Corrigendum and Addendum
to
« Multifunctions on Abstract Measurable Spaces
and Applications to Stochastic Decision Theory » (*) ().**

C. J. HIMMELBERG - F. S. VAN VLECK (Lawrence, Kansas)

Summary. – *This note points out that the « only if » part of Theorem 1' (i) of the above paper [Annali di Matematica pura ed applicata, (IV), **101** (1974), pp. 229-236] is false as is part (ii) of Theorem 2. Counterexamples are given and a weak form of the « only if » part of Theorem 1' (i) is established.*

The purpose of this note is to point out an error, and its consequences, in the paper [HV-1] mentioned in the title, to give appropriate counterexamples, and to establish a correct, though weaker, result. The notation used here is that of the original paper.

First, the « only if » part of Theorem 1' (i) is not true. This part of the theorem asserted that an $(\mathcal{A}, \mathcal{C})$ -measurable multifunction F with complete values from an arbitrary measurable space $(T, \mathcal{A})$ to a separable metric space X has a Castaing representation, i.e. there is a countable set U of measurable selectors of F such that $F(t) = \overline{U(t)}$ for all $t \in T$. The « proof » does not follow from the « only if » part of the proof of [C, Theorem 5.4] as we stated. In fact, the following counterexample, due to NISHIURA [N], shows that the statement is false even if the measurable space $(T, \mathcal{A})$ is complete and X is a Polish space.

Before giving Nishiura's counterexample we should point out that this incorrect result would imply, among other things, that $(\mathcal{A}, \mathcal{C})$ -measurability is equivalent to weak measurability under the hypotheses of Theorem 1' (i). This result, which is false, has been reported, based upon our error, in the survey by D. WAGNER [W, Theorem 4.2 (e) ($xi \Rightarrow ix$)]. Another consequence would be that the intersection of countably many closed-valued, weakly measurable multifunctions is weakly measurable. That too is false; a counterexample will be included in [HPV].

EXAMPLE 1 [Nishiura]. – Let T be the set of irrational numbers and let $A \in \mathcal{A}$ iff A is residual (cf. [K, p. 201]) in T or coresidual in T . Let $\mu(A) = 1$ if A is residual

(*) Entrata in Redazione il 16 novembre 1977.

(**) Partially supported by National Science Foundation Grant MCS 76-24436.

and $\mu(A) = 0$ if A is coresidual. Then $(T, \mathcal{A}, \mu)$ is a finite, complete measure space. Let $X = T$ and let $F: T \rightarrow X$ be the identity function. Then $F^{-1}(K)$ is coresidual for each compact $K \subset X$ and hence F is $(\mathcal{A}, \mathcal{C})$ -measurable. But F is not measurable and so can have no measurable selectors and, hence, no Castaing representation.

We can, however, establish the following weaker result if we assume the continuum hypothesis. (Recall from [HV-2] that $F: T \rightarrow X$ is $(\mathcal{R}, \mathcal{C})$ measurable iff $F^{-1}(B) \in \mathcal{R}$ whenever B is compact.)

THEOREM. — Assume the continuum hypothesis. Let T be a set with a σ -ring $\mathcal{R}$, let X be separable metric, and let $F: T \rightarrow X$ be a multifunction with closed, non-empty values. If F is $(\mathcal{R}, \mathcal{C})$ -measurable, then there is a countable family U of $(\mathcal{R}, \mathcal{C})$ -measurable selectors for F such that $F(t) = \overline{U(t)}$ for each $t \in T$.

PROOF. — Let $D = \{x_m | m = 1, 2, \dots\}$ be dense in X and let $B_{mn} = B(x_m, 1/n)$ be the closed ball of radius $1/n$ about x_m . Let $u_{mn}: T \rightarrow X$ be an $(\mathcal{R}, \mathcal{C})$ -measurable selector for F such that $u_{mn}(t) \in B_{mn}$ whenever $B_{mn} \cap F(t) \neq \emptyset$. (That such functions u_{mn} exist is established in the lemma below.) Then $U = \{u_{mn} | m, n = 1, 2, \dots\}$ is the desired family. For let $t \in T$, $x \in F(t)$ and $\varepsilon > 0$. Choose n and m such that $2/n < \varepsilon$ and $d(x_m, x) < 1/n$. Then $x \in B_{mn} \subset B(x, \varepsilon)$ and $u_{mn}(t) \in B_{mn} \cap F(t)$; so $d(u_{mn}(t), x) < \varepsilon$. Thus $U(t)$ is dense in $F(t)$.

LEMMA. — Assume the continuum hypothesis. Let B be a closed ball in a separable metric space X , let $(T, \mathcal{R})$ be a set with a σ -ring $\mathcal{R}$, and let $F: T \rightarrow X$ be an $(\mathcal{R}, \mathcal{C})$ -measurable multifunction with closed values. Then there exists an $(\mathcal{R}, \mathcal{C})$ -measurable selector f for F such that $f(t) \in B$ whenever $F(t) \cap B \neq \emptyset$.

