the uniform continuity of $\beta, \hat\eta$ and the convergence pointwise and in measure of the Σ^j. Also

$$\lim_{j\to\infty} \int_{\Omega\beta_j - \Omega\alpha_j} \{|W_{p_i}(p) - W_{p_i}(q)||\hat\zeta_i| - \varkappa|u - v|\hat\eta\}\,dx = 0\,.$$

We conclude

$$\lim_{j\to\infty} \int_{\Omega\alpha_j} \{[W_{p_i}(p) - W_{p_i}(q)]\hat\zeta_i + \varkappa(u - v)\hat\eta\}\,dx = 0$$

and hence, using (12), the same relation with $\hat\eta, \hat\zeta$ replaced by η, ζ. From Lemma 3 now follows both $u \leqslant v$ and $v \leqslant u$ almost everywhere, thus establishing the uniqueness.

To prove the minimizing property, let $v(x) \in Q(\Omega)$ and set $\eta(x) = v(x) - u(x)$. We verify as in the proof of Lemma 2 that

$$\int_{\Omega^j} (W_{p_i}\zeta_i + \varkappa u\eta)\,dx - \oint_{\Sigma^j} \beta\eta\,d\sigma = \frac{d}{d\lambda} E_j[u + \lambda\eta]\Big|_{\lambda=0}, \tag{13}$$

with

$$E_j = \int_{\Omega^j} \left(W + \frac{\varkappa}{2} u^2 \right) dx - \oint_{\Sigma^j} \beta u \, d\sigma .$$

Thus, given $\varepsilon > 0$, we will have $|(d/d\lambda)E_j|_{\lambda=0} < \varepsilon$ for a suitable sequence Ω^j, for all $j > j_0(\varepsilon)$. But as in the proof of Lemma 3, we find

$$E_j(v) - E_j(u) = \frac{dE_j}{d\lambda}\bigg|_{\lambda=0} + \int_{\Omega^j} P[u; v] \, dx$$

with $P[u; v] > 0$ almost everywhere unless $u \equiv v$. Choosing

$$\varepsilon < \int_{\Omega^{j_0}} P[u; v] \, dx$$

for some fixed j_0, we obtain the result.

5. – Here and in the remainder of this paper we assume $\varkappa > 0$; for our principal results this condition is necessary, cf. [17, Corollaries 3.2, 3.3], also [4, pp. 136-7].

Let $\Sigma \in C^2$ and let B_δ be a ball of radius δ, with $\Omega \cap B_\delta = \Omega_\delta \neq \emptyset$. The boundary of Ω_δ consists of piecewise smooth portions $\Sigma_\delta \subset \Sigma$ and $\Gamma_\delta \subset \partial\Omega_\delta \cap \Omega$. Let β be continuous on Σ, $0 \leqslant |\beta| < \beta_0 < 1$, and let $u(x)$ be the minimizing function for $E(f)$ in Ω. By [1, 2], $u(x) \in C^2(\Omega) \cap H^{1,1}(\Omega)$ and by Lemma 2, $u(x)$ satisfies the variational condition (4) in Ω. Since clearly $\Omega \subset ISC_{\delta'}$ for some $\delta' > 0$ whenever $\Sigma \in C^2$, there holds, by Corollary 1, $u(x) \in L^\infty(\Omega)$. Hence $u(x) \in Q(\Omega)$. Denote by ν the exterior normal to Σ_δ or to Γ_δ.

LEMMA 5. – *Under the above conditions, let $\hat{\beta} = \nu \cdot Tu$ on Γ_δ. Then $u(x)$ satisfies* (4) *in Ω_δ, with data β on Σ_δ and $\hat{\beta}$ on Γ_δ.*

PROOF. – Choose $h > 0$. Take the origin of coordinates at the center of B_δ, and set

$$\lambda(r) = \begin{cases} 1, & r \leqslant \delta \\ 1 - \dfrac{r-\delta}{h}, & \delta \leqslant r \leqslant \delta + h \\ 0, & r \geqslant \delta + h \end{cases}$$

If $\eta \in H^{1,1}(\Omega)$, set $\eta_\delta^h = \lambda\eta$. Then $\eta_\delta^h \in H^{1,1}(\Omega)$, and (4) becomes, for the domain Ω,

$$\oint_{\Sigma_{\delta+h}} \eta_\delta^h \beta \, d\sigma = \int_{\Omega_\delta} (W_{p_i} \zeta_i + \varkappa u \eta) \, dx + \int_{A_\delta} [W_{p_i}(\lambda\eta)_{x_i} + \varkappa \lambda u \eta] \, dx$$

where $\mathcal{A}_\delta$ is an annular region of width h (see Figure 2). We obtain the stated result by letting $h \to 0$. (No contributions can appear from neighborhoods of the intersection sets $\Sigma \cap \Gamma_\delta$, since $|W_{p_i}| < 1$ for all values of its arguments.)