PROOF. — Let $S = F^{-1}(B)$ and define $F_1: S \rightarrow X$ by $F_1(t) = F(t) \cap B$ for each $t \in S$. Then F_1 is $(\mathcal{R}_1, \mathcal{C})$ -measurable (where $\mathcal{R}_1 = \{A \in \mathcal{R} | A \subset S\}$), since, for each compact $K \subset X$, $F_1^{-1}(K) = F^{-1}(B \cap K)$. Now let $f_1: S \rightarrow X$ be an $(\mathcal{R}_1, \mathcal{C})$ -measurable selector for $F_1|_S$, and let $f_2: T \rightarrow X$ be an $(\mathcal{R}, \mathcal{C})$ -measurable selector for F . Define $f: T \rightarrow X$ by

$$f(t) = \begin{cases} f_1(t) & \text{if } t \in S, \\ f_2(t) & \text{if } t \in T - S. \end{cases}$$

Then f is the desired selector. It is $(\mathcal{R}, \mathcal{C})$ -measurable; for if K is compact, then

$$f^{-1}(K) = f_1^{-1}(K \cap B) \cup f_2^{-1}(K \cap (X - B)),$$

which belongs to $\mathcal{R}$, since $f_1^{-1}(K \cap B) \in \mathcal{R}_1 \subset \mathcal{R}$ by the compactness of $K \cap B$, and $f_2^{-1}(K \cap (X - B)) \in \mathcal{R}$ by the σ -compactness of $K \cap (X - B)$.

The «if» part of Theorem 1' (i) is true since weak measurability, and hence $(\mathcal{A}, \mathcal{C})$ -measurability, follows from a Castaing representation by measurable func-

tions. However, the converse of the theorem of this note is not true as the following counterexample shows.

EXAMPLE 2. — Let $\{r_n | n = 1, 2, \dots\}$ be the set of non-negative rational numbers, and let T and X be as in Example 1. For each n , define a function $u_n: T \rightarrow X$ by

$$u_n(t) = t + r_n \quad \text{for } t \in T.$$

Define $F: T \rightarrow X$ by

$$F(t) = \cup \{u_n(t) | n = 1, 2, \dots\}.$$

Then each u_n is $(\mathcal{A}, \mathbb{C})$ -measurable, but F is not since, for example, $F^{-1}(\{x\})$ is the segment $(-\infty, x] \cap T$, which is neither residual nor coresidual.

One of the consequences of Theorem 1' (i) in [HV-1] was Theorem 2 (ii), which asserted that $F_1 + F_2$ is an $(\mathcal{A}, \mathbb{C})$ -measurable multifunction when both F_1 and F_2 are $(\mathcal{A}, \mathbb{C})$ -measurable. This is also false as the following example shows.

EXAMPLE 3. Let T and X be as in Example 1 and embed X as a closed subset of a Banach space E using, for example, the Kuratowski-Wojdyslawski embedding theorem (cf. [Hu, p. 81]). (We can get X closed in E , by giving X a complete metric before embedding it in E .) Let $F_1: T \rightarrow E$ be as in Example 1. Let $0 \neq b \in E$ and define $F_2: T \rightarrow E$ by

$$F_2(t) = \begin{cases} -F_1(t) & \text{for } t \in T^- = \{t | t < 0\} \\ b - F_1(t) & \text{for } t \in T^+ = \{t | t > 0\}. \end{cases}$$

(The algebraic operations in the definition of F_2 are those of E and not those of X) Then, for each compact $K \subset E$,

$$\begin{aligned} F_2^{-1}(K) &= \{t | F_2(t) \in K\} = \{t < 0 | -F_1(t) \in K\} \cup \{t > 0 | b - F_1(t) \in K\} \\ &= [T^- \cap F_1^{-1}(-K)] \cup [T^+ \cap F_1^{-1}(b - K)]. \end{aligned}$$

Since K compact implies $-K$ and $b - K$ are compact, $F_2^{-1}(K)$ is a union of two sets each of which is coresidual since each is a subset of a coresidual set. Thus F_2 is $(\mathcal{A}, \mathbb{C})$ -measurable. The function $F_1 + F_2$ is not $(\mathcal{A}, \mathbb{C})$ -measurable since it is the step function whose value for $t < 0$ is zero and whose value for $t > 0$ is b .

REMARK. — This example can not be simplified by regarding the function $F_1 = F: T \rightarrow X$ of Example 1 as a function from T into R^1 and then proceeding as in Example 3 with E replaced by R^1 . The trouble is that F is $(\mathcal{A}, \mathbb{C})$ -measurable as a map into X , but not as a map into R^1 .

The remaining theorems in [HV-1] which appear to depend on Theorem 1' (i) are true. For instance, Theorem 3 (ii) is true. The proof follows from [Hi, Corol-

lary 4.3]. Theorem 4 (ii) is also true (assuming the continuum hypothesis). The only substantive change that is necessary to complete the proof is to use either the Theorem of this paper or [HV-2, Theorem 3] to obtain a $(\mathcal{A}, \mathcal{C})$ -measurable selector for $K \cap I$.

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