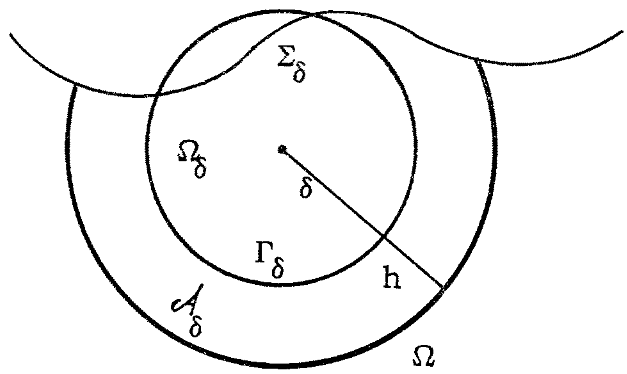

Figure 2

LEMMA 6. – *Let Ω^j be a family of domains such that $\Sigma^j \in C^2$ and $\Omega^j \subset ISC_{\delta,\gamma_0}$. Let u^j be the unique minimizing function for $E(f)$ in Ω^j, with data γ^j such that $|\gamma^j - \pi/2| < |\gamma_0 - \pi/2|$. Then $|u^j(x)| \leqslant n/\varkappa\delta + \delta$, uniformly in j and in Ω^j.*

PROOF. – In what follows we suppress the index j. Let $x \in \Omega$, let B_{δ,γ_0} be a ball containing x such that a lower hemisphere S (represented as the graph of a function v over B_{δ,γ_0}) meets the cylinder over Σ in angles $\gamma_S \leqslant \gamma_0$: Then $\Omega_\delta = \Omega \cap \cap B_{\delta,\gamma_0} \neq \emptyset$, and $\partial\Omega_\delta = \Sigma_\delta \cup \Gamma_\delta$, with $\Sigma_\delta \subset \Sigma$, $\Gamma_\delta \subset \Omega \cap \partial\Omega_\delta$. In Ω_δ, v satisfies the relation

$$\operatorname{div} Tv = \frac{n}{\delta}, \tag{14}$$

with $\beta_S = \nu \cdot Tv \geqslant \cos\gamma_0$ on Σ_δ, $\beta_S = 1$ on Γ_δ. We may choose v so that $v(x_0) = n/\varkappa\delta$ at the center x_0 of B_{δ,γ_0}. Then $v(x) < n/\varkappa\delta + \delta$ in Ω_δ. Applying Lemma 5, we obtain after an integration by parts

$$\int_{\Omega_\delta} \nabla\eta \cdot (Tu - Tv)\, dx - \oint_{\partial\Omega_\delta} (\beta^* - \beta_S)\eta \, d\sigma + \varkappa \int_{\Omega_\delta} \eta \left(u - \frac{n}{\varkappa\delta} \right) dx = 0 \tag{15}$$

for any $\eta \in H^{1,1}(\Omega_\delta)$, where

$$\beta^* = \begin{cases} \beta\,, & \text{on } \Sigma_\delta \\ \hat{\beta}\,, & \text{on } \Gamma_\delta \end{cases}$$

(cf. Lemma 5). Choosing $\eta = \max(u - v, 0)$ we conclude as in the proof of Lemma 3

$$u(x) \leqslant v(x) < \frac{n}{\varkappa\delta} + \delta$$

since $\beta^* - \beta \leqslant 0$. Similarly, consideration of an upper hemisphere leads to the estimate from below.

6. – THEOREM 1. – *Let Ω, β be admissible, with $|\gamma - \pi/2| \leqslant |\gamma_0 - \pi/2|$ on Σ. Then there exists a unique minimizing function $u \in Q(\Omega)$ for $E(f)$, in the sense implied by Lemma 4, and $u(x)$ can be chosen to be real analytic. If $\Sigma \in C^1$ and there exists a strict solution v of (2) in Ω with $\nu \cdot Tv = \beta$ on Σ, then $u \equiv v$ in Ω.*

PROOF. – Let Ω^j be an admissible family exhausting Ω and satisfying a uniform condition $\Omega^j \subset ISC_{\delta,\gamma_0}$. Let u^j denote the corresponding solutions, with data $\gamma(x)$ on Σ^j, whose existence is proved in [1]. By Lemma 2, each u^j satisfies the variational condition (4). Since $\Omega^j \subset ISC_{\delta,\gamma_0}$, each $x \in \Omega^j$ is in some B_{δ,γ_0}, and we obtain, from Lemma 6, $|u^j(x)| < n/\varkappa\delta + \delta$ in Ω^j. In particular, $u^j(x) \in Q(\Omega^j)$, all j. By Lemma 2,

$$\oint_{\Sigma^j} \eta\beta \, d\sigma = \int_{\Omega^j} (W_{p_i} \zeta_i + \varkappa u^j \eta) \, dx \tag{16}$$

for any $\eta \in H^{1,1}(\Omega^j)$. Choosing $\eta = u^j(x)$ yields

$$\int_{\Omega^j} \{W(p^j) + \varkappa |u^j|^2\} \, dx < \left(\frac{n}{\varkappa\delta} + \delta\right) |\Sigma^j| + |\Omega^j| \,, \tag{17}$$

since $W \geqslant 1$ for all values of its arguments. Thus $\hat{E}(u^j)$ is bounded independent of j.

We may now either repeat (essentially) the procedure of [1] or we may apply to the u^j the result at the end of [10] together with general results on elliptic equations with divergence structure [11]. We obtain the existence of a subsequence of the Σ^j (not relabled) such that the corresponding u^j converge, uniformly in any fixed Ω^{j_0}, to a solution $u(x)$ of (2) in Ω.

We now note any $\eta \in H^{1,1}(\Omega)$ is in $H^{1,1}(\Omega^j)$ for every j. Since (16) holds for each j, since any $\eta \in H^{1,1}(\Omega)$ can be approximated in $H^{1,1}$ norm by functions uniformly continuous in Ω (see [8, 9]), and since $|W_{p_i}| < 1$ for any choice of its arguments, we conclude (16) holds also for the limit function $u(x)$, in the sense of a limit as $j \to \infty$, for any $\eta \in H^{1,1}(\Omega)$. Using (17) we find without difficulty

$$\lim_{j\to\infty} \int_{\Omega^j} W(p^j) \, dx = \int_{\Omega} W \, dx$$

$$\lim_{j\to\infty} \int_{\Omega^j} (u^j)^2 \, dx = \int_{\Omega} u^2 \, dx \,.$$

The uniqueness and minimizing property are obtained from Lemma 4, and the final statement of the theorem from Lemma 4 and the observation that (10) holds for $v(x)$ in Ω, with $\hat{\beta} = \nu \cdot Tv$ on Σ.

7. – We consider some examples:

i) *if $\Sigma \in C^2$ the solution of Emmer exists for any continuous $\gamma(\sigma)$ on Σ, with* $0 \leqslant \gamma(\sigma) \leqslant \pi$. In fact, we may in this case shorten the procedure by dealing directly with the Emmer solution in Ω, corresponding to a sequence of data $\gamma^j \to \gamma$, $0 < \gamma^j < \pi$:

ii) if $\Sigma \in C^1$ the result of Emmer yields the existence of a solution for any γ, $0 < \gamma < \pi$. In this case the corresponding Ω satisfies the following variant of the internal sphere condition:

Let Ω be a bounded C^1-domain. Then, for any choice of γ_0, $0 < \gamma_0 \leqslant \pi/2$, Ω can be covered by a finite number of balls $B^j_{\delta_j}$ such that $B^j_{\delta_j}$ meets the cylinder over Σ in angles γ^j_S, with $\gamma^j_S \leqslant \gamma_0$, and $\delta_j \geqslant \delta > 0$ all j.

PROOF. – Let $x_0 \in \Sigma$, and suppose that in a neighborhood of x_0, Σ can be specified to be the graph of a C^1-function φ

$$\varphi\colon |x'| < r \to \mathbf{R}_+ , \qquad x = (x', x^n) , \qquad (x', \varphi(x')) \in \Sigma$$

and $x_0 = (0, x_0^n)$, $x_0^n > 0$, and $0 \in \Omega$. Let v be a lower hemisphere over a ball $B_\delta(y_0)$ containing x_0

$$v(x) = -(\delta^2 - |x - y_0|^2)^{\frac{1}{2}} .$$

Then v meets the cylinder over Σ in angles γ_S satisfying

$$\cos \gamma_S = -\sum_{i=1}^{n-1} \frac{x^i - y_0^i}{|x - y_0|} \cdot \frac{D_i\varphi}{W} + \frac{x^n - y_0^n}{|x - y_0|} \cdot \frac{1}{W} \tag{20}$$

at points $x = (x', x^n) \in \Sigma$, where $W = (1 + |\nabla\varphi|^2)^{\frac{1}{2}}$.

We may assume $|\nabla\varphi(0)| = 0$. Thus, to given $\varepsilon > 0$ there exists δ_0 such that

$$|\nabla\varphi(x')| \leqslant \varepsilon \quad \text{ for all } \quad |x - x_0| \leqslant 2\delta_0 .$$

Let ε, $0 < \varepsilon < 1$, be given and choose δ_0, $0 < \delta_0 < x_0^n$, appropriately. Then consider the ball $B_\delta(y_0)$, with $y_0 = (0, x_0^n - \delta/\lambda)$, $\lambda = 1 - \varepsilon$, $\delta = \delta_0$, and the corresponding lower hemisphere v.

Let $x \in B_\delta(y_0)$. Then

$$|x - x_0| \leqslant |x_0 - y_0| + |x - y_0| \leqslant \frac{\delta}{\lambda} + \delta < 2\delta . \tag{21}$$

Hence we have $|\nabla\varphi(x')| \leqslant \varepsilon$ for all $x \in \Sigma \cap B_\delta(y_0)$, and

$$|x^n - x_0^n| \leqslant \varepsilon \cdot |x' - x_0'| \leqslant \varepsilon \cdot |x - x_0| \leqslant 2\varepsilon\delta \tag{22}$$

for those x. In view of these relations and in view of the estimate

$$x^n - y_0^n \geqslant x_0^n - y_0^n - |x^n - x_0^n| \geqslant \frac{\delta}{\lambda} - 2\varepsilon\delta \tag{23}$$

we deduce from (20)

$$\cos\gamma_S \geqslant -\frac{|x'-y_0'|}{|x-y_0|}\cdot\frac{\varepsilon}{\sqrt{1+\varepsilon^2}} + \frac{\delta/\lambda - 2\varepsilon\delta}{|x-y_0|}\cdot\frac{1}{\sqrt{1+\varepsilon^2}} \geqslant \frac{1}{\sqrt{1+\varepsilon^2}}\cdot\left\{\frac{1}{\lambda} - 2\varepsilon - \varepsilon\right\} \geqslant \frac{1-3\varepsilon}{\sqrt{1+\varepsilon^2}} \tag{24}$$

or all $x \in \Sigma \cap B_\delta(y_0)$. Since a finite number of such balls covers Σ, the assertions of the theorem are proved.

This result establishes a new proof (cf. [6]) for the boundedness of the Emmer solution in this case.

iii) if $\Sigma \in C^1$ then it is a Lipschitz boundary for any constant $L > 0$, hence by Emmer's theorem there is a unique minimizing function for any γ, $0 < \gamma < \pi$. We show that *for $0 \leqslant \gamma \leqslant \pi$ there is still a solution in the sense of the variational condition* (4). To do so, set $\beta = \cos\gamma$, $\beta_0 = \max\{\beta, -1+\varepsilon_0\}$, $0 < \varepsilon_0 < 1$, and $\beta_0^j = \min\{\beta_0, 1-\varepsilon^j\}$, $\varepsilon^{j+1} < \varepsilon^j < 1$, $\varepsilon^j \to 0$. We consider the solution u_0^j of Emmer in the (fixed) domain Ω, with data β_0^j on Σ. For each u_0^j we have

$$\oint_\Sigma \beta_0^j \eta \, d\sigma - \int_\Omega [W_{p_i}\zeta_i + \varkappa u_0^j \eta]\, dx = 0 \tag{25}$$

for any $\eta \in H^{1,1}(\Omega)$, by Lemma 2, and also $\hat{E}(u_0^j) < \infty$, each j. For any compact $K \subset\subset \Omega$, we have $K \subset ISC_\delta$ for some $\delta > 0$, hence $|u_0^j(x)| < n/\varkappa\delta + \delta$, all $x \in K$. Thus, a subsequence of the u_0^j (again denoted by u_0^j) can be found, which converges, uniformly on any $K \subset\subset \Omega$, to a strict solution $u_0(x)$ of (2) in Ω.

We examine the transition in (25), for fixed η. We may suppose $\eta \geqslant 0$, since any η can be expressed as the sum of its positive and negative parts. Since $\beta_0^j \to \beta_0$ uniformly, we have

$$\lim_{j\to\infty} \oint_\Sigma \beta_0^j \eta \, d\sigma = \oint_\Sigma \beta_0 \eta \, d\sigma .$$

Also, since $|W_{p_i}| < 1$ and $\eta \in H^{1,1}(\Omega)$,

$$\lim_{j\to\infty} \int_\Omega W_{p_i}(p_0^j)\zeta_i \, dx = \int_\Omega W_{p_i}(p_0)\zeta_i \, dx .$$

It follows that

$$\lim_{j\to\infty} \int_\Omega \varkappa u_0^j \eta \, dx = L_0$$

exists, and $L_0 \neq \pm\infty$.

We now apply Lemma 3. Since $\beta_0^{j+1} \geqslant \beta_0^j$, there follows $u_0^{j+1} \geqslant u_0^j$ in Ω. We conclude from the monotone convergence theorem that

$$\lim_{j\to\infty} \int_\Omega \varkappa u_0^j \eta \, dx = \int_\Omega \varkappa u_0 \eta \, dx$$

and hence (25) holds with the superscript deleted.

Now, let $\varepsilon_0 \to 0$. We obtain a nonincreasing sequence of solutions, for which we again pass to the limit under the integral sign, obtaining the stated existence.

We note the corollary to the method of construction of $u(x)$, that $u \in L^1(\Omega) \cap C^2(\Omega)$.

iv) The boundary illustrated in Figure 3 satisfies a Lipschitz condition with constant $L = \cot\alpha$. Without changing this constant, the corners can be rounded so that Σ has a uniquely defined normal at each point.

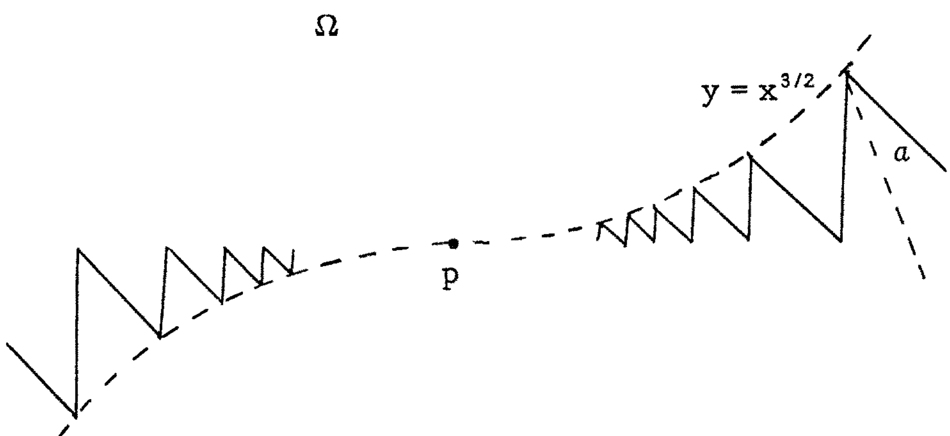

Figure 3

One verifies readily that if a closed ball B_δ contains the point p, then there exist boundary points at which a lower hemisphere over B_δ meets the cylinder over Σ in angles θ satisfying $|\pi/2 - \theta| < \varepsilon$ for any $\varepsilon > 0$; hence there will be points on Σ at which $|\tan\theta|$ is arbitrarily large.

On the other hand, Emmer's theorem assures the existence of a unique solution $u(x)$ of (2) in Ω, for any data γ satisfying $|\tan\gamma| > \cot\alpha$. Thus Emmer's result applies to situations for which the sphere condition fails, so that Ω will not be admissible in the sense of the present paper.

v) In the situation just discussed, Lemma 1 cannot be applied to obtain a bound for the solution. We do have $\hat{E}(u) < \infty$, as follows from the procedure in [1]. The solution is in fact bounded; this follows from the results of [6].

vi) The method of iii) can be applied to the above situation, yielding the existence of a solution in the sense of (4) under the (weakened) hypothesis $|\tan\gamma| \geqslant L$.

vii) In the domain of Figure 4, Emmer's method fails if $\cot\alpha \geqslant |\tan\gamma|$. Nevertheless, Theorem 1 yields the existence of a unique solution for any choice of γ. Thus, the present result is not included in that of Emmer, even if $|\cos\gamma| \neq 1$.

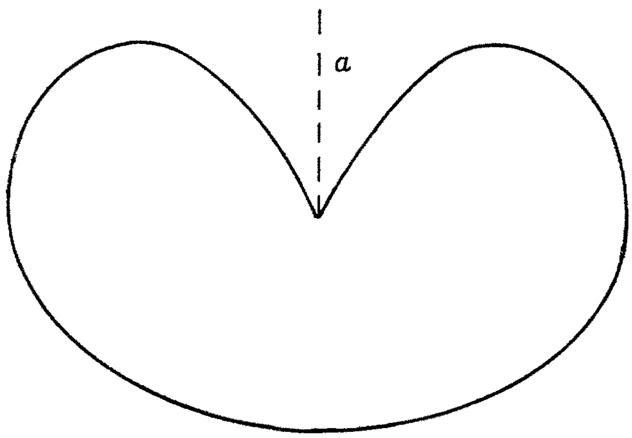

Figure 4

viii) In Figure 5, the two sides of the «spine» are not identified if $n = 2$, while if $n > 2$ the spine becomes a singular (lower dimensional) part of the boundary. The domain Ω is however admissible in the sense of this paper for any $n \geqslant 2$, and hence a unique solution exists for any γ, $0 \leqslant \gamma \leqslant \pi$.

If $n \geqslant 3$, the «spine» can be enclosed within Ω by a surface of arbitrarily small area. We may thus apply the general maximum principle of [5] to conclude that the solution is identical to the one obtained when the spine is removed (Figure 6).

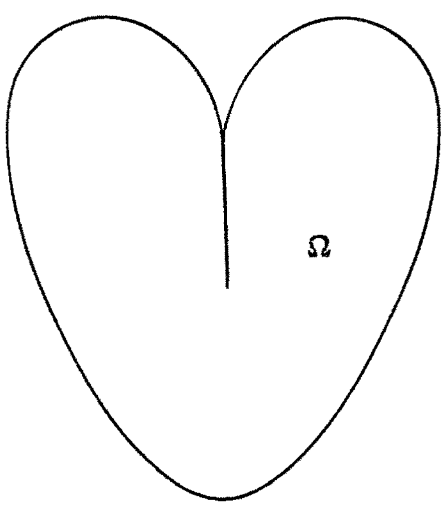

Figure 5

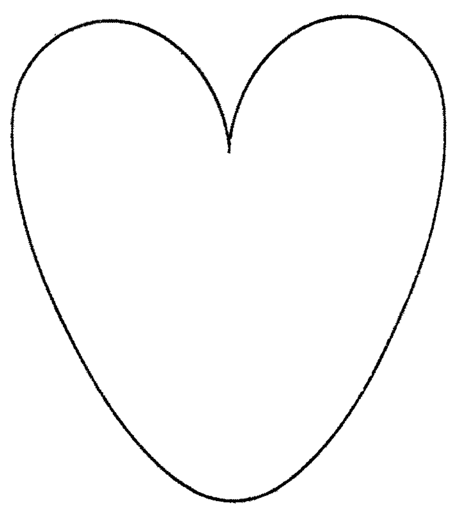

Figure 6

In particular, the data γ on the spine will not in general be achieved strictly in this case.

ix) We consider the domain of Figure 7, with a conical boundary singularity of half angle α. If $|\gamma-\pi/2|\leqslant\alpha$, then $\Omega\subset ISC_{\delta,\gamma}$, in the sense that it can be approximated from the interior by domains with smooth boundaries and in this class. Thus, Theorem 1 assures the existence of a unique solution. (Emmer's theorem requires $|\gamma-\pi/2|<\alpha$.) If $|\gamma-\pi/2|>\alpha$, Theorem 5 in [5] shows that no solution with finite energy integral can exist, and thus Emmer's method cannot be applied directly. Nevertheless, a solution does exist in the sense of the variational condition (4). To show this, we approximate Ω as in Figure 8 and solve the corresponding problem in Ω^j with data $\beta=1$ on Γ^j. Since each $\Omega^j\subset ISC_{\delta^j}$ for some $\delta^j>0$, the solutions interior to any fixed Ω^{j_0} are uniformly bounded. By Lemma 3, the sequence u^j is monotonically decreasing, hence by the estimates of [10, 11] we find convergence to a strict solution $u(x)$ of (2), uniformly in any $K\subset\subset\Omega$.

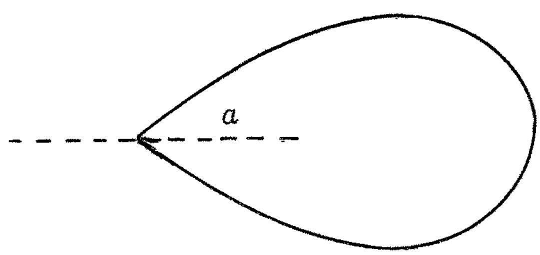

Figure 7

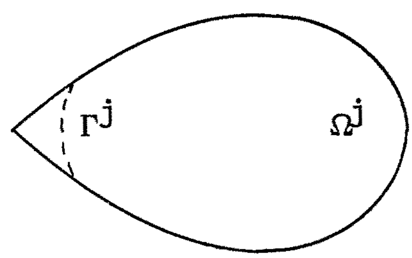

Figure 8

For each j, the variational condition (4) holds in Ω^j for any $\eta\in Q(\Omega^j)$, hence in particular for any $\eta\in Q(\Omega)$. We have also $\delta^j>r\tan\alpha$, where r is the minimum distance in Ω to the vertex, so that $|u^j|\leqslant(n)/(kr\tan\alpha)+r\tan\alpha$. It follows that $|u^j|$ is bounded by an integrable function in (4), as $j\to\infty$.

The limiting transition is now immediate, and we obtain the variational condition (4) for $u(x)$, for any $\eta\in Q(\Omega)$.

Since $u(x)\notin H^{1,1}(\Omega)$, the uniqueness does not follows directly from Lemma 4. To prove it, we use again that $\Omega^j\subset ISC_{\delta^j}$, hence any solution $v(x)$ is bounded in Ω^j. Integrating over a fixed Ω^{j_0}, we find

$$\int_{\Omega^{j_0}}\left(W-\frac{1}{W}+\varkappa v^2\right)dx=\oint_{\partial\Omega^{j_0}}v(\nu\cdot Tv)\,d\sigma$$

the surface integral being understood as a limit of integrals over interior boundaries. Since $|Tv|<1$, $|W|>1$, there follows

$$\int_{\Omega^{j_0}}(W+\varkappa v^2)\,dx<C_0<\infty\,;$$

thus $v\in Q(\Omega^{j_0})$.

Given two solutions u, v, we find

$$\int_\Omega \{[W_{p_i}(p) - W_{p_i}(q)]\,\zeta_i + \varkappa(u - v)\eta\}\, dx = 0$$

for any $\eta \in Q(\Omega)$. If, for some M, the set $\{x \in \Omega;\ 0 < u - v < M\}$ has positive measure, we proceed as follows: Choose $\varepsilon > 0$, and set

$$\lambda(r) = \begin{cases} 0, & r < \varepsilon \\ \dfrac{r - \varepsilon}{\varepsilon}, & \varepsilon \leqslant r \leqslant 2\varepsilon \\ 1, & r \geqslant 2\varepsilon . \end{cases}$$

Let r denote distance from the vertex, and choose

$$\eta = \begin{cases} 0, & u - v < 0, \\ \lambda(u - v), & 0 \leqslant u - v \leqslant M, \\ \lambda M, & M < u - v . \end{cases}$$

Then $\eta \in Q(\Omega)$ for each ε, and a formal estimation, using the bound $|W_{p_i}| < 1$, leads to a contradiction.

x) The notion of internal sphere condition can be introduced for general Lipschitz domains, by requiring that the condition be satisfied at all boundary points for which a normal vector is defined. We show now that *if Ω is a Lipschitz domain and if $\Omega \subset ICS_{\delta,\gamma_0}$, then for any $\hat{\gamma} > \gamma_0$, Σ can be approximated from within Ω by boundaries $\hat{\Sigma} \in C^\infty$, which bound domains $\hat{\Omega} \subset ICS_{\delta,\hat{\gamma}}$.*

PROOF. – If $\Omega \subset ISC_{\delta,\gamma_0}$, then Ω can be covered by a finite number of balls B_δ with the property: if we choose the origin at the center of any B_δ and let $F(x_1, \ldots, x_n) = 0$ be a representation of Σ within B_δ, there holds

$$\frac{x_i}{\delta}\frac{F_{x_i}}{|\nabla F|} = \cos\gamma \geqslant \cos\gamma_0$$

at almost all points $x \in \Sigma$ in B_δ: In particular, $r = |x| \geqslant \delta\cos\delta_0$, hence r is bounded from zero.

For any such x at which the normal to Σ is defined, let α be the angle between the radius vector from the origin to x, and the normal to Σ at x. Introducing now local spherical coordinates φ and representing Σ in the form $r = r(\varphi)$ near x, we find

$$\tan\alpha = \frac{1}{r}|\nabla r| = \frac{\sqrt{1 - \cos^2\alpha}}{\cos\alpha}.$$

But

$$\cos\alpha = \frac{x_i}{r}\frac{F_{x_i}}{|\nabla F|} = \frac{\delta}{r}\cos\gamma$$

hence

$$\frac{1}{r}|\nabla r| = \frac{\sqrt{(r/\delta)^2 - \cos^2\gamma}}{\cos\gamma} < \frac{\sqrt{(r/\delta)^2 - \cos^2\gamma_0}}{\cos\gamma_0}$$

which bounds $|\nabla r|$ uniformly in terms of the sphere condition.

The above relation yields

$$\cos^2\gamma = \left(\frac{r}{\delta}\right)^2 \frac{r^2}{r^2 + |\nabla r|^2}.$$

Set $|\nabla r| = \lambda$. For fixed λ, there holds

$$\left|\frac{\partial \cos^2\gamma}{\partial r^2}\right| \leqslant \frac{1}{\delta^2}. \tag{26}$$

Given $x \in \Sigma$ and $\varepsilon > 0$, denote by $V_{x;\varepsilon}$ the closed neighborhood $|\xi - x| \leqslant \delta^2\varepsilon$. We choose ε sufficiently small that $V_{x;\varepsilon} \subset B_\delta$, and set

$$\lambda_{x;\varepsilon} = \underset{\substack{|\xi - x| \leqslant \delta^2\varepsilon \\ \xi \in \Sigma}}{\text{l.u.b.}} \lambda(\xi).$$

In view of the above estimate for $|\nabla r|$ we conclude

$$\lambda_{x;\varepsilon} \leqslant \frac{(r + \delta^2\varepsilon)\sqrt{((r + \delta^2\varepsilon)/\delta)^2 - \cos^2\gamma_0}}{\cos\gamma_0},$$

hence

$$\cos^2\gamma_0 \leqslant \left(\frac{r + \delta^2\varepsilon}{\delta}\right)^2 \frac{(r + \delta^2\varepsilon)^2}{(r + \delta^2\varepsilon)^2 + \lambda_{x;\varepsilon}^2}$$

so that by (26)

$$\cos^2\gamma_0 - \varepsilon \leqslant \left(\frac{r}{\delta}\right)^2 \frac{r^2}{r^2 + \lambda_{x;\varepsilon}}$$

at all values of r in $V_{x;\varepsilon}$.

We may now choose a mollifier Q and φ-neighborhoods $U_{x;\varepsilon} \subset V_{x;\varepsilon}$ of the points $x \in \Sigma$, such that $\operatorname{supp} Q \subset U_{x;\varepsilon}$ and

$$\left|r(\varphi) - \int_{U_{x;\varepsilon}} Qr\, d\varphi\right| < \tfrac{1}{2}\delta^2\varepsilon$$

throughout $U_{x;\varepsilon/2}$. We set

$$\hat{r}(\varphi) = \int\limits_{U_{x;\varepsilon}} Qr\, d\varphi - \tfrac{1}{2}\delta^2 \varepsilon$$

in $U_{x;\varepsilon/2}$. The image of Σ under this mapping then lies interior to Ω, and

$$\nabla\hat{r} = -\int r \nabla Q\, d\varphi = \int Q \nabla r\, d\varphi$$

so that

$$|\nabla\hat{r}| \leqslant \sup_{U_{x;\varepsilon}} |\nabla r| \leqslant \lambda_{x;\varepsilon}\, .$$

It follows that on the image of $\Sigma \cap U_{x;\varepsilon/2}$ there holds

$$\cos^2\hat{\gamma} = \left(\frac{\hat{r}}{\delta}\right)^2 \frac{\hat{r}^2}{\hat{r}^2 + |\nabla\hat{r}|^2} \geqslant \left(\frac{\hat{r}}{\delta}\right)^2 \frac{\hat{r}^2}{\hat{r}^2 + \lambda_{x;\varepsilon}} \geqslant \cos^2\lambda_0 - \varepsilon\, .$$

A finite number of such neighborhoods $U_{x;\varepsilon}$ will cover Σ, and the proof is completed by piecing together the local mappings, using a partition of unity.

We conclude from the above result that *any Lipschitz domain Ω satisfying a sphere condition is admissible, and hence that Theorem* 1 *applies.* The criterion for existence of a solution is obtained not from the Lipschitz constant (as in the result of Emmer) but from the angle γ_0.

xi) For any Lipschitz domain satisfying a sphere condition, the procedure of this paper permits a simplification of Emmer's existence proof, in the sense that his result need be demonstrated only for domains with smooth boundary. In this situation, the technical difficulties arising in the proof of Emmer's basic Lemma 1.1 can be avoided, cf. the remarks in [4, pp. 133-4].

xii) We remark finally that if γ is smooth in some smooth neighborhood $\mathcal{N}$ on Σ, then γ is achieved in the strict sense in $\mathcal{N}$. We refer the reader to recent important contributions by URALTSEVA [13], by SPRUCK [12], by SIMON and SPRUCK [14] and by GERHARDT [15].

